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WIENER INDEX OF THE TENSOR PRODUCT OF CYCLES

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ABSTRACT. The *Wiener index,* denoted by $W(G)$, of a connected graph G is the sum of all pairwise distances of vertices of the graph, that is, $W(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} d(u,v)$. In this paper, we obtain the Wiener index of the tensor product of two cycles.

Key words : Tensor product, Wiener index.
AMS SUBJECT : 05C12, 05C76.

1. INTRODUCTION

For two graphs G and H their *tensor product,* denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2$is an edge in G and $h_1h_2$is an edge in H. The *Cartesian product* of two graphs G and H is the graph, denoted by $G \,\Box\, H$, whose vertex set is $V(G \,\Box\, H) = V(G) \times V(H)$ and (g_1, h_1) is adjacent to (g_2, h_2) in $G \,\Box\, H$ if and only if $g_1 = g_2$ and $h_1h_2 \in E(H)$ or $g_1g_2 \in E(G)$ and $h_1 = h_2$.

Let G and H be simple graphs with vertex sets $V(G) = \{x_1, x_2, \ldots, x_m\}$ and $V(H) = \{y_1, y_2, \ldots, y_n\}$, respectively. Then $V(G \times H) = V(G) \times V(H)$ and for our convenience, we write $V(G \times H) = \bigcup_{i=1}^{m} X_i$, where $X_i = \{x_i\} \times V(H)$; we may also write $V(G \times H) = \bigcup_{j=1}^{n} Y_j$, where $Y_j = V(G) \times \{y_j\}$. We shall denote the vertices of X_i by $\{x_{i,j} \mid 1 \leq j \leq n\}$ and the vertices of Y_j by $\{x_{i,j} | 1 \leq i \leq m\}$, where $x_{i,j}$ stands for the vertex (x_i, y_j). We shall call $X_i, 1 \leq i \leq m$, the i^{th} *layer* of $G \times H$ and $Y_j, 1 \leq j \leq n$, the j^{th} *column* of $G \times H$; see Fig.1. For two disjoint subsets A and B of $V(G)$, $E(A, B)$ denotes the set of edges of G from A to B. Let C_r denote a cycle of length r. Let $V(C_r) = \{x_1, x_2, \ldots, x_r\}$ and $V(C_s) = \{y_1, y_2, \ldots, y_s\}$. For terms not defined here see [3] or [12].

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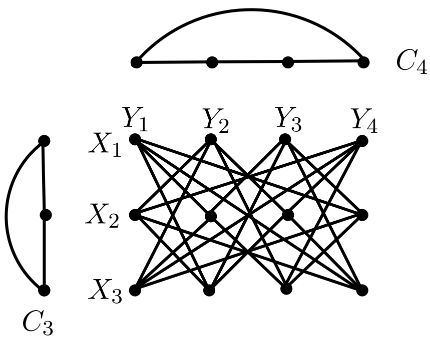

Fig.1 Tensor Product of C_3 and C_4

The *Wiener index* of a graph G, $W(G)$, is defined as $\frac{1}{2}\sum_{u,v\in V(G)} d(u,v)$, where d is the distance function on G. The Wiener index has important applications in chemistry. The graphical invariant $W(G)$ has been studied by many researchers under different names such as distance, transmissions, total status and sum of all distances; see [5, 10, 13, 14]. The chemist Harold Wiener was the first to point out in 1947 that $W(G)$ is well correlated with certain physico-chemical properties of the organic compound.

Besides applications in chemistry, there are many situations in communication, facility location, cryptology, etc., that are effectively modeled by a connected graph G satisfying certain restrictions. Because of cost restraints one is often interested in finding a spanning tree of G that is optimal with respect to one or more properties. Finding a spanning tree T of G that has minimum Wiener index is proved to be important, see [11].

The tensor product of graphs has been extensively studied in relation to the areas such as graph colorings, graph recognition and decomposition, graph embeddings, matching theory, see [1, 7, 12, 15]. Also it is related to design theory, see [2]. Du and Zhou [8] have obtained the minimum Wiener indices of trees and unicyclic graphs of given matching number. Further, the same authors also have obtained the Wiener indices of unicyclic graphs [9]. Very recently Balakrishnan et al. have given a sharp lower bound for the Wiener index of the arbitary graph G in terms of the order, size and diameter of G [6]. In [16], the Wiener index of the tensor product of a path and a cycle has been obtained. In this paper, we compute the exact Wiener index of $C_r \times C_s$, where r or s (or both) are odd. Since if G and H are connected graphs, then $G \times H$ is connected only if atleast one of the graph is nonbipartite, see [12]. Hence the graph $C_r \times C_s$ is disconnected, when r and s are both even. The notation $d(x,S)$ denotes the sum of the distances from x to all the vertices of S, that is, $d(x,S) = \sum_{y\in S} d(x,y)$.

2. The Weiner Index of $C_{2m+1} \times C_{2n+1}$.

We quote the following lemma which is not difficult to prove.

Lemma 1. For $r \geq 3$,

$$W(C_r) = \begin{cases} \frac{n(n+1)(2n+1)}{2} \; if \; r = 2n+1, \\ n^3 \; if \; r = 2n. \end{cases}$$

Proof of Lemma 1 is given in [18]. For an odd integer $r = 2n+1 \geq 3$, it is known [13, p183] that $C_r \times C_r \cong C_r \,\Box\, C_r$; further, it is known [4] that $W(C_r \,\Box\, C_r) = 2r^2 W(C_r)$, and hence $W(C_r \times C_r) = n(n+1)(2n+1)^3$, by Lemma 1. Thus we have

Lemma 2. $W(C_{2n+1} \times C_{2n+1}) = n(n+1)(2n+1)^3$.

We use the following observations implicitly while finding distances among the vertices of $C_r \times C_s$.

Observation 3. *Let $H = C_r \times C_s - (E(Y_1, Y_s) \cup E(X_1, X_r))$; there are two components H_1 and H_2 in H. The vertices in one of the components, say H_1,(resp. H_2) are those (i,j) with i and j are of same(resp. different) parity. By the nature of the graph $C_r \times C_s$, in any shortest path between a pair of distinct vertices, consecutive vertices of the path are either in different layers or different columns and hence the length of a shortest path between the vertices is either the number of layers the path visits minus one or number of columns it visits minus one. Further, finding a shortest path, in $C_r \times C_s$, from $x_{1,1}$ to a vertex in H_2, the path has to either use the first edge $x_{1,1}x_{2,s}$ or $x_{1,1}x_{r,2}$.*

The following observation is helpful in finding a shortest path between a pair of distinct vertices in $C_r \times C_s$:

Observation 4. *A path of length k exists between (u,v) and (x,y) in $G \times H$ only if there exists in G a walk of length k between u and x and a walk of length k between v and y in H.*

The observation 4 is explained in a different context in [18, p273]. As the tensor product is commutative, $C_r \times C_s \cong C_s \times C_r$. Hence, in the sequel, we assume that $s \geq r$ in $C_r \times C_s$.

Theorem 5. *If $r = 2m+1 \geq 3$ and $s = 2n+1 \geq 3$ with $s > 2r$, then for $G = C_r \times C_s$, $W(G) = \frac{(2m+1)^2(2n+1)}{6}\Big(3n(n+1) + 4m(m+1)\Big)$.*

Proof. As G is vertex transitive, it is enough to find the distances from $x_{1,1}$ to all other vertices of G. We compute the sum of the distances from $u = x_{1,1}$ to all other vertices of G.

$$\sum_{u,v \in V(G)} d_G(u,v) = \sum_{v \in X_1} d_G(u,v) + 2\Big(\sum_{v \in X_i, 2 \leq i \leq m+1} d_G(u,v)\Big), \qquad (1)$$

where X_i denotes the vertices of the i^{th} layer of G, the multiplication factor 2 in one of the terms in (1) appears as the distances from u to all the vertices of the layer X_i is same as the distances from $u = x_{1,1}$ to all the vertices of $X_{2m-i+3},\ 2 \le i \le m+1$; this is true because the length of a shortest path that descends to a vertex from $u = x_{1,1}$ to a vertex of $X_i,\ 2 \le i \le m+1$, is same as the length of a shortest path that goes from $x_{1,1}$ to a vertex of X_{2m+1} and then ascending to a vertex in X_{2m-i+3} are the same.

We complete the proof in two steps, namely, **(A)** and **(B)**. In **(A)**, we find the distances from $x_{1,1}$ to all other vertices of the layer X_1. In **(B)**, we find the distances from $x_{1,1}$ to all the vertices of $\bigcup\limits_{i=2}^{m+1} X_i$.

Assume that n is odd.

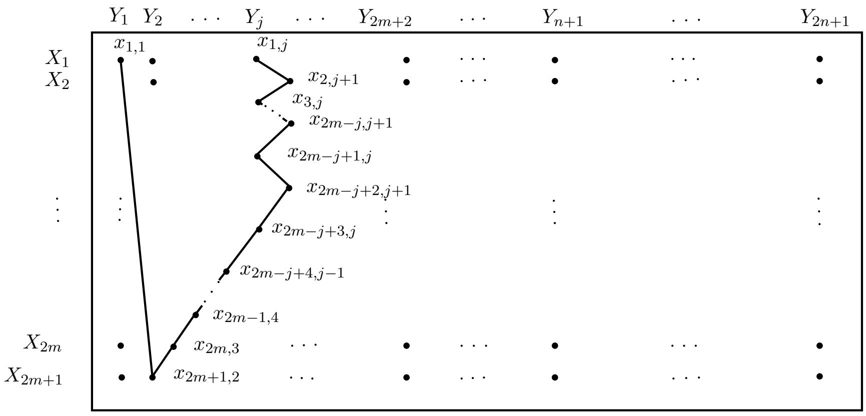

Fig.2 Vertices of $C_r \times C_s$

(A): First we calculate the sum of the distances from $u = x_{1,1}$ to all other vertices of X_1.

$$\sum_{v \in X_1} d_G(u,v) = 2\Big(\underbrace{(2m+1) + (2m+1) + \ldots + (2m+1)}_{(m+1)\ times} + (2m+3) + (2m+5) + \ldots + n\Big) + 2\Big(2 + 4 + \ldots + (n-1)\Big),$$

since $d_G(u, x_{1,j}) = 2m+1$, for $j = 2, 4, \ldots, 2m+2$, since the path traverses through $2m+1$ rows with its origin and terminus at X_1, see Fig.2.

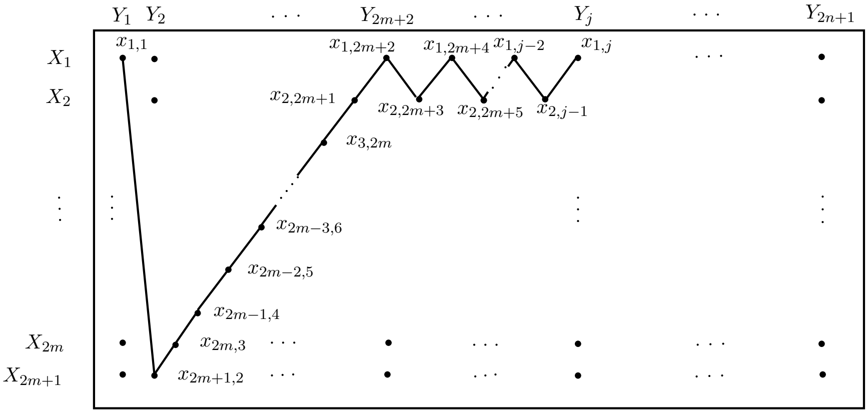

Fig.3 Vertices of $C_{\times}C_r$

The distances from u to the vertices $x_{1,2m+4}, x_{1,2m+6}, x_{1,2m+8}, \ldots, x_{1,n+1}$ are $2m+3, 2m+5, 2m+7, \ldots, n$, respectively, see Fig.3; the distance is easy to calculate as the path contains vertices from j columns, one in each column, and hence the distance is $j-1$. Further, the distances from u to the vertices $x_{1,3}, x_{1,5}, x_{1,7}, \ldots, x_{1,n}$ are $2, 4, 6, \ldots, n-1$, respectively, see Fig.4. Hence

$$\begin{aligned}\sum_{v \in X_1} d_G(u,v) &= 2(2m+1)(m+1) + 2\Big((2m+3) + (2m+5) + \ldots + n\Big) \\ &+ 2\Big(2 + 4 + 6 + \ldots + (n-1)\Big) \\ &= 2(m+1)m + n(n+1). \qquad (2)\end{aligned}$$

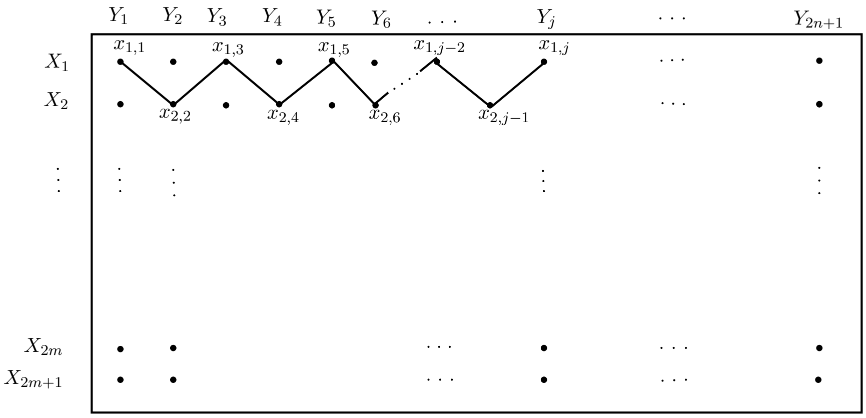

Fig.4 Vertices of $C_r \times C_s$

(B): Next we shall calculate the sum of the distances from $u = x_{1,1}$ to the vertices of X_i, $2 \le i \le m+1$. For this, we compute $\sum_{v \in X_i} d_G(u,v)$, for the single layer X_i. First we compute the distances from $u = x_{1,1}$ to all the vertices in a

layer with odd suffix, that is, $\sum\limits_{v\in X_{2k+1}} d_G(u,v)$, for some $2k+1,\ 3\le 2k+1\le m+1$.
For i odd,

$$\sum_{v\in X_i} d_G(u,v)=\begin{cases}(i-1)+2\Big(\underbrace{(i-1)+\cdots+(i-1)}_{\frac{i-1}{2}\ times}+(i+1)+\ldots+(n-1)\Big)\\ +2\Big(\underbrace{2m-i+2)+\ldots+(2m-i+2)}_{\frac{2m-i+3}{2}\ times}+(2m-i+4)+\ldots+n\Big)\end{cases} \quad (3)$$

Explanations for the terms in (3) are as follows:
$d_G(u,x_{i,1})=i-1$ as the path contains i vertices from i layers, see Fig.5, and also $d_G(u,x_{i,j})=i-1,\ j=3,5,\ldots,i$, see Fig.6. The distances from u to the vertices $x_{i,i+2},x_{i,i+4},x_{i,i+6},\ldots,x_{i,n}$ are $i+1,i+3,i+5,\ldots,n-1$, respectively, see Fig.7. Further, $d_G(u,x_{i,j})=2m-i+2$, for $j=2,4,\ldots,2m-i+3$, see Fig.9, and the distances from u to the vertices $x_{i,2m-i+5},x_{i,2m-i+7},x_{i,2m-i+9},\cdots,x_{i,n+1}$ are $2m-i+4,2m-i+6,2m-i+8,\ldots,n$, respectively, see Fig.10.

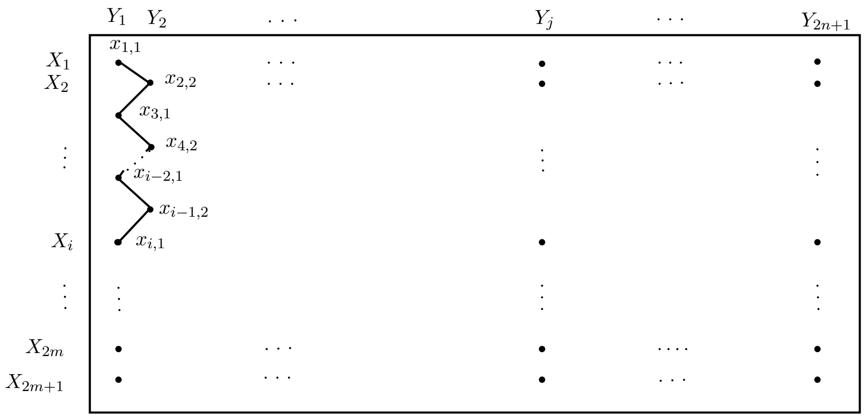

Fig.5 Vertices of $C_r\times C_s$

The multiplication factor 2 appears in all the terms except the first term of the sum (3) because $d_G(u,x_{i,j})=d_G(u,x_{i,2n-j+3}),\ 2\le j\le n+1$, due to the "symmetry" of the graph.
The summation of the terms of (3) gives

$$\begin{aligned}\sum_{v\in X_i} d_G(u,v) &= (i-1)i+2\Big((i+1)+(i+3)+\ldots+(n-1)\Big)+(2m-i+2)\\ &\quad (2m-i+3)+2\Big((2m-i+4)+(2m-i+6)+\ldots+n\Big)\\ &= \Big(\frac{(2m-i+2)^2}{2}+n(n+1)+\frac{(i-1)^2}{2}-\frac{1}{2}\Big). \qquad (4)\end{aligned}$$

Next we compute the distances from $u = x_{1,1}$ to all the vertices in a layer with even suffix, that is $\sum\limits_{v\in X_{2k}} d_G(u,v)$, for some $2k,\ 2 \le 2k \le m+1$.
For an even i,

$$\sum_{v\in X_i} d_G(u,v) = \begin{cases} (2m-i+2) + 2\{\underbrace{(2m-i+2)+\ldots+(2m-i+2)}_{\frac{(2m-i+2)}{2}\ times} + (2m-i+4) \\ +\ldots+(n-1)\} + 2\{\underbrace{(i-1)+\cdots+(i-1)}_{\frac{i}{2}\ times} + (i+1)+\ldots+n\} \end{cases} \tag{5}$$

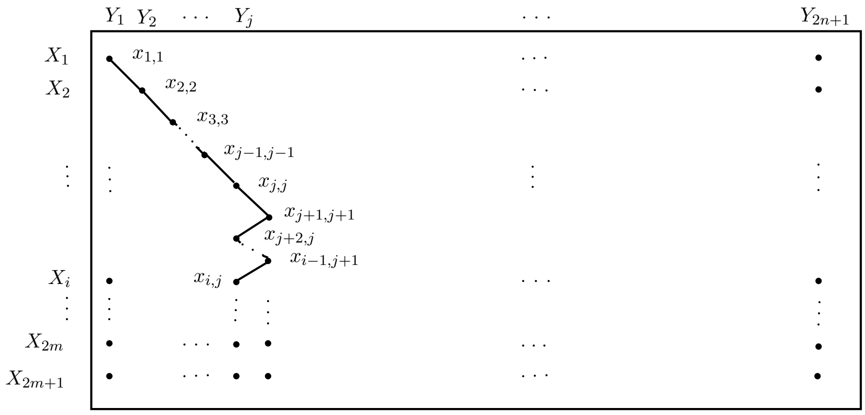

Fig.6 Vertices of $C_r \times C_s$

Explanations for the terms of (5) are given below:
$d_G(u, x_{i,1}) = 2m - i + 2$, see Fig.8, and $d_G(u, x_{i,j}) = 2m - i + 2$, for $j = 3, 5, \ldots, 2m - i + 3$, see Fig.9.

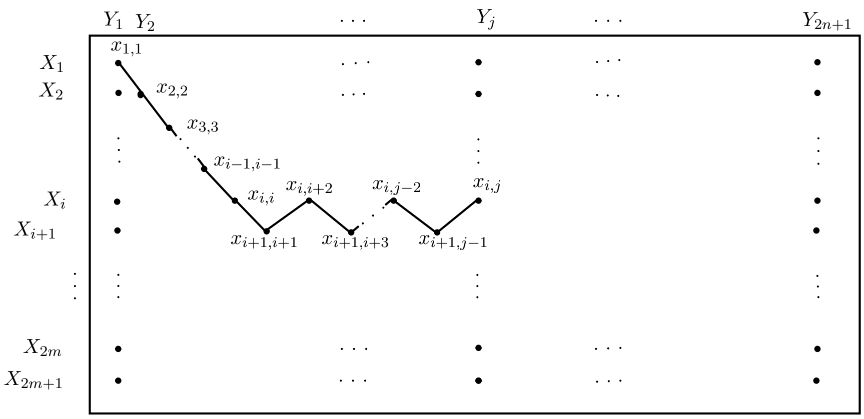

Fig.7 Vertices of $C_r \times C_s$

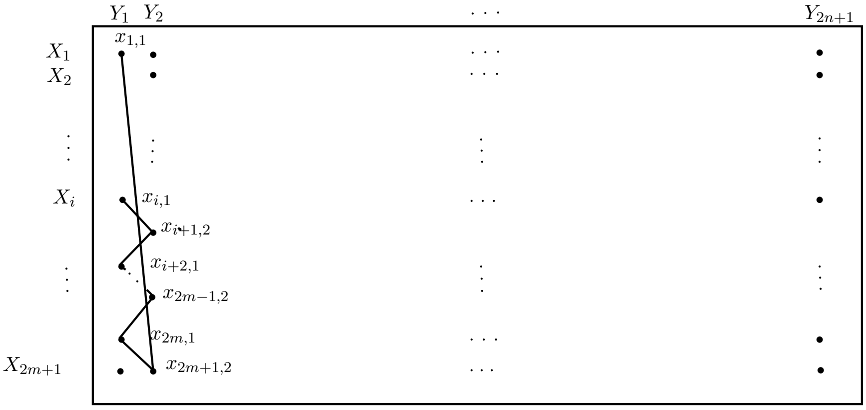

Fig.8 Vertices of $C_r \times C_s$

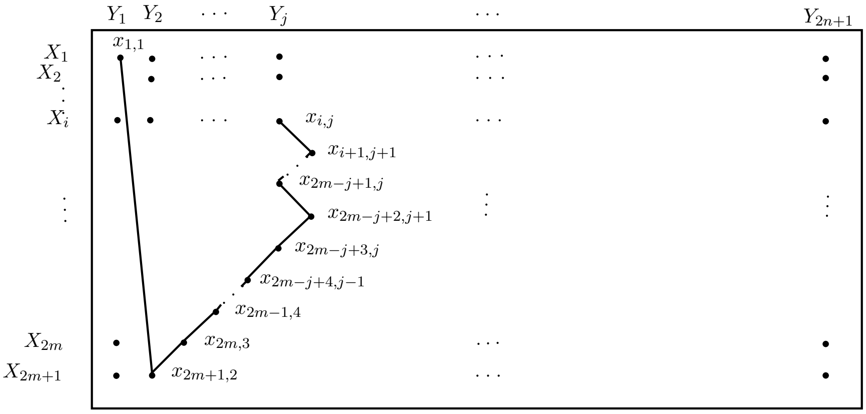

Fig.9 Vertices of $C_r \times C_s$

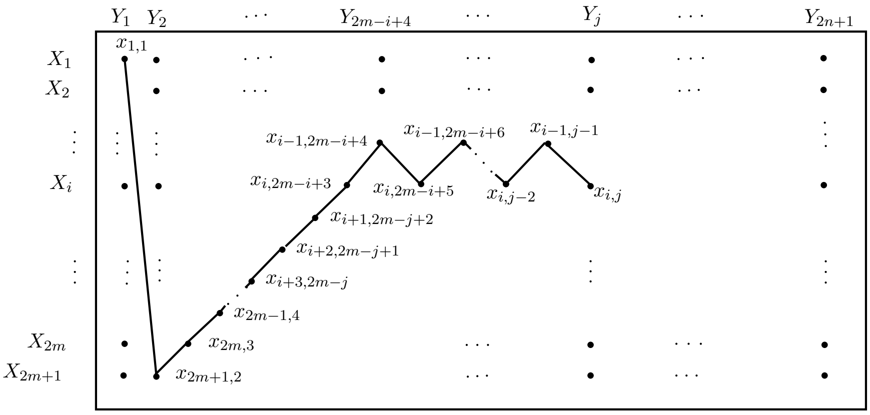

Fig.10 Vertices of $C_r \times C_s$

The distances from u to the vertices $x_{i,2m-i+5}, x_{i,2m-i+7}, x_{i,2m-i+9}, \ldots, x_{i,n}$ are $2m-i+4, 2m-i+6, 2m-i+8, \ldots, n-1$, respectively, see Fig.10. Further,

$d_G(u, x_{i,j}) = i - 1,\ j = 2, 4, \ldots, i$, see Fig.6, and the distances from u to the vertices $x_{i,i+2}, x_{i,i+4}, x_{i,i+6}, \ldots, x_{i,n+1}$ are $i+1, i+3, i+5, \ldots, n$, respectively, see Fig.7. The multiplication factor 2 appears in all the terms except the first term in the sum (5) because $d_G(u, x_{i,j}) = d_G(u, x_{i,2n-j+3}),\ 2 \leq j \leq n+1$, due to the "symmetry" of the graph G. The summation of the terms of (5) gives

$$\begin{aligned}\sum_{v \in X_i} d_G(u,v) &= (2m-i+2)(2m-i+3) + 2\Big((2m-i+4) + (2m-i+6) \\ &+ \ldots + (n-1)\Big) + (i-1)i + 2\Big((i+1) + (i+3) + \ldots + n\Big) \\ &= \Big(\frac{(2m-i+2)^2}{2} + n(n+1) + \frac{(i-1)^2}{2} - \frac{1}{2}\Big). \qquad (6)\end{aligned}$$

From (4) and (6) we see that irrespective of the parity of i, $\sum_{v \in X_i} d_G(u,v)$ is same. Thus

$$\begin{aligned}\sum_{2 \leq i \leq m+1} d_G(u,v) &= \sum_{2 \leq i \leq m+1} \Big(\frac{(2m-i+2)^2}{2} + n(n+1) + \frac{(i-1)^2}{2} - \frac{1}{2}\Big) \\ &= mn(n+1) + \frac{m}{3}(4m^2 + 3m - 1). \qquad (7)\end{aligned}$$

Substituting the values obtained in (2) and (7) in (1), we have

$$\begin{aligned}\sum_{u,v \in V(G)} d_G(u,v) &= \sum_{v \in X_1} d_G(u,v) + 2\Big(\sum_{\substack{v \in X_i \\ 2 \leq i \leq m+1}} d_G(u,v)\Big) \\ &= \Big(2(m+1)m + n(n+1)\Big) + 2\Big(mn(n+1) + \frac{m}{3}(4m^2 + 3m - 1)\Big) \\ &= \frac{2m+1}{3}\Big(3n(n+1) + 4m(m+1)\Big). \qquad (8)\end{aligned}$$

The proof is similar when n is even and in this case also $\sum_{u,v \in V(G)} d_G(u,v)$ is found to be the same as (8); we omit the details.

As the graph G is vertex transitive, the sum of the distances from $u = x_{1,1}$ to all other vertices of G is same as the sum of the distances from $x_{i,j}$ to all other vertices of G, for all $i,\ j,\ 1 \leq i \leq 2m+1,\ 1 \leq j \leq 2n+1$. Hence

$$\begin{aligned}W(G) &= \frac{|V(G)|}{2}\Big(\sum_{u,v \in V(G)} d_G(u,v)\Big) \\ &= \frac{(2m+1)(2n+1)}{2}\Big(\frac{2m+1}{3}(3n(n+1) + 4m(m+1))\Big), \ \textit{by } (8) \\ &= \frac{(2m+1)^2(2n+1)}{6}\Big(3n(n+1) + 4m(m+1)\Big).\end{aligned}$$

In Theorem 5 we have assumed that $r = 2m+1,\ s = 2n+1$ and $s > 2r$. In the next theorem we consider the case $r < s < 2r$.

Theorem 6. *If* $r = 2m+1$ *and* $s = 2n+1$ *with* $r < s < 2r$, *then for* $G = C_r \times C_s$, $W(G) = \frac{(2m+1)(2n+1)}{3}\Big(n^3+2n-4m^3+12m^2n-3mn^2+9mn+m\Big)$.

Proof. As in the above theorem, it is enough to find the sum of the distances from the vertex $u = x_{1,1}$ to all other vertices of G.

$$\begin{aligned}\sum_{u,v\in V(G)} d_G(u,v) &= \sum_{v\in X_1} d_G(u,v) + 2\Big(\sum_{\substack{v\in X_i\\ 2\le i\le 2m-n}} d_G(u,v) \\ &+ \sum_{\substack{v\in X_i\\ 2m-n+1\le i\le m+1}} d_G(u,v)\Big), \qquad (9)\end{aligned}$$

since the distances from u to all the vertices of the layer X_i is same as the distances from u to all the vertices of the layer $X_{2m-i+3},\ 2 \le i \le m+1$; this is true because the length of a shortest path that descends to a vertex from $u = x_{1,1}$ to a vertex of $X_i,\ 2 \le i \le m+1$, is same as the length of a shortest path that goes from $x_{1,1}$ to X_{2m+1} and then ascending to a vertex in X_{2m-i+3} are the same. The distances from u to the vertices in $X_i,\ 2 \le i \le 2m-n$, is different from u to the vertices in $X_i,\ 2m-n+1 \le i \le m+1$, as $s < 2r$, reaching a vertex from u to $H_2 \cap X_i,\ 2 \le i \le 2m-n$, by using the first edge $u = x_{1,1}x_{2,2n+1}$ is shorter than finding a path which uses the first edge $u = x_{1,1}x_{2m+1,2}$.

Therefore, we find the sum of distances from u to all the vertices in $X_i,\ 2 \le i \le 2m-n$, and from u to all the vertices in $X_i,\ 2m-n+1 \le i \le m+1$, separately. We complete the proof in three steps, namely, **(A)**, **(B)** and **(C)**. In **(A)**, we find the distances from $x_{1,1}$ to all other vertices of the layer X_1, in **(B)**, we find the distances from $x_{1,1}$ to all the vertices of $\bigcup\limits_{i=2}^{2m-n} X_i$; in **(C)**, we fine the distances from $x_{1,1}$ to all the vertices of $\bigcup\limits_{i=2m-n+1}^{m+1} X_i$.

Assume that n is odd.

(A): First we obtain the sum of the distances from $u = x_{1,1}$ to all other vertices of X_1.

$$\begin{aligned}\sum_{v\in X_1} d_G(u,v) &= 2\Big(\underbrace{(2m+1)+(2m+1)+\ldots+(2m+1)}_{(n-m)\ times} + 2m + (2m-2) \\ &+ (2m-4)+\ldots+(n+1)\Big) + 2\Big(2+4+\ldots+(n-1)\Big) \\ &= 2(n-m)(2m+1) + 2m(m+1) \\ &= 2(2mn+n-m^2). \qquad (10)\end{aligned}$$

Explanations for the terms involved in the above equation are as follows: $d_G(u, x_{1,j}) = 2m+1$ for $j = 2, 4, \ldots, 2n-2m$, see Fig.2, and the distances from u to the vertices $x_{1,2n-2m+2}, x_{1,2n-2m+4}, x_{1,2n-2m+6}, \ldots, x_{1,n+1}$ are $2m, 2m-2, 2m-4, \ldots, n+1$, respectively, the paths are similar to the one described in Fig.3. Further, $d_G(u, x_{i,j}) = j-1$ for $j = 3, 5, \ldots, n$, see Fig.4. The multiplication factor 2 appears in all the terms of the sum because $d_G(u, x_{1,j}) = d_G(u, x_{1,2n-j+3})$, $2 \leq j \leq n+1$, due to the "symmetry" of the graph G.

(B): Next we compute the sum $\sum\limits_{v \in X_i} d_G(u, v)$, $2 \leq i \leq 2m-n$. First we compute the distances from $u = x_{1,1}$ to all the vertices in a layer with odd suffix, that is, $\sum\limits_{v \in X_{2k+1}} d_G(u, v)$, for some $2k+1$, $3 \leq 2k+1 \leq m+1$.

For i odd, $3 \leq i \leq 2m-n$,

$$\sum_{v \in X_i} d_G(u, v) = \begin{cases} (i-1) + 2\Big(\underbrace{(i-1) + \ldots + (i-1)}_{\frac{i-1}{2}\ times} + (i+1) + (i+3) + \ldots + (n-1)\Big) \\ +2\Big(\underbrace{(2m-i+2) + \ldots + (2m-i+2)}_{\frac{2n-2m+i-1}{2}\ times} \\ +(2m-i+1) + (2m-i-1) + (2m-i-3) + \ldots + (n+1)\Big) \end{cases} \tag{11}$$

Explanations for the terms involved in the above equation are given below: $d_G(u, x_{i,1}) = i-1$, see Fig.5, and $d_G(u, x_{i,j}) = i-1$, $j = 3, 5, \ldots, i$, see Fig.6. The distances in G from u to the vertices $x_{i,i+2}, x_{i,i+4}, x_{i,i+6}, \ldots, x_{i,n}$ are $i+1, i+3, i+5, \ldots, n-1$, respectively, see Fig.7. Further, $d_G(u, x_{i,j}) = 2m-i+2$, $j = 2, 4, \ldots, 2n-2m+i-1$, see Fig.9, and the distances from u to the vertices $x_{i,2n-2m+i+1}, x_{i,2n-2m+i+3}$, $x_{i,2n-2m+i+5}, \ldots, x_{i,n+1}$ are $2m-i+1, 2m-i-1, 2m-i-3, \ldots, n+1$, respectively; the path is similar to the one shown in Fig.10. The multiplication factor 2 appears in all the terms except the first term of the sum because $d_G(u, x_{1,j}) = d_G(u, x_{1,2n-j+3})$, $2 \leq j \leq n+1$, due to the "symmetry" of the graph G.

The summation of the terms of (11) gives

$$\begin{aligned} \sum_{v \in X_i} d_G(u, v) &= (i-1)i + 2\{(i+1) + (i+3) + \ldots + (n-1)\} + (2m-i+2) \\ &\quad (2n-2m+i-1) + 2\Big((2m-i+1) + (2m-i-1) \\ &\quad +(2m-i-3) + \ldots + (n+1)\Big) \\ &= \frac{(2m-i+2)(4n-2m+i)}{2} + \frac{(i-1)^2}{2} - \frac{1}{2}. \end{aligned} \tag{12}$$

For even $i,\ 2 \le i \le 2m-n$,

$$\sum_{v\in X_i} d_G(u,v) = \begin{cases} (2m-i+2)+2\Big(\underbrace{(2m-i+2)+\ldots+(2m-i+2)}_{\frac{2n-2m+i-2}{2}\ times}+(2m-i+1) \\ +(2m-i-1)+(2m-i-3)+\ldots+(n+2)\Big) \\ +2\Big(\underbrace{(i-1)+\ldots+(i-1)}_{\frac{i}{2}\ times}+(i+1)+(i+3)+\ldots+n\Big) \end{cases} \quad (13)$$

Explanations for the terms involved in the above equation are given below: $d_G(u,x_{i,1}) = 2m-i+2$, see Fig.8, and $d_G(u,x_{i,j}) = 2m-i+2,\ j = 3,5,\ldots,2n-2m+i-1$, see Fig.9. The distances in G from u to the vertices $x_{i,2n-2m+i+1}, x_{i,2n-2m+i+3}$, $x_{i,2n-2m+i+5},\ldots,x_{i,n}$ are $2m-i+1, 2m-i-1, 2m-i-3,\ldots,n+2$, respectively, see Fig.10. Further, $d_G(u,x_{i,j}) = i-1,\ j = 2,4,\ldots,i$, see Fig.6 and the distances in G from u to the vertices $x_{i,i+2}, x_{i,i+4}, x_{i,i+6},\ldots,x_{i,n+1}$ are $i+1, i+3, i+5,\ldots,n$, respectively, see Fig.7. The multiplication factor 2 appears in all the terms except the first term of the sum because $d_G(u,x_{1,j}) = d_G(u,x_{1,2n-j+3}),\ 2 \le j \le n+1$, due to the "symmetry" of the graph G.
The summation of the terms of (13) gives

$$\begin{aligned} \sum_{v\in X_i} d_G(u,v) &= (2m-i+2)(2n-2m+i-1)+2\Big((2m-i+1)+(2m-i-1) \\ &+ (2m-i-3)+\ldots+(n+2)\Big)+(i-1)i \\ &+ 2\Big((i+1)+(i+3)+\ldots+n\Big) \\ &= \frac{(2m-i+2)(4n-2m+i)}{2}+\frac{(i-1)^2}{2}-\frac{1}{2}. \qquad (14) \end{aligned}$$

From (12) and (14) we see that regardless of the parity of i, $\sum\limits_{v\in X_i} d_G(u,v)$ is same. Thus

$$\begin{aligned} \sum_{\substack{v\in X_i \\ 2\le i\le 2m-n}} d_G(u,v) &= \sum_{2\le i\le 2m-n}\Big(\frac{(2m-i+2)(4n-2m+i)}{2}+\frac{(i-1)^2}{2}-\frac{1}{2}\Big) \\ &= 2m^2n+mn^2+3mn-n^3-3n^2-2n. \qquad (15) \end{aligned}$$

(C): Here we compute the distances in G from $u = x_{1,1}$ the vertices of $X_i,\ 2m-n+1 \le i \le m+1$; as the distances from $x_{1,1}$ to all the vertices of $X_i,\ 2m-n+1 \le i \le m+1$, are the same as in the **(B)** of the proof of Theorem

2.3, we use same distances as in Theorem 2.3 to calculate the following sum.

$$\begin{aligned}\sum_{\substack{v\in X_i\\2m-n+1\le i\le m+1}} d_G(u,v) &= \sum_{2m-n+1\le i\le m+1}\Big(\frac{(2m-i+2)^2}{2}+\frac{(i-1)^2}{2}+n(n+1)-\frac{1}{2}\Big)\\ &= \frac{1}{3}\Big(4n^3+9n^2+5n-4m^3+6m^2n+3m^2-6mn^2-6mn+m\Big).\end{aligned} \tag{16}$$

Substituting the values obtained in (10), (15) and (16) in (9), we have

$$\begin{aligned}\sum_{u,v\in V(G)} d_G(u,v) &= \sum_{v\in X_1} d_G(u,v)+2\Big(\sum_{\substack{v\in X_i\\2\le i\le 2m-n}} d_G(u,v)+\sum_{\substack{v\in X_i\\2m-n+1\le i\le m+1}} d_G(u,v)\Big)\\ &= 2(2mn+n-m^2)+2\Big(2m^2n+mn^2+3mn-n^3-3n^2-2n\Big)\\ &+ 2\Big(\frac{1}{3}(4n^3+9n^2+5n-4m^3+6m^2n+3m^2-6mn^2-6mn+m)\Big)\\ &= \frac{2}{3}\Big(n^3+2n-4m^3+12m^2n-3mn^2+9mn+m\Big).\end{aligned} \tag{17}$$

It has been verified that when n is even $\sum\limits_{u,v\in V(G)} d_G(u,v)$ is same as (17). Hence regardless of the parity of n, the sum $\sum\limits_{u,v\in V(G)} d_G(u,v)$ is the same.

As the graph G is vertex transitive, the sum of the distances from $u=x_{1,1}$ to all other vertices of G is same as the sum of the distances from $x_{i,j}$ to all other vertices of G, for all $i,\ j,\ 1\le i\le 2m+1,\ 1\le j\le 2n+1$. Hence

$$\begin{aligned}W(G) &= \frac{|V(G)|}{2}\Big(\sum_{u,v\in V(G)} d_G(u,v)\Big)\\ &= \frac{(2m+1)(2n+1)}{2}\Big(\frac{2}{3}(n^3+2n-4m^3+12m^2n-3mn^2+9mn+m)\Big)\\ &= \frac{(2m+1)(2n+1)}{3}\Big(n^3+2n-4m^3+12m^2n-3mn^2+9mn+m\Big).\end{aligned}$$

3. The Weiner Index of $C_{2m}\times C_{2n+1}$.

In Theorems 5 and 6 we considered the tensor product of two odd cycles. Here we consider the case where one cycle is of odd length and the other is of even length.

Theorem 7. *If $r=2m$ and $s=2n+1$, then for $G=C_r\times C_s$,* $W(G)=\frac{m^2(2n+1)}{3}\Big(12n^2+12n+m^2+2\Big)$.

Proof. As in the proof of the previous theorems, it is enough to compute the sum of the distances, in G, from $u=x_{1,1}$ to all other vertices of G.

$$\sum_{u,v\in V(G)} d_G(u,v)=\sum_{v\in X_1} d_G(u,v)+2\Big(\sum_{v\in X_i,2\le i\le m} d_G(u,v)\Big)+\sum_{v\in X_{m+1}} d_G(u,v) \tag{18}$$

since the distances from u to all the vertices of X_i is identical with the distances from u to all the vertices of $X_{2m-i+2},\ 2 \le i \le m$; this is true because the length of a shortest path that descends to a vertex from $u = x_{1,1}$ to a vertex of $X_i,\ 2 \le i \le m$, is same as the length of a shortest path that goes from $x_{1,1}$ to X_{2m} and then ascending to a vertex in X_{2m-i+2} are the same. We shall calculate the sum of the terms of (18), separately.

If n is odd, then

$$\sum_{v \in X_1} d_G(u,v) = 2(2+4+\ldots+2n) = 2n(n+1), \tag{19}$$

Explanations for the terms involved in the above equation are as follows: $d_G(u, x_{1,j}) = j-1$, for $j = 3,5,7,\ldots,n$; see Fig.4, and the distances from u to the vertices $x_{1,2}, x_{1,4}, x_{1,6}, \ldots, x_{1,n+1}$ are $2n, 2n-2, 2n-4, \ldots, n+1$, respectively, see Fig.11. The multiplication factor 2 appears in all the terms of the sum in (19) because $d_G(u, x_{1,j}) = d_G(u, x_{1,2n-j+3}),\ 2 \le j \le n+1$, due to the "symmetry" of the graph G.

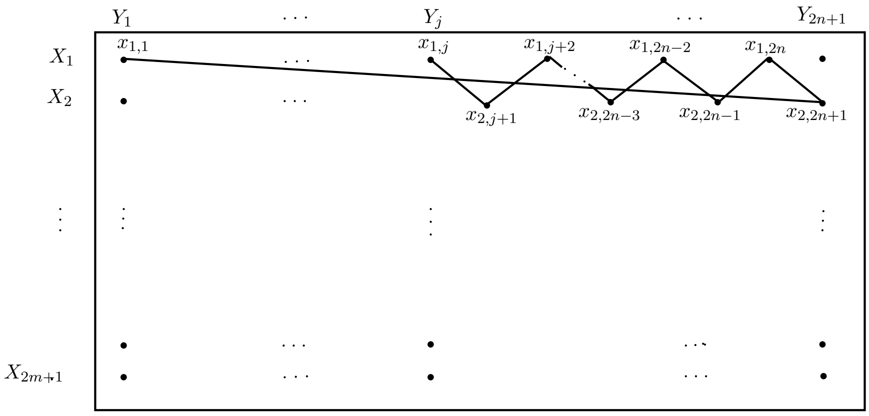

Fig.11 Vertices of $C_r \times C_s$

To compute the sum of the second term of the equation (18), we need $\sum_{v \in X_i} d_G(u,v)$ for each $i,\ 2 \le i \le m$. First we compute it.

$$\sum_{v \in X_i} d_G(u,v) = \begin{cases} (2n+1) + \{(2n-1) + (2n-3) + \ldots + (n+2)\} \\ +2\{\underbrace{(i-1) + \ldots + (i-1)}_{\frac{i}{2}\ times} + (i+1) + (i+3) + \ldots + n\} \text{ if } i \text{ is even,} \\ (i-1) + 2\{\underbrace{(i-1) + \ldots + (i-1)}_{\frac{i-1}{2}\ times} + (i+1) + (i+3) + \ldots + n - 1\} \\ +\{(2n-1) + (2n-3) + \ldots + (n+1)\} \text{ if } i \text{ is odd,} \end{cases} \tag{20}$$

Explanations for the terms involved in the above equation are given below: If i is even, then $d_G(u, x_{i,1}) = 2n+1$, see Fig.12, and the distances from u to the vertices $x_{i,3}, x_{i,5}, x_{i,7}, \ldots, x_{i,n}$ are $2n-1, 2n-3, 2n-5, \ldots, n+2$, respectively, see Fig.12. Further, $d_G(u, x_{i,j}) = i-1,\ j = 2, 4, \ldots, i$; the pattern of the path is similar to the one shown in Fig.6, and the distances from u to the vertices $x_{i,i+2}, x_{i,i+4}, x_{i,i+6}, \ldots, x_{i,n+1}$ are $i+1, i+3, i+5, \ldots, n$, respectively; the pattern of the path is similar to the one shown in Fig.7. If i is odd, then $d_G(u, x_{i,1}) = i-1$; the pattern of the path is similar to the one shown in Fig.5, and $d_G(u, x_{i,j}) = i-1,\ j = 3, 5, \ldots, i$; the pattern of the path is similar to the one shown in Fig.6.

The distances from u to the vertices $x_{i,i+2}, x_{i,i+4}, x_{i,i+6}, \ldots, x_{i,n}$ are $i+1, i+3, i+5, \ldots, n-1$, respectively; the pattern of the path is similar to the one shown in Fig.7. The distances from u to the vertices $x_{i,3}, x_{i,5}, x_{i,7}, \ldots, x_{i,n+1}$ are $2n, 2n-2, 2n-4, \ldots, n+1$, respectively, see Fig.12. The multiplication factor 2 appears in all the terms except the first term of the sum because $d_G(u, x_{1,j}) = d_G(u, x_{1,2n-j+3}),\ 2 \le j \le n+1$, due to the "symmetry" of the graph G.

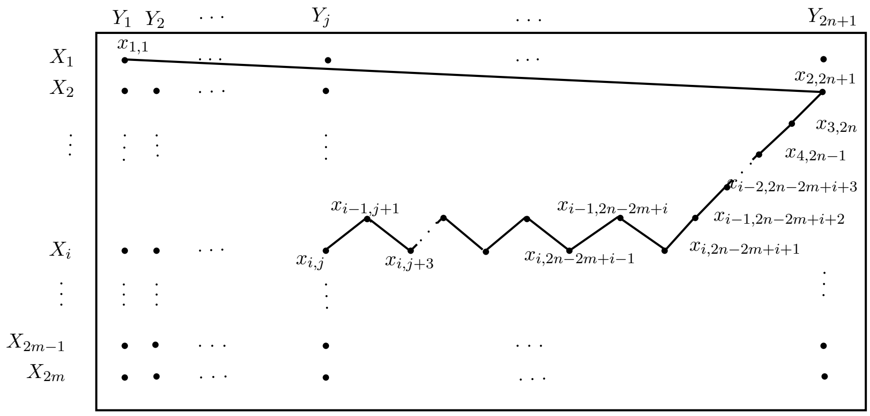

Fig.12 Vertices of $C_r \times C_s$

The summation of the terms of (20) gives

$$\sum_{v \in X_i} d_G(u, v) = \begin{cases} \frac{(i-1)^2}{2} + 2n(n+1) \text{ if } i \text{ is odd,} \\ \frac{(i-1)^2}{2} + 2n(n+1) + \frac{1}{2} \text{ if } i \text{ is even.} \end{cases} \tag{21}$$

Next we compute $\sum\limits_{v\in X_i,\ 2\leq i\leq m} d_G(u,v)$. For this, first we assume m is odd.
For odd m,

$$\begin{aligned}\sum_{v\in X_i,\ 2\leq i\leq m} d_G(u,v) &= \sum_{2\leq i\leq m}\Big(\frac{(i-1)^2}{2}+2n(n+1)\Big)+\sum_{i=2,4,\ldots,m-1}\frac{1}{2}\\ &= \frac{(m-1)}{12}\Big((2m^2-m+3)+24n(n+1)\Big) \qquad (22)\end{aligned}$$

For an even m,

$$\begin{aligned}\sum_{v\in X_i,\ 2\leq i\leq m} d_G(u,v) &= \sum_{2\leq i\leq m}\Big(\frac{(i-1)^2}{2}+2n(n+1)\Big)+\sum_{i=2,4\ldots,m}\frac{1}{2}\\ &= \frac{m}{12}(2m^2-3m+4)+2n(n+1)(m-1) \qquad (23)\end{aligned}$$

Next we compute the distances from u to all the vertices of X_{m+1}.

$$\sum_{v\in X_{m+1}} d_G(u,v)=\begin{cases} m+2\Big(\underbrace{m+\ldots+m}_{\frac{m}{2}\ times}+(m+2)+(m+4)+\ldots+2n\Big) \text{ if } m \text{ is even,}\\ (2n+1)+2\Big(\underbrace{m+\ldots+m}_{\frac{m+1}{2}\ times}+(m+2)\\ +(m+4)+\ldots+(2n-1)\Big) \text{ if } m \text{ is odd.}\end{cases}$$

After summing the above terms, we get

$$\sum_{v\in X_{m+1}} d_G(u,v) = \begin{cases}\frac{m^2}{2}+2n(n+1) \text{ if } m \text{ is even,}\\ \frac{m^2-1}{2}+2n(n+1)+1 \text{ if } m \text{ is odd.}\end{cases} \qquad (24)$$

Substituting the values obtained in (19), (23) and (24) in (18), we have
If m is even, then

$$\begin{aligned}\sum_{u,v\in V(G)} d_G(u,v) &= \sum_{v\in X_1} d_G(u,v)+2\Big(\sum_{v\in X_i,2\leq i\leq m} d_G(u,v)\Big)+\sum_{v\in X_{m+1}} d_G(u,v)\\ &= 2n(n+1)+\Big(\frac{m}{12}(2m^2-3m+4)+2n(n+1)(m-1)\Big)\\ &\quad +\Big(\frac{m^2}{2}+2n(n+1)\Big)\\ &= \frac{m}{3}\Big(12n^2+12n+m^2+2\Big), \qquad (25)\end{aligned}$$

If m is odd, again, $\sum\limits_{u,v\in V(G)} d_G(u,v)$ is found to be the same as (25). Hence $\sum\limits_{u,v\in V(G)} d_G(u,v)=\frac{m}{3}(12n^2+12n+m^2+2)$, irrespective of m is odd or even.

The proof is similar when n is even and in this case also $\sum_{u,v\in V(G)} d_G(u,v)$ is found to be same as (25); we omit the details.

As the graph G is vertex transitive, the sum of the distances from $u = x_{1,1}$ to all other vertices of G is same as the sum of the distances from $x_{i,j}$ to all other vertices of G, for all $i,\ j,\ 1 \le i \le 2m,\ 1 \le j \le 2n+1$. Hence

$$\begin{aligned} W(G) &= \frac{|V(G)|}{2}\Big(\sum_{u,v\in V(G)} d_G(u,v)\Big) \\ &= \frac{2m(2n+1)}{2}\Big(\frac{m}{3}(12n^2+12n+m^2+2)\Big),\ \textit{by}\ (25) \\ &= \frac{m^2(2n+1)}{3}\Big(12n^2+12n+m^2+2\Big). \end{aligned}$$

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