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ELEMENTARY CALCULUS IN CHEVALLEY GROUPS OVER RINGS

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ABSTRACT. The article studies structure theory of Chevalley groups over commutative rings. Main results of the article are relative dilation and local-global principles. and an economic set of generators of relative elementary subgroup. These statements proved by computations with elementary unipotents (hence the title) are very important in further development of the subject. No restrictions on the ground ring or the root system Φ are imposed except that the rank of Φ is not less than 2. The results improve previous results in the area. The article contains a brief survey of the subject, some gaps in proofs or incorrect references are discussed. Proofs of some known related results are substantially simplified.

Key words: Chevalley groups, principal congruence subgroup, local global principle, dilation principle.
AMS SUBJECT: 20G35.

In memory of my friend and excellent mathematician Oleg Izhboldin

Важно не только, что доказано,
но, также, кто и кому это доказал.
Нет никакого сомнения, что Воеводский
доказал Суслину гипотезу Милнора.
talk at POMI, O.Izhboldin

Ну что, закончил свою статью?
Давай найду ошибку.
O.Izhboldin

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Introduction

The article is to show which computations are really necessary to build up structure theory of Chevalley groups over rings. We prove important technical statements including generators of relative elementary subgroup, dilation principle (clearing denominators), and Suslin's local-global principle. The proof of relative dilation principle is a little bit more carefull than in [5]. This allows us to remove unnecessary assumptions on invertibility of structure constants. Theorem 3.4 is a common generalization of the results of L. Vaserstein [35], W. van der Kallen [18], and H. Apte, P. Chattopadhyay, and R. Rao [4] on generators of the relative elementary subgroup. The article contains a survey of previous results on the subject.

The relative dilation principle is extremely important for such applications as multiple commutator formulas and nilpotent structure of relative K_1^G. This is shown in author's recent work [27] without further computations with individual elements of Chevalley groups. Using methods of the current article and [27] we plan to obtain similar results for quasi-split groups defined by congruences, e. g. general unitary groups or congruence subgroups corresponding to admissible pairs in the sense of E. Abe and K. Suzuki [3]. Theorem 3.4 seems to be an important tool for stabilization and prestabilization results for lower K-functors modeled on Chevalley groups.

By elementary computations we mean computations in the Steinberg group of a Chevalley group. In other words, elementary computations use only the Chevalley commutator formula. There are (at least) two more questions in the theory of Chevalley groups over rings that require elementary computations. First is a proof that elementary subgroup is perfect and similar results. Second are level computations, i. e. finding the largest subgroup generated by root unipotents inside a given normal subgroup. This kind of computations was carried out in the paper [25] of M. Stein and were further developed in [3, 2, 16]. Since we cannot improve this kind of elementary computations, we do not touch them in the current article.

The first section contains preliminary results on properties of root systems. All other sections contain results mentioned above with detailed and simplified proofs followed by a survey. We also discuss gaps in the proofs in related articles. This is done to avoid copying mistakes in future research. Some of the results have been already published, some of them several times, but we believe that our simplified proofs will be useful for further developments. Proofs of published statements are given only in case they differ from the published ones. Everything is proved without any assumptions of invertibility of structure constants. *The only blanket assumption of the current article is that the root system has rank at least* 2. In the survey parts we also mention results on isotropic reductive group as they are direct generalization of Chevalley groups.

But we do not comment the proofs of this results as technically they are much more difficult.

In sections 2 and 3 the functor E can be replaced by the Steinberg group functor $\mathrm{St}(\Phi, \_)$. The only place where we use the inclusion $E \hookrightarrow G$ is Corollary 2.2. But the proof of this statement can be extracted directly from the proof of Lemma 2.1 without a reference to the inclusion.

Notation. All rings in the article are assumed to be commutative with unity and all ring homomorphisms preserve the unity. For an ideal $\mathfrak{a}$ of a ring R by $\mathfrak{a}^m$ we denote the ideal in R, generated by all products $r_1 \dots r_m$, where $r_1, \dots, r_m \in \mathfrak{a}$. In a sequel we need to distinguish between $\mathfrak{a}^m$ and $\mathfrak{a}^{\boxed{m}}$, that denotes the ideal generated by r^m for all $r \in \mathfrak{a}$. The reduction homomorphism $R \to R/\mathfrak{a}$ is denoted by $\rho_{\mathfrak{a}}$.

For a multiplicative subset S of A we denote by $S^{-1}A$ the localization of A at S. The localization homomorphism $A \to S^{-1}A$ is denoted by λ_S. In the current article we mostly use two kinds of localizations.

- Principal localizations, where $S = \{s, s^2, \dots\}$ for some element $s \in A$. Principal localization at S is denoted by A_s and the corresponding localization homomorphism by λ_s.
- Localizations at a maximal ideal. This means that $S = A \setminus \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of A. In this case localization is denoted by $A_{\mathfrak{m}}$ and the localization homomorphism by $\lambda_{\mathfrak{m}}$.

Let G be an algebraic group and $\varphi : R \to R'$ a ring homomorphism. By abuse of notation the induced group homomorphism $G(\varphi) : G(R) \to G(R')$ is usually denoted by φ. This cannot lead to a confusion as one can distinguish between different meanings of φ by the type of its argument.

Let F and H be subgroups of an abstract group G. We follow standard group-theoretical notation.

- If $a, b \in G$, then $[a, b] = a^{-1}b^{-1}ab$ denotes their commutator whereas $a^b = b^{-1}ab$ stands for the conjugate to a by b.
- F^H denotes the normal closure of F by H, i. e. the smallest subgroup, containing F and normalized by H. It is generated by all elements f^h, $f \in F$, $h \in H$.
- $[F, H]$ is the mixed commutator subgroup, i. e. the subgroup generated by commutators $[a, b]$ for all $a \in F$ and $b \in H$.

Throughout the article Φ denotes a reduced irreducible root system of rank greater than 1. By $W = W(\Phi)$ we denote the Weyl group of Φ. Let $G = \mathrm{G}_P(\Phi, \_)$ denote a Chevalley-Demazure group scheme with a root system Φ and a weight lattice P and E its elementary subgroup functor. We usually suppress the weight lattice from the notation as it makes no difference in the current considerations.

Denote by Φ_ℓ (resp. Φ_s) the set of all long (resp. short) roots in Φ (if Φ is simply laced, all roots are called long). Clearly, Φ_ℓ is a subsystem whereas Φ_s is not a closed subset of roots.

A proper closed subset $\Delta \subset \Phi$ is called parabolic if $\Delta \cup (-\Delta) = \Phi$. In this case $\Delta = \Lambda \cup \Sigma$, where $\Lambda = -\Lambda$ is the symmetric (reductive) part of Δ whereas Σ is its special part, i. e. $\Lambda \cap (-\Lambda) = \varnothing$. It is well known that Δ contains the set of positive roots Φ^+ for a unique ordering on Φ and that $\Sigma \subseteq \Phi^+$ is generated by a set of simple roots. It easy to see that Σ is an ideal in Δ, i. e. $\Phi \cap (\Sigma + \Delta) \subseteq \Sigma$.

Fix an ordering on Φ and let Π be the set of simple roots. By $\operatorname{ht}\alpha = \operatorname{ht}_\Pi \alpha$ we denote the height of a root $\alpha \in \Phi^+$ with respect to Π. By convection $\operatorname{ht}(-\alpha) = \operatorname{ht}\alpha$. For roots $\alpha \in \Pi$ and $\beta \in \Phi$ we write $m_\alpha(\beta)$ to denote the coefficient at α in decomposition of β as a linear combination of simple roots. Sometimes we need to include the ordering on Φ to this notation. In this case we write $m^\Sigma_\alpha(\beta)$ instead of $m_\alpha(\beta)$, where Σ is a set of positive roots.

From the very beginning we fix a split maximal torus T of G. All root subgroups and parabolic subgroups are assumed to correspond to this torus. Note that we do not fix an ordering on the root system. Thus, different parabolic subgroup may contain different Borel subgroups. Throughout the text all parabolic subgroups are assumed to be proper. For a parabolic subgroup P we denote by L_P (resp. U_P) the Levi subgroup (resp. unipotent radical) of P. The opposite parabolic subgroup is denoted by P^- and $U_P^- = U_{P^-}$. The elementary subgroup of L_P is denoted by EL_P.

If an ordering on Φ is chosen and α is a simple root, then by P_α we denote the maximal parabolic subgroups corresponding to the subsystem $\Pi \setminus \{\alpha\}$, i. e. the parabolic subgroup, containing all root subgroups X_β with $m_\alpha(\beta) \geqslant 0$. The Levi subgroup (unipotent radical) of P_α is denoted by L_α (resp. U_α). Put $P_\alpha^- = P_{-\alpha}$ and $U_\alpha^- = U_{-\alpha}$. Clearly, $X_\beta \leqslant L_\alpha$ iff $m_\alpha\beta = 0$ and $X_\beta \leqslant U_\alpha$ iff $m_\alpha\beta > 0$.

1. Some properties of root systems

Before developing important technical tools we obtain some simple properties of root systems. Let Φ be a reduced irreducible root system.

Lemma 1.1. *Let V be a real vector space spanned by an irreducible root system Φ. Then Φ is not contained in the union of two proper subspaces of V.*

Proof. Suppose in contrary that there exist $V_1, V_2 \leqslant V$ such that $\Phi \subseteq V_1 \cup V_2$. Put $\Phi_1 = \Phi \cap V_1$ and $\Phi_2 = \Phi \setminus \Phi_1$. Let U_i be the span of Φ_i $(i = 1, 2)$. Then $\Phi \subseteq U_1 \cup U_2$. Since Φ is irreducible, $U_1 \not\perp U_2$. It follows that there exists $\beta \in \Phi_1$ that is not orthogonal to U_2. Hence, $w_\beta(U_2) \neq U_2$. Since Φ_2 spans U_2, there exists $\alpha \in \Phi_2$ such that $w_\beta(\alpha) \notin U_2$. Therefore, $w_\beta(\alpha) = \alpha - 2\frac{(\beta|\alpha)}{(\beta|\beta)}\beta \in U_1$ and $\alpha \in U_1$. But this contradicts the choice of Φ_2. □

Lemma 1.2. *Given $\alpha \neq \beta \in \Phi$ there exists an ordering on Φ such that $\alpha \in \Phi^+$ and $\beta \in \Phi^-$.*

Proof. A hyperplane, orthogonal to an internal vector of the fundamental chamber, separates positive and negative roots. The space spanned by Φ is divided in a union of the Weyl chambers. Therefore, any hyperplane that does not contain roots separate positive and negative roots for some choice of an ordering on Φ. Thus, it suffices to take an appropriate hyperplane separating two given roots. □

Lemma 1.3. *Suppose that $\Phi \neq A_1$. Fix an ordering on Φ and let Γ be a proper subset of Φ, containing all positive roots. Then there exists $\alpha \in \Phi \setminus \Gamma$ and a parabolic set Δ of roots (possibly with respect to another ordering) such that α belongs to its symmetric part whereas its special part is contained in Γ.*

Proof. For $\beta \in \Phi$ denote by $s_\beta \in W$ the reflection through the hyperplane orthogonal to β. Recall that if β is a simple root, then $s_\beta(\Phi^+) = \Phi^+ \setminus \{\beta\} \cup \{-\beta\}$ [10, ch.VI, § 6, Corollary 1 of Proposition 17].

Denote by C the fundamental chamber. Let $\gamma \in \Phi \setminus \Gamma$. There exists a chamber C' such that $-\gamma$ is a simple root with respect C'. We claim that there exists a sequence of roots $\beta_1, \ldots, \beta_n$ satisfying the following properties.

- Let $w_k = \prod_{i=1}^{k} s_{\beta_i}$. The root β_{k+1} is a simple root with respect to the chamber $w_k(C)$, where $k = 0, \ldots, n-1$.
- $w_n(C) = C'$.

Indeed, the reflection $s_{\beta_{k+1}}$ is a reflection through an arbitrary wall of the chamber $w_k(C)$. Clearly, by a sequence of such reflections we can move one given chamber to another.

Put $\beta_{n+1} = -\gamma$. Then $-\beta_{n+1} \notin \Gamma$. Let j be the smallest number such that $-\beta_{j+1} \notin \Gamma$. Using the remark above one shows by induction on i that $\beta_i\big(w_{i-1}(\Phi^+)\big) \subseteq \Gamma$ for all $i \leqslant j$. In particular $\Sigma = w_j(\Phi^+) \subseteq \Gamma$. Let $\beta \neq \beta_{j+1}$ be a simple root with respect to the chamber $w_j(C)$ (here we employ the condition $\Phi \neq A_1$). Take $\alpha = -\beta_{j+1}$ and $\Delta = \{\delta \in \Phi \mid m^\Sigma_\beta \delta \geqslant 0\}$. Clearly $m^\Sigma_\beta \alpha = 0$, hence α belongs to the symmetric part of Δ. On the other hand, the special part of Δ lies in $\Sigma \subseteq \Gamma$ as required. □

2. Span of unipotent radicals of opposite parabolic subgroups

Next statement is the only place of the article where we use the classification of root systems.

Lemma 2.1. *Let P be a parabolic subgroup and $\mathfrak{a}$, $\mathfrak{b}$ ideals of a ring R. Let $\mathfrak{c} = \mathfrak{a}^{2}\mathfrak{b} + 2\mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{b}^{2}$ if $\Phi = C_l$, and $\mathfrak{c} = \mathfrak{a}\mathfrak{b}$ otherwise. Then $\mathrm{E}(\mathfrak{c}) \leqslant \langle U_P(\mathfrak{a}), U^-_P(\mathfrak{b})\rangle$.*
In any case, if $\mathfrak{a} = R$, then $\mathfrak{c} = \mathfrak{b}$.

Proof. Put $H = \langle U_P(\mathfrak{a}), U_P^-(\mathfrak{b})\rangle$. We must show that $x_\alpha(r) \in H$ for all $r \in \mathfrak{c}$ and $\alpha \in \Phi$. Let $\Lambda = \{\beta \in \Phi \mid X_\beta \leqslant L_P\}$ and $\Sigma = \{\beta \in \Phi \mid X_\beta \leqslant U_P\}$, so that Φ is the disjoint union of $-\Sigma$, Σ, and Λ. There is nothing to prove if $\alpha \in \pm\Sigma$.

Let $\alpha \in \Lambda$. If $\Phi \neq C_l$, then Φ_ℓ is an irreducible subsystem of the same rank as Φ. The span of subsystem Λ is a proper subspace in the span of Φ as well as $\alpha^\perp$. By Lemma 1.1, Φ_ℓ is not contained in the union of these subspaces, i.e. there exists a long root $\beta \in \pm\Sigma$ such that $(\beta \mid \alpha) > 0$. Then, $\gamma = \alpha - \beta \in \Phi$, so that $\alpha = \beta + \gamma$. On the other hand, $\beta - \gamma \notin \Phi$, otherwise $|\beta \pm \gamma| \leqslant |\beta|$, i.e. the length of the median is not less than the lengths of both adjacent sides, that is impossible. This shows that the structure constant of the corresponding simple Lie algebra $c_{\beta,\gamma} = \pm 1$, therefore $N_{\beta,\gamma,1,1} = c_{\beta,\gamma} = \pm 1$ (see [11]). Without loss of generality we may assume that $\beta \in \Sigma$. Since Λ normalizes $\pm\Sigma$ (i.e. $\Phi \cap (\Lambda \pm \Sigma) = \pm\Sigma$), we have $\gamma \in -\Sigma$. By Chevalley commutator formula we obtain

$$[x_\beta(\pm p), x_\gamma(r)] = x_\alpha(pr) \prod x_{i\beta+j\gamma}(N_{\beta,\gamma,i,j} p^i r^j),$$

where the product is taken over all $i, j > 0$ such that $i+j > 2$ and $i\beta + j\gamma \in \Phi$. If $p \in \mathfrak{a}$ and $r \in \mathfrak{b}$, then the left-hand side of the above formula belongs to H. Since Φ is reduced, $i\beta + j\gamma \notin \Phi$ for $i = j > 1$. Therefore, $i\beta + j\gamma = i\alpha + (j - i)\gamma \in \pm\Sigma$. It follows that each factor of the product belongs to H, hence $x_\alpha(pr) \in H$.

It remains to consider the case $\Phi = C_l$. By Lemma 1.1 there exists $\beta \in \Phi$ outside $\langle\Lambda\rangle \cup \alpha^\perp$. As well as in the previous paragraph we may assume that $(\beta \mid \alpha) > 0$ and $\beta \in \Sigma$. Put $\gamma = \alpha - \beta \in -\Sigma$ and take arbitrary $p \in \mathfrak{a}$ and $r \in \mathfrak{b}$. If both β and α are short, then the angle between them is $\pi/3$ and $x_\alpha(pr) = [x_\beta(\pm p), x_\gamma(r)] \in H$. If β is long, then we can argue as in the previous paragraph to get $x_\alpha(pr) \in H$. In the remaining case $\alpha \in \Phi_\ell$ and $\beta \in \Phi_s$ they generate subsystem of type C_2. Then $x_\alpha(2pr) = [x_\beta(\pm p), x_\gamma(r)] \in H$. Clearly, $\beta - \gamma \in \Sigma$ is a long root. Therefore, $x_\alpha(pr^2) = [x_{\beta-\gamma}(\pm p), x_\gamma(r)]x_\beta(\pm pr) \in H$ and similarly $x_\alpha(p^2 r) \in H$. □

Corollary 2.2. *With the notation of the previous lemma we have*

$$\mathrm{E}(\mathfrak{c}) \leqslant [U_P(\mathfrak{a}), U_P^-(\mathfrak{b})] U_P(\mathfrak{ab}) U_P^-(\mathfrak{ab}).$$

Proof. Clearly, for subgroups K, L of a group H we have $\langle K, L\rangle = [K, L]KL$. By the lemma any element of $c \in \mathrm{E}(\mathfrak{c})$ can be written as $c = [a', b']ab$ for some $a, a' \in U_P(\mathfrak{a})$ and $b, b' \in U_P^-(\mathfrak{b})$. Since $[G(R, \mathfrak{a}), G(R, \mathfrak{b})] \leqslant G(R, \mathfrak{ab})$ (see [38] for the case $G = \mathrm{GL}_n$ and [5] or [27] for the general case) and $\mathfrak{c} \subseteq \mathfrak{ab}$, we have $ab \in G(R, \mathfrak{ab})$. It follows that $\rho_{\mathfrak{ab}}(a)\rho_{\mathfrak{ab}}(b) = 1$, hence $\rho_{\mathfrak{ab}}(a) = \rho_{\mathfrak{ab}}(b) = 1$ as $U_P \cap U_P^-$ is trivial. This means that $a \in U_P(R) \cap G(R, \mathfrak{ab}) = U_P(\mathfrak{ab})$ and, similarly, $b \in U_P^-(\mathfrak{ab})$. □

Elementary calculations similar to the proof of Lemma 2.1 was developed in all articles cited below. One of the first source of elementary calculus in Chevalley groups over rings is paper [25] by M. Stein. However, Stein computations have different goals – level computation and perfectness of the elementary subgroup. In some sense they are more precise than computations in Lemma 2.1. In particular, there are more exceptional cases there: root systems C_2 and G_2 over a ring, having a residue field of 2 elements. Perfectness of the elementary subgroup of an isotropic reductive group was obtained by A. Luzgarev and A. Stavrova in [21] under natural assumptions.

Recently, there were new developments in elementary calculus due to R. Hazrat, N. Vavilov, Zhang Zuhong, and the author [15, 26, 14, 16, 17]. In particular, in HVZrelachev the authors proved the inclusion

$$E(R,\mathfrak{a}\mathfrak{b})\leqslant[E(R,\mathfrak{a}),E(R,\mathfrak{b})].$$

In our terminology this means that $[E(R,\mathfrak{a}),E(R,\mathfrak{b})]=EE(R,\mathfrak{a},\mathfrak{b})$ (see section 4 for the definition of EE). A new exception occurs in this result: $\Phi=C_l$, if 2 is not invertible in R.

3. Generators of relative elementary subgroup

Let $\mathfrak{a}$ be an ideal of R. Put $z_\alpha(p,r)=x_\alpha(p)^{x_{-\alpha}(r)}$. For $\Sigma\subseteq\Phi$ denote by $H^\Sigma(\mathfrak{a})$ the group generated by $E(\mathfrak{a})$ and all $z_\alpha(p,r)$, where α ranges over Σ, $p\in\mathfrak{a}$, and $r\in R$. Put $H(\mathfrak{a})=H^\Phi(\mathfrak{a})$. If an ordering on Φ is chosen, then we denote $H^+(\mathfrak{a})=H^{\Phi^+}(\mathfrak{a})$.

In this section we prove that $H^\Sigma(\mathfrak{a})=E(R,\mathfrak{a})$ provided that Σ is the special part of a parabolic set of roots. The following lemma is a preparation for this result.

Lemma 3.1.

(i) *If $\beta\neq-\alpha\in\Phi$, then $x_\alpha(p)^{x_\beta(r)}\in E(\mathfrak{a})$ for all $p\in\mathfrak{a}$, and $r\in R$.*
(ii) *$EL_P(R)$ normalizes $U_P(\mathfrak{a})$ and $U_P^-(\mathfrak{a})$.*
(iii) *$E(\mathfrak{a})^{x_\gamma(r)}\leqslant H^+(\mathfrak{a})$ for all $\gamma\in\Phi^-$ and $r\in R$.*
(iv) *Let Σ be the special part of a parabolic set of roots. The group $H^\Sigma(\mathfrak{a})$ is normalized by $X_\alpha(R)$ for all $\alpha\in-\Sigma$. In particular, $U^-(R)$ normalizes $H^+(\mathfrak{a})$.*

Proof. The first assertion follows easily from Chevalley commutator formula and immediately implies items (ii) and (iii).

Let $p\in\mathfrak{a}$ and $r,s\in R$. To prove item (iv) it suffices to show that the conjugate to each generator of $H^\Sigma(\mathfrak{a})$ by $x_\beta(s)$ lies in $H^\Sigma(\mathfrak{a})$ for all $\beta\in-\Sigma$. For generators $x_\alpha(p)$ of $E(\mathfrak{a})$ the statement follows from (i).

Let $\alpha\in\Sigma$, $p\in\mathfrak{a}$, $r,s\in R$, and n the height of the maximal root. We prove by induction on $n-\operatorname{ht}\beta$ that $z_\alpha(p,r)^{x_\beta(s)}=x_\alpha(p)^{x_{-\alpha}(r)x_\beta(s)}\in H^\Sigma(\mathfrak{a})$.

Note that it is nothing to prove if $\beta = -\alpha$. Otherwise, $x_{-\alpha}(r)x_\beta(s) = x_\beta(s)x_{-\alpha}(r)\prod_\gamma x_\gamma(r_\gamma)$, where for each factor of the product $\operatorname{ht}\gamma > \operatorname{ht}\beta$ and $r_\gamma \in R$. If β is the maximal negative root (base of induction), then the product is empty, otherwise, by induction hypothesis the product normalizes $H^+(\mathfrak{a})$. In both cases it suffices to show that $x_\alpha(p)^{x_\beta(s)x_{-\alpha}(r)} \in H^+(\mathfrak{a})$, where $\beta \neq -\alpha$. But this follows immediately from items (i) and (ii). □

The next statement is weaker than the main result of this subsection but it is a step towards the proof of the latter.

Proposition 3.2 (Vaserstein [35])**.** *The relative elementary group $E(R, \mathfrak{a})$ is generated by $z_\alpha(p, r)$ for all $\alpha \in \Phi$, $p \in \mathfrak{a}$, and $r \in R$.*

Proof. With the notation given at the beginning of this subsection we must prove that $E(R, \mathfrak{a}) = H(\mathfrak{a})$. Obviously, $E(\mathfrak{a}) \leqslant H(\mathfrak{a}) \leqslant E(R, \mathfrak{a})$. Therefore it suffices to show that $H(\mathfrak{a})$ is normal in $E(R)$, i. e. $z_\alpha(p, r)^{x_\beta(s)} \in H(\mathfrak{a})$ for all $\alpha, \beta \in \Phi$, $p \in \mathfrak{a}$ and $r, s \in R$. If $\beta \neq \alpha$, then by Lemma 1.2 there exists an ordering of Φ such that $\alpha \in \Phi^+$ and $\beta \in \Phi^-$. In this case the result follows from item (iv) of the previous lemma.

If $\beta = \alpha$, choose a system of simple roots Π containing α. Since $\operatorname{rank}\Phi > 1$, there exists $\gamma \in \Phi$ distinct from α. By Corollary 2.2

$$x_\alpha(p) \in [U_\gamma(\mathfrak{a}), U_\gamma^-(R)]U_\gamma(\mathfrak{a})U_\gamma^-(\mathfrak{a})$$

. Since $\gamma \neq \alpha \in \Pi$, we have $x_{-\alpha}(r)x_\alpha(s) \in EL_\gamma(R)$, hence it normalizes subgroups $U_\gamma^\pm(\mathfrak{a})$ and $U_\gamma^\pm(R)$. Therefore, $z_\alpha(p, r)^{x_\alpha(s)} = x_\alpha(p)^{x_{-\alpha}(r)x_\alpha(s)} \in [U_\gamma(\mathfrak{a}), U_\gamma^-(R)]U_\gamma(\mathfrak{a})U_\gamma^-(\mathfrak{a}) \leqslant [H^+(\mathfrak{a}), U^-(R)]E(\mathfrak{a})$. Finally, item (iv) of the previous lemma shows that the latter group is contained in $H^+(\mathfrak{a}) \leqslant H(\mathfrak{a})$. □

The elementary subgroup $E(\mathfrak{a})$ usually is strictly smaller than the (normal) relative elementary subgroup $E(R, \mathfrak{a})$. The next statement shows that it still contains some relative elementary subgroup.

Corollary 3.3 (Vaserstein [35])**.** *If $\Phi \neq C_l$, then $E(R, \mathfrak{a}^2) \leqslant E(\mathfrak{a})$, otherwise, $E(R, \mathfrak{a}\mathfrak{a}^{\boxed{2}}) \leqslant E(\mathfrak{a})$.*

Proof. Let $\mathfrak{c} = \mathfrak{a}^2$ if $\Phi \neq C_l$ and $\mathfrak{c} = \mathfrak{a}\mathfrak{a}^{\boxed{2}}$ otherwise. By Proposition 3.2 it suffices to show that $x_\alpha(p)^{x_{-\alpha}(r)} \in E(\mathfrak{a})$ for all $\alpha \in \Phi$, $p \in \mathfrak{c}$, and $r \in R$. Choose a system Π of simple roots, containing α, and let $\alpha \neq \beta \in \Pi$. Then $X_{-\alpha}(R) \leqslant EL_\beta(R)$. By Lemma 2.1 $x_\alpha(p) \in \langle U_\beta(\mathfrak{a}), U_\beta^-(\mathfrak{a})\rangle$. By Lemma 3.1(ii) $x_{-\alpha}(r)$ normalizes both $U_\beta(\mathfrak{a})$ and $U_\beta^-(\mathfrak{a})$, which implies the result. □

Theorem 3.4. *Let Σ be the special part of a parabolic set of roots. The relative elementary group $E(R, \mathfrak{a})$ is generated by $E(\mathfrak{a})$ and $z_\alpha(p, r)$ for all $\alpha \in \Sigma$, $p \in \mathfrak{a}$, and $r \in R$.*

Proof. We have to prove that $E(R,\mathfrak{a}) = H^{\Sigma}(\mathfrak{a})$. The idea of the proof is due to W. van der Kallen [18, Lemma 2.2]. Let Ψ be a parabolic set of roots and Ω its special part. First, we show that $H^{\Psi}(\mathfrak{a}) = H^{\Omega}(\mathfrak{a})$. It suffices to prove that $z_\alpha(p,r) \in H^{\Omega}(\mathfrak{a})$ for all $p \in \mathfrak{a}$, $r \in R$, and $\alpha \in \Lambda$, where Λ is the symmetric part of Ψ. Let P be a parabolic subgroup, corresponding to Ψ (i. e. $X_\beta \leqslant P$ iff $\beta \in \Psi$). Note that the unipotent radical U_P is generated by X_β, were β ranges over Ω. By Corollary 2.2 $x_\alpha(p) \in [U_P(\mathfrak{a}), U_P^-(R)]U_P(\mathfrak{a}), U_P^-(\mathfrak{a})$. By item (ii) of the previous lemma $x_{-\alpha}(r)$ normalizes subgroups $U_P(\mathfrak{a})$, $U_P^-(\mathfrak{a})$, and $U_P^-(R)$. Therefore,

$$z_\alpha(p,r) = x_\alpha(p)^{x_{-\alpha}(r)} \in [U_P(\mathfrak{a}), U_P^-(R)]E(\mathfrak{a}),$$

and the latter group is contained in $H^{\Omega}(\mathfrak{a})$ by item (iv) of the previous lemma.

Let Δ be the set of all $\alpha \in \Phi$ such that $H^{\{\alpha\}}(\mathfrak{a}) \leqslant H^{\Sigma}(\mathfrak{a})$. In other words, Δ is the largest subset of Φ such that $H^{\Delta}(\mathfrak{a}) \leqslant H^{\Sigma}(\mathfrak{a})$. Clearly $\Sigma \subseteq \Delta$ and by the first paragraph of the proof Δ contains a parabolic set of roots. Choose an ordering on Φ such that $\Phi^+ \subseteq \Delta$. Suppose that Δ is a proper subset. By Lemma 1.3 there exists a root $\alpha \notin \Delta$ and a parabolic subset Γ such that α lies in the symmetric part of Γ whereas the special part Σ' of Γ is contained in Δ. Now it follows from the first paragraph of the proof that $H^{\{\alpha\}}(\mathfrak{a}) \leqslant H^{\Sigma'}(\mathfrak{a})$. Since by assumption $H^{\Sigma'}(\mathfrak{a}) \leqslant H^{\Sigma}(\mathfrak{a})$, we have $H^{\{\alpha\}}(\mathfrak{a}) \in H^{\Sigma}(\mathfrak{a})$, i. e. $\alpha \in \Delta$. The contradiction shows that $\Delta = \Phi$ that means that $H^{\Sigma}(\mathfrak{a}) = H(\mathfrak{a})$. Finally, by the previous proposition we have $E(R,\mathfrak{a}) = H^{\Sigma}(\mathfrak{a})$. □

The following statement was shown in [18] to be useful for stabilization results.

Corollary 3.5. *Let P be a parabolic subgroup of G. Then*

$$\langle E(\mathfrak{a}), U_P(R)\rangle \bigcap G(R,\mathfrak{a}) = E(R,\mathfrak{a}).$$

Corollary 3.3 was stated for all Chevalley groups (without a proof) by J. Tits in 1976 in [32]. Note that it is a direct implication of Proposition 3.2 but this one was not mentioned in [32]. In the same year a proof of this theorem for GL_n modulo Proposition 3.2 due to L. Vaserstein was published in [36]. It is really amazing that generators of relative elementary subgroup were not even mentioned in [36] although W. van der Kallen in [18] and N. Vavilov in [37] cite Proposition 3.2 as “mentioned” or obtained in [36].

Five years later L. Vaserstein included statements and proofs of Proposition 3.2 and Corollary 3.3 for the general linear group over a noncommutative ring to his paper [34]. For all Chevalley groups the statements with detailed proofs appeared in [35]. Method of the proof of Proposition 3.2 in this article was suggested by a referee. Actually, the idea of the referee translated to the language of parabolic subgroups was a starting point of our method. A counterpart of Proposition 3.2 for the relative elementary subgroup, corresponding to an admissible pair, was obtained by E. Abe in [2]. A natural generating set of the group $[E(R,\mathfrak{a}),E(R,\mathfrak{b})]$ was found in HVZrelachev. Corollary 3.5 for $G=\mathrm{GL}_n$ and $P=P_1$ was obtained in [18, Lemma 2.2] (by P_1 we denote the maximal parabolic subgroup, corresponding to the first simple root in notation of [10]. For classical group and $P=P_1$ this corollary was stated in [4, Lemma 3.6] as an important step of the proof of relative local-global principle. The proof of this statement was extremely short:

"The proof goes on the similar lines as in the linear case. Replacing e_{ij} by ge_{ij} works."

with a reference to van der Kallen's proof in [18] (here e_{ij} are root unipotents of GL_n whereas ge_{ij} are those of a classical group).

4. Splitting principle

We call $\mathfrak{a}$ a splitting ideal if $A=R\oplus\mathfrak{a}$ as additive groups, where R is a subring of A. Of course, in this case $R\cong A/\mathfrak{a}$. Equivalently, $\mathfrak{a}$ is a splitting ideal iff it is a kernel of a retraction $A\to R\subseteq A$. For example, if $A=R[t]$ is a polynomial ring, then tA is a splitting ideal. For ideals $\mathfrak{a}$ and $\mathfrak{b}$ of a ring R denote

$$EE(R,\mathfrak{a},\mathfrak{b})=E(A,\mathfrak{a}\mathfrak{b})[E(A,\mathfrak{a})E(A,\mathfrak{b})].$$

This group appears naturally in the following theorem and in the multicommutator formula in [27]. In this article it will be used in section 7 for the proof of relative local-global principle and it also plays an important role in

The proof of the following statement is quite easy. It can be found in [5, Lemma 2.2].

Theorem 4.1. *Suppose that $\mathfrak{a}$ is a splitting ideal of A so that $A=R\oplus\mathfrak{a}$ for some subring R of A. Let $\mathfrak{b}'$ be an ideal of R. Put $\mathfrak{b}=A\mathfrak{b}'$. Then*

$$\mathrm{E}(A,\mathfrak{b})\cap\mathrm{G}(A,\mathfrak{a})=EE(R,\mathfrak{a},\mathfrak{b}).$$

In particular, if $\mathfrak{b}=A$, then $\mathrm{E}(A)\cap\mathrm{G}(A,\mathfrak{a})=E(A,\mathfrak{a})$.

It seems that the idea to use splitting in the theory of linear groups over rings is due to A. A. Suslin. However, in his work [28] he uses another consequence of splitting: $\mathrm{GL}_n(R)\cap\mathrm{E}_n(A)=E_n(R)$, where R is a retraction of a ring A. This formula does not depend on two functors from an arbitrary category to the category of sets. The absolute splitting principle $E(R,\mathfrak{a})=E(R)\cap G(R,mfa)$ for a splitting ideal $\mathfrak{a}$ of R appeared first in [1, Proposition 1.6]. It was reproved in [4] with a more difficult proof. Recently it was generalized for isotropic reductive groups by V. Petrov and A. Stavrova in [22]. The relative version presented in the current article will appear in the joint paper [5] by H. Apte and the author.

5. Dilation and local-global principles

The following statement is a key technical point in localization procedure. First we state the lemma which helps to get around non-injectivity of a localization homomorphism.

Lemma 5.1. *Let H be an algebraic group scheme. Let S be a multiplicative subset in R and $g(t), h(t) \in H(R[t], tR[t])$. If $\lambda_S(g) = \lambda_S(h)$. Then there exists $s \in S$ such that $g(s^m t) = h(s^m t)$.*

The proof is fairly easy. It can be found in [28], [22, Lemma 14], or [5].

Theorem 5.2. *Let S be a multiplicative subset of R and $g = g(t) \in \mathrm{E}(R_S[t], tR_S[t])$. Then there exists $s \in S$ such that $g(st) \in \lambda_S\big(\mathrm{E}(R[t], tR[t])\big)$.*

Proof. Let u be another independent variable and $R' = R_S[t, u]$. Put $f(t, u) = g(tu)$. Then $f(t, u) \in E(R', uR')$ and $f(t, u^3) \in E(R', (uR')^{\boxed{3}} \leqslant E(uR')$ by Corollary 3.3. Replacing u by a suitable element $s_1 \in S$ we can clear denominators, i. e. $g(ts_1^3) = f(t, s_1^3) = \lambda_S(h)$ for some $h' \in E(R[t])$. Put $s = s_1^3$.

By the splitting principle $h' = h''h$, where $h'' \in E(R)$ and $h \in E(R[t], tR[t])$. Now, $\lambda_S(h'') = g(st)\lambda_S(h^{-1}) \in E(R_S[t], tR_S[t])$. On the other hand, $\lambda_S(h'') \in E(R_S)$. Since the intersection $G(R_S[t], tR_S[t]) \cap G(R_S)$ is trivial, $\lambda_S(h'') = 1$ and $g(st) = \lambda_S(h)$. □

Corollary 5.3. *Let S be a multiplicative subset of R and $g = g(t) \in \mathrm{G}(R[t], tR[t])$. Suppose that $\lambda_S(g) \in \mathrm{E}(R_s[t])$. Then there exists $s \in S$ such that $g(st) \in \mathrm{E}(R[t], tR[t])$.*

Proof. By Theorem 4.1 $\lambda_S(g) \in E(R_S[t], tR_S[t])$ and by the previous theorem there exists $s_1 \in S$ such that $\lambda_S\big(g(s_1 t)\big) = \lambda_S\big(h(t)\big)$ for some $h(t) \in E(R[t], tR[t])$. By the lemma of the current subsection there exists $s_2 \in S$ such that $g(s_1 s_2 t) = h(s_2 t) \in E(R[t], tR[t])$. □

Theorem 5.4. *Let R be a commutative ring and $g = g(y) \in G(R[y], yR[y])$. Suppose that $\lambda_{R\setminus\mathfrak{m}}(g) \in E(R_\mathfrak{m}[y])$ for every maximal ideal $\mathfrak{m}$ of R. Then, $g \in E(R[y], yR[y])$.*

Proof. Let t denote another independent variable. Consider the element

$$h(y, t) = g(y)g(y - yt)^{-1} \in G(R[y, t]).$$

Since $h(y, 0) = 1$, we have $h(y, t) \in G(R[y, t], tR[y, t])$. Evaluation homomorphism $y \mapsto y - yt$ commutes with a localization homomorphism, hence $\lambda_{R\setminus\mathfrak{m}}(h(y, t)) \in E(R_\mathfrak{m}[y, t])$ for every maximal ideal $\mathfrak{m}$ of R. By Corollary 5.3 there exists $s \in R \setminus \mathfrak{m}$ such that $h(y, ts) \in E(R[y, t])$. Therefore, the set $I = \{s \in R \mid h(y, ts) \in E(R[y, t])$ in not contained in any maximal ideal $\mathfrak{m}$ of R. We claim that I is an ideal of R. Indeed, if $s, s' \in I$, then $h(y, trs)$

and $h(y - yts, ts')$ belong to $E(R[y,t])$ for all $r \in R$. Thus, $rs \in I$ and $h\big(y, t(s+s')\big) = h\big(y, ts\big)h\big(y - yts, ts'\big) \in E(R[y,t])$, i.e. $s + s' \in R$.

Since I is an ideal that is not contained in a maximal ideal, it coincides with R. Hence, $1 \in I$, i.e. $h(y,t) \in E(R[y,t])$ and $g(y) = g(y)g(0)^{-1} = h(y,1) \in E(R[y])$. Finally, by the splitting principle $g \in E(R[y], yR[y])$. □

Local-global principle for K_0 was obtained by D. Quillen in [23] for the proof of Serre's problem. A version of this principle for K_1 was proved by A. Suslin. It was an important step in his proof of K_1-anlogue of Serre's problem. For orthogonal groups Suslin's local-global principle was obtained by V. Kopeiko and A. Suslin in [30]. For symplectic groups it was announced by V. Kopeiko in [20]. A proof appeared later in the paper [12] by F. Grunewald, J. Mennicke, and L. Vaserstein.

The dilation principle seems to be a necessary tool for the proof of Suslin's local-global principle. It was named "Quillen's lemma", "Q-axiom" or "clearing denominators". The expression "dilation principle" is due to R. Rao. The history of the dilation principle coincides with the history of the local-global principle described above except for the result of D. Quillen.

For Chevalley groups both principles were obtained by E. Abe in [1] under additional condition (P), although it is missed in the statement of [1, Lemma 1.11] (this lemma is the same as our dilation principle 5.1). Indeed, in the proof of Lemma 1.11 E. Abe claims without any explanation that the image of the matrix

$$z = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \begin{pmatrix} 1 & s^n tp \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix} = e + \begin{pmatrix} 1 \\ r \end{pmatrix} s^n tp \begin{pmatrix} -r & 1 \end{pmatrix}$$

under the homomorphism $\varphi_\alpha : \mathrm{SL}_2(A_s[t]) \to G(A_s[t])$ lies in $\lambda_s\big(E(A[t])\big)$ when n is large enough. Here A is an arbirary commutative ring, $s \in A$, $p, r \in A_s[t]$, $\alpha \in \Phi$ and φ_α is a homomorphism induced by inclusion of root systems $A_1 \cong \{\pm\alpha\} \hookrightarrow \Phi$. This claim is a direct consequence of Condition (P). If α lies in a subsystem of type A_2, then it follows from the Whitehead–Vaserstein lemma. But it seems that this assertion is still unknown for $\Phi = C_2$, $\Phi = G_2$, long root $\alpha \in C_l$, or short root $\alpha \in B_l$. It worth mentioning that normality of the elementary subgroup does not help to prove the claim or condition (P).

An incorrect reference to [1, Lemma 1.11] was already given by M. Wendt in [39]. He cites [1, Lemma 1.11] for the proof of [39, Proposition 4.1] Then, for the proof of [39, Proposition 4.4] M. Wendt writes about [1, Lemma 3.7]:

"*Note that although this result appears in Section 3 of [Abe83] where a property (P') is assumed, it does not depend on this property, and is in fact a direct consequence of 4.1 which holds for arbitrary Chevalley groups.*"

But as we have just seen, [1, Lemma 1.11] already uses condition (P) (condition (P') is the same as (P) for localizations of a given ring). Note that this is only incorrect reference but not a gap in the proof in [39] as the dilation principle as well as the local-global principle is proved by V. Petrov and A. Stavrova in [22] for all isotropic reductive groups without extra conditions.

There is one more gap in the proof of homotopy invariance of K_1^G in [39], which again is filled by A. Stavrova in her recent preprint [24], where she proves the homotopy invariance for isotropic reductive groups. Namely, in the proof of [39, Proposition 4.7] M. Wendt claims:

"*. . . the space of maximal ideals of* $(R[t_1,\dots,t_n])_{\mathfrak{m}}[t_{n+1}]$ *is noetherian of dimension 1. . .* "

, where R is a Dedekind ring and $\mathfrak{m}$ is a maximal ideal of $R[t_1,\dots,t_n]$. But in fact it is easy to see that even if R is a field, dimension of maximal spectrum equals $n+1$. It is noticed by H. Bass in [9, Part I, Proposition 3.13] that Bass–Serre dimension of $R[t_1,\dots,t_n])_{\mathfrak{m}}[t_{n+1}]$ is not greater than n (see [6] and [13] for the definition and properties of Bass–Serre dimension). But this result do not help to repair the proof of M. Wendt.

6. Bak's key lemma and normality of the elementary subgroup

Lemma 6.1. *Let R be a ring and S a multiplicative subset in R. Given $g \in E(R_S)$ and $s' \in S$ there exists $s \in S$ such that*

$$\lambda_S\big(E(R,sR)\big)^g \leqslant \lambda_S\big(E(R,s'R)\big).$$

Proof. Over the polynomial ring $R_S[t]$ consider the element $x_\alpha(t)^g \in E(R_S[t], tR_S[t])$, where $\alpha \in \Phi$. By dilation principle there exists $s(\alpha) \in S$ such that $x_\alpha(s(\alpha)t)^g \in \lambda_S\big(E(R[t],tR[t])\big)$. Let $s = s' \prod_{\alpha\in\Phi} s(\alpha)$. Sending t to $s/s(\alpha)$, from the above inclusion we get $x_\alpha(s)^g \in \lambda_S\big(E(R,s'R)\big)$ for all $\alpha \in \Phi$, which implies the result. □

Theorem 6.2. *$E(R)$ is normal in $G(R)$.*

Proof. For $\alpha \in \Phi$ and $g \in G(R)$ consider the element $h(t) = x_\alpha(t)^g \in G(R[t]$. If $\mathfrak{m}$ is a maximal ideal of R, then $\lambda_{\mathfrak{m}}(g) = da$ for some $d \in T(R_{\mathfrak{m}})$ and $a \in E(R_{\mathfrak{m}})$. Since the torus normalizes the elementary group, we have $\lambda_{R\setminus\mathfrak{m}} \in E(R_{\mathfrak{m}}[t])$. By Theorem 5.4 $h(t) \in E(R[t])$. Sending t to an arbitrary element $r \in R$ we get $x_\alpha(r)^g$, which implies the result. □

Bak's key lemma was obtained for all Chevalley groups by G. Taddei in [31] as a main lemma for normality of the elementary subgroup. For the general linear group over a quasi-finite ring the result was proved by A. Bak in [6] (a quasi-finite ring is a direct limit of rings finitely generated as modules over their centers). In the latter work it was entitled "Key Lemma" and was a key point of the proof of nilpotency of $\mathrm{SK}_1(R)$ over a Noetherian ring R of finite

Bass–Serre dimension (the latter is closed to the notion of dimension of the maximal spectrum).

Normality of the elementary subgroup in the general linear group was proved by A. Suslin in [28] in 1977. His proof is based on the Suslin lemma about solution of a linear equation over a commutative ring with unimodular row of coefficients.

Afterwards, the results was generalized by different methods to certain classes of noncommutative rings by A. Suslin [33], L. Vaserstein [34], A. Bak [6] and S. Khlebutin [19]. Normality of the elementary subgroup of classical groups was obtained in a series of works by A. Suslin and V. Kopeiko [30, 29] and in [7, 8]. A general result about normality of the elementary subgroup in all Chevalley groups was obtained by G. Taddei in [31]. Recently V. Petrov and A. Stavrova [22] proved normality of the elementary subgroups in an isotropic reductive group of local isotropic rank at least 2.

7. Relative dilation and local-global principles

In this section $\mathfrak{b}$ is an ideal of a ring R and t is an independent variable. Put $\mathfrak{b}[t] = \mathfrak{b}R[t]$. Denote by $E(tR[t], t\mathfrak{b}[t])$ the normal closure of $E(t\mathfrak{b}[t])$ in $E(tR[t])$.

Lemma 7.1. $[E(R[t], \mathfrak{b}[t]), E(R[t], t^{27}R[t])] \leqslant E(tR[t], t\mathfrak{b}[t])$.

Proof. $[E(R[t], \mathfrak{b}[t]), E(R[t], t^{27}R[t])] \leqslant [E(\mathfrak{b}[t]), E(R[t], t^{27}R[t])]^{E(R[t])}$ and the latter group is contained in $[E(\mathfrak{b}[t]), E(t^9R[t])]^{E(R[t])}$ by Corollary 3.3. We claim that the latter group is contained in $E(R[t], t^3\mathfrak{b}[t])$. Recall that if $h, g_1, \dots, g_m$ are elements of a group H, then $[h, g_1 \dots g_m] = \prod_{i=1}^m [h, g_i]^{f_i}$ for some $f_1, \dots, f_m \in H$. Therefore it suffices to show that $[x_\alpha(p), x_\beta(t^9r)] \in E(R[t], t^3\mathfrak{b}[t])$ for all $\alpha, \beta \in \Phi$, $p \in \mathfrak{b}[t]$, and $r \in R[t]$.

If $\alpha \neq -\beta$, then Chevalley commutator formula shows that $[x_\alpha(p), x_\beta(t^9r)] \in E(t^9\mathfrak{b}[t])$. If $\alpha = -\beta$, choose a parabolic subgroup P such that $X_\alpha \in EL_P$ as in the proof of Corollary 3.3. By Lemma 2.1 $x_\beta(t^9r) \in \langle U_P(t^3R[t]), U_P^-(t^3R[t])\rangle$, in particular, $x_\beta(t^9r)$ is a product of root elements $x_\gamma(t^3s)$ with $\gamma \neq -\alpha$. Again, each commutator $[x_\alpha(p), x_\gamma(t^3s)]$ belongs to $E(t^3\mathfrak{b}[t])$, hence $[x_\alpha(p), x_\beta(t^9r)] \in E(R[t], t^3\mathfrak{b}[t])$.

By Proposition 3.2 the group $E(R[t], t^3\mathfrak{b}[t])$ is generated by $z_\alpha(pt^3, r)$ for all $\alpha \in \Phi$, $p \in \mathfrak{b}[t]$, and $r \in R[t]$. Take a parabolic subgroup P such that $X_\alpha \leqslant EL_P$. By Corollary 2.2

$$x_\alpha(pt^3) \in [U_P(t\mathfrak{b}[t]), U_P^-(tR[t])]U_P(t\mathfrak{b}[t])U_P^-(t\mathfrak{b}[t])$$

and by Lemma 3.1(ii) $x_{-\alpha}(r)$ normalizes all the subgroups from the above inclusion. Thus, $E(R[t], t^3\mathfrak{b}[t]) \leqslant E(tR[t], t\mathfrak{b}[t])$ which completes the proof. □

Theorem 7.2. *Let S be a multiplicative subset of R and $\mathfrak{b}$ an ideal. Let $g = g(t) \in \mathrm{E}(R_S[t], \mathfrak{b}R_S[t])$ be such that $g(0) = 0$. Then there exists $s \in S$ such that $g(st) \in \lambda_S\big(\mathrm{E}(R[t], \mathfrak{b}[t])\big)$.*

The proof is essentially the same as for Theorem 5.2.

Theorem 7.3. *Let $\mathfrak{b}$ be an ideal of a ring R and $a \in G(R[t], tR[t])$. Suppose that $\lambda_{\mathfrak{m}}(a) \in E(R_{\mathfrak{m}}[t], \mathfrak{b}R_{\mathfrak{m}}[t])$ for any maximal ideal $\mathfrak{m}$ of R. Then $a \in EE(R[t], \mathfrak{b}[t], tR[t])$.*

Proof. Let u denote another independent variable. Put

$$h(t, u) = a(t)a(t - tu)^{-1} \in G(R[t, u]).$$

Since $h(t, 0) = e$, we have $h(t, u) \in G(R[t, u], uR[t, u])$. The evaluation homomorphism $t \mapsto t - tu$ commutes with a localization homomorphism, hence $\lambda_{\mathfrak{m}}(h(t, u)) \in E(R_{\mathfrak{m}}[t, u], \mathfrak{b}R_{\mathfrak{m}}[t, u])$ for any maximal ideal $\mathfrak{m}$ of R. By dilation principle 5.3 there exists $s \in R \setminus \mathfrak{m}$ such that $h(t, us) \in E(R[t, u], \mathfrak{b}[t, u])$. It follows that the set $\mathfrak{s} = \{s \in R \mid h(t, us) \in E(R[t, u])$ is not contained in a maximal ideal $\mathfrak{m}$ of R. We claim that $\mathfrak{s}$ is an ideal of R. Indeed, if $s, s' \in \mathfrak{s}$, then $h(t, urs)$, and $h(t - tus, us')$ belong to $E(R[t, u], \mathfrak{b}[t, u])$ for all $r \in R$. Thus $rs \in \mathfrak{s}$ and

$$h\big(t, u(s + s')\big) = h\big(t, us\big)h\big(t - tus, us'\big) \in E(R[t, u], \mathfrak{b}R[t, u]),$$

i.e. $s + s' \in \mathfrak{s}$.

Since the ideal $\mathfrak{s}$ is not contained in a maximal ideal of R, it must coincide with R. Thererfore $1 \in \mathfrak{s}$, i.e. $h(t, u) \in E(R[t, u])$ and $a(t) = a(t)a(0)^{-1} = h(t, 1) \in E(R[t], \mathfrak{b}[t])$. Finally, by splitting principle 4.1 $a \in EE(R[t], \mathfrak{b}[t], tR[t])$. □

The relative version of the local-global principle for classical groups appeared in [4] by H. Apte, P. Chattopadhyay, R. Rao, but their proof is incomplete (see section 3). A complete proof of the relative case for all Chevalley groups was obtained in [5] by H. Apte and the author provided that for $\Phi = C_2, G_2$ the ground ring does not have residue fields of two elements.

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