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SOME CHARACTERIZATIONS OF SEMIGROUPS IN TERMS OF INTUITIONISTIC FUZZY INTERIOR IDEALS

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Abstract. The importance of semigroups and their fuzzy subsystems is evident from their applications and significant role in several applied disciplines like computer sciences, control engineering, error-correcting codes and fuzzy automata theory. In this paper, we give generalizations of intuitionistic fuzzy interior ideals of semigroups and introduced the notions of intuitionistic fuzzy interior ideals of type $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ and $(\overline{\in}, \overline{\in})$ of semigroups. The important mile stone of the present paper is to link ordinary intuitionistic fuzzy interior ideals, $(\overline{\in}, \overline{\in})$-intuitionistic fuzzy interior ideals and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideals. Moreover semigroups are characterized by the properties of these notions.

Key words : Interior ideal; Intuitionistic fuzzy interior ideal; Intuitionistic fuzzy interior ideal; $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy interior ideal; $(\overline{\in}, \overline{\in})$-fuzzy interior ideal.
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1. Introduction

Fuzzy set theory by Zadeh [22] is a useful tool to handle situations whenever it is not clear to decide whether a certain point belongs to a certain set or not by giving a grade of membership to each point but there is no instrument to attribute the grade of non-membership to each point. Keeping this point in view, Atanassov [1] generalized the concept given in [22] and introduced the

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notion of intuitionistic fuzzy sets. Later on, Biswas [3] applied the concept of intuitionistic fuzzy set in algebraic structure and proposed a more general form of Rosenfeld's fuzzy subgroup [17] known as intuitionistic fuzzy subgroups. In addition, Bustince and Burillo [4] initiate the idea of intuitionistic fuzzy relations, where Kim et al. [12] gave the concept of intuitionistic fuzzy subquasigroups. Meanwhile, Kim and Jun [13, 14] characterized semigroups by the properties of intuitionistic fuzzy (interior, left, right, bi-) ideals.

Pu and Liu [16] introduced the notions of "belongs to relation" ($\in$) and "quasi-coincidence with relation" (q) between a fuzzy point and fuzzy set. Meanwhile, Bhakat and Das [2] used the idea of [16] and introduced $(\in, \in \vee q)$-fuzzy subgroup as another generalization of Rosenfeld's fuzzy subgroups [17]. Further, Jun and Song [7] gave the concept of an (α, β)-fuzzy interior ideal, which is a generalization of a fuzzy interior ideal, where Davvaz and Khan [6] studied some characterizations of regular ordered semigroups in terms of (α, β)-fuzzy generalized bi-ideals. In addition, Khan et. al. [10] introduced $(\in, \in \vee \mathrm{q}_k)$-intuitionistic fuzzy interior ideals of ordered semigroups. Kazanci and Yamak [9] introduced generalized fuzzy bi-ideal in semigroups and gave some properties of fuzzy bi-ideals in terms of $(\in, \in \vee \mathrm{q})$-fuzzy bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}})$-fuzzy bi-ideals. Further, Shabir et al. [19, 20] characterized regular semigroups by the properties of $(\in, \in \vee \mathrm{q})$-fuzzy ideals, bi-ideals and quasi-ideals and introduced more general forms of (α, β)-fuzzy ideals and defined $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideals of semigroups. Moreover, the notion of $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}})$-fuzzy interior ideal of semigroup is introduced by Zhan and Jun [23]. In addition Shabir et al. [21] studied $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}}_k)$-fuzzy ideals, generalized bi-ideals and quasi-ideals of a semigroup and characterized regular semigroups by the properties of these ideals.

In continuation of [21, 23], we introduce the notion of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy subsystems of semigroups namely $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideals and $(\overline{\in}, \overline{\in})$-intuitionistic fuzzy interior ideals of semigroups. Moreover, ordinary intuitionistic fuzzy interior ideals are linked with intuitionistic fuzzy interior ideals of type $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ and $(\overline{\in}, \overline{\in})$ by using level subsets.

2. Preliminaries

In what follows S will represent a semigroup unless otherwise stated. A non-empty subset A of S is called a *subsemigroup* of S if $A^2 \subseteq A$ [7].

A non-empty subset A of a semigroup S is called an *interior ideal* of S if:
(i) $A^2 \subseteq A$,
(ii) $SAS \subseteq A$. (see [7]).

A non-empty subset A of a semigroup S is called left (right) *ideal* of S if $SA \subseteq A$ $(AS \subseteq A)$ [15].

A function $f : S \to [0,\ 1]$ is called a fuzzy set in S [15].

If $f(xy) \geq \min\{f(x),\ f(y)\}$ for all $x, y \in S$, then f is called a fuzzy subsemigroup of S [7].

A fuzzy set f in S is called a fuzzy interior ideal if the following hold for all $x, y, a \in S$:

(iii) $f(xy) \geq \min\{f(x),\ f(y)\}$,

(iv) $f(xay) \geq f(a)$ (see [7]).

A fuzzy set f in S is called a fuzzy left (right) ideal of S if $f(xy) \geq\ f(y)$ $(f(xy) \geq\ f(x))$ for all $x, y \in S$ [15].

Next, some basic concepts of intuitionistic fuzzy set theory are given in the following lines.

Definition 1. [1] *An intuitionistic fuzzy subset (briefly IFS) A in a nonempty set X is an object having the form* $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in X\}$, *where the function* $\mu_A : X \longrightarrow [0, 1]$ *and* $\gamma_A : X \longrightarrow [0, 1]$ *denote the degree of membership (namely* $\mu_A(x)$*) and the degree of non-membership (namely* $\gamma_A(x)$*) of each element* $x \in X$ *to the set A, respectively and* $0 \leq \mu_A(x)+\gamma_A(x) \leq 1$ *for all* $x \in X$*. For the sake of simplicity we shall use the symbol* $A = \langle x, \mu_A, \gamma_A\rangle$ *for the intuitionistic fuzzy set* $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in X\}$.

Note: An intuitionistic fuzzy set $A = \langle x, \mu_A, \gamma_A\rangle$ has two membership functions namely μ_A (used for the degree of membership) and γ_A(used for the degree of non-membership). Hence in the definitions of intuitionistic fuzzy subsemigroup, ideals etc. we always use two conditions one is used for the degree of membership and other is used for the degree of non-membership.

Definition 2. [14] *Let* $A = \langle x, \mu_A, \gamma_A\rangle$ *be an IFS of S, then A is called an intuitionistic fuzzy subsemigroup of S if for all* $x, y \in S$

$$(\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \text{ and } \gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}).$$

Definition 3. [14] *An intuitionistic fuzzy subset* $A = \langle x, \mu_A, \gamma_A\rangle$ *of a semigroup S is called an intuitionistic fuzzy left (right) ideal of a semigroup S if the following condition holds:*

(v) $\mu_A(xy) \geq \mu_A(y)$ and $\gamma_A(xy) \leq \gamma_A(y)\}$

$(\mu_A(xy) \geq \mu_A(x)$ and $\gamma_A(xy) \leq \gamma_A(x)\})$ for all $x, y, a \in S$.

Whereas, if $A = \langle x, \mu_A, \gamma_A\rangle$ is *an intuitionistic fuzzy left as well as intuitionistic fuzzy right ideal*, then it is called *an intuitionistic fuzzy ideal.*

Definition 4. [13] *An intuitionistic fuzzy subset* $A = \langle x, \mu_A, \gamma_A\rangle$ *of a semigroup S is called an intuitionistic fuzzy interior ideal of S if the following conditions hold for all* $x, y, a \in S$*:*

(vi) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,

(vii) $\mu_A(xay) \geq \mu_A(a)$ and $\gamma_A(xay) \leq \gamma_A(a)$.

Definition 5. [13] *The μ_A-level cut and γ_A-level cut of an intuitionistic fuzzy subset $A = \langle x, \mu_A, \gamma_A \rangle$ of a semigroup S respectively are denoted and defined as*

$$U(\mu_A; t) = \{x \in S \mid \mu_A(x) \geq t\} \text{ and } L(\gamma_A; s) = \{x \in S \mid \gamma_A(x) \leq s\},$$

where $t \in (0, 1]$ and $s \in [0, 1)$. And the (μ_A, γ_A)-level (t, s)-cut is defined as

$$C_{(t,s)}(A) = \{x \in S \mid \mu_A(x) \geq t \text{ and } \gamma_A(x) \leq s\}.$$

It is clear that $C_{(t,s)}(A) = U(\mu_A; t) \cup L(\gamma_A; s)$.

Definition 6. [11] *For any IFS $A = \langle x, \mu_A, \gamma_A \rangle$ of S, $t \in (0, 1]$ and $s, k \in [0, 1)$, the following two level subsets are defined as:*

$$\begin{aligned} Q^k_{(t,s)}(A) \quad &: \quad = \{x \in S \mid [x; (t, s)]\, q_k A\} \\ &\quad\; \text{and} \\ [A]^k_{(t,s)} \quad &: \quad = \{x \in S \mid [x; (t, s)] \in \vee q_k A\}. \end{aligned}$$

It is obvious that $[A]^k_{(t,s)} = C_{(t_1,s_1)}(A) \cup Q^k_{(t,s)}(A)$.

Proposition 1. [13] *Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy subset of S. Then $A = \langle x, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy interior ideal of S if and only if $(\forall t \in (0, 1], s \in [0, 1))\, C_{(t,s)}(A)\, (\neq \phi)$ is an interior ideal of S.*

Definition 7. [5] *An intuitionistic fuzzy set $A = \langle x, \mu_A, \gamma_A \rangle$ of the form*

$$\mu_A(y) := \begin{cases} t, & \text{if} \quad x = y, \\ 0, & \text{if} \quad x \neq y, \end{cases} \quad \text{and } \gamma_A(y) := \begin{cases} s, & \text{if} \quad x = y, \\ 1, & \text{if} \quad x \neq y, \end{cases}$$

is called an intuitionistic fuzzy point (IFP for short) in X and is denoted by $[x; (t, s)]$ where $t \in (0, 1]$ and $s \in [0, 1)$ represent the degree of membership and the degree of non-membership of $[x; (t, s)]$ and $x \in X$ is the support of $[x; (t, s)]$.

An intuitionistic fuzzy point $[x; (t, s)]$ is said to be belong to an intuitionistic fuzzy set $A = \langle x, \mu_A, \gamma_A \rangle$ and is denoted by $[x; (t, s)] \in A$, if $\mu_A(x) \geq t$ and $\gamma_A \leq s$. On the other hand $[x; (t, s)]$ is said to be quasai-coincident with $A = \langle x, \mu_A, \gamma_A \rangle$ denoted by $[x; (t, s)]qA$, if $\mu_A(x) + t > 1$ and $\gamma_A + s < 1$. If $[x; (t, s)] \in A$ or $[x; (t, s)]qA$, then we write $[x; (t, s)] \in \vee qA$ and if $[x; (t, s)] \in A$ and $[x; (t, s)]qA$ then it is dented by $[x; (t, s)] \in \wedge qA$). If $[x; (t, s)]\alpha A$ do not hold, then we write $[x; (t, s)]\overline{\alpha} A$.

Definition 8. [8] *For a non-empty subset A of S the characteristic function $\chi_A =< x, \mu_{\chi_A}(x), \gamma_{\chi_A}(x) >$ of A is defined by*

$$\begin{aligned} \mu_{\chi_A}(x) \quad &: \quad S \longrightarrow [0, 1], x \mapsto \mu_{\chi_A}(x) := \begin{cases} 1, & \text{if} \quad x \in A, \\ 0, & \text{if} \quad x \notin A, \end{cases} \\ \gamma_{\chi_A}(x) \quad &: \quad S \longrightarrow [0, 1], x \mapsto \gamma_{\chi_A}(x) := \begin{cases} 0, & \text{if} \quad x \in A, \\ 1, & \text{if} \quad x \notin A. \end{cases} \end{aligned}$$

The following proposition links classical subsystems and intuitionistic fuzzy subsystems on which the whole theory is defined and therefore is a very important one.

Proposition 2. [18] *A non-empty subset A of an ordered semigroup S is an interior ideal of S if and only if the intuitionistic characteristic function χ_A of A is an intuitionistic fuzzy interior ideal of S.*

3. $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-INTUITIONISTIC FUZZY INTERIOR IDEALS

Throughout in this paper S will represent a semigroup and by $[x;(t,s)]\overline{q}_k A$ we mean $\mu_A(x)+t+k \leq 1$ and $\gamma_A+s+k \geq 1$. In this section, by using generalized quasai-coincident with relation $(\overline{q}_k)$ the concepts of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal and $(\overline{\in}, \overline{\in})$-intuitionistic fuzzy interior ideal are defined in semigroups. Further, a suitable example has been constructed to support the defined concept. Moreover, some characterization results of semigroups in terms of intuitionistic fuzzy interior ideals of types $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ and $(\overline{\in}, \overline{\in})$.

Definition 9. *Consider an intuitionistic fuzzy subset $A =< x, \mu_A(x), \gamma_A(x) >$ of S. Let $x, y, a \in S$, $t, t_1, t_2 \in (0,1]$, $s, s_1, s_2 \in [0,1)$ and $k \in [0,1)$. Then we say that A is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S if the following conditions are satisfied:*

(C_1) $[xy; \min\{t_1,t_2\}, \max\{s_1,s_2\}]\overline{\in} A \rightarrow [x;(t_1,s_1)]\overline{\in} \vee \overline{q}_k A$ or $[y;(t_2,s_2)]\overline{\in} \vee \overline{q}_k A$,

(C_2) $[xay;(t,s)]\overline{\in} A \rightarrow [a;(t,s)]\overline{\in} \vee \overline{q}_k A$.

Example 1. *Consider a semigroup $S = \{a, b, c, d, e\}$ with the multiplication given below in Table 1:*

Table: 1

.	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

Define an IFS $A =< x, \mu_A, \gamma_A >$ by

$$\mu_A(x) = \begin{cases} 0.45, & \text{if} \quad x = a, \\ 0.50, & \text{if} \quad x = b, \\ 0.45, & \text{if} \quad x = c, \\ 0.60, & \text{if} \quad x = d, \\ 0.50, & \text{if} \quad x = e, \end{cases} \quad \text{and } \gamma_A(x) = \begin{cases} 0.50, & \text{if} \quad x = a, \\ 0.40, & \text{if} \quad x = b, \\ 0.35, & \text{if} \quad x = c, \\ 0.35, & \text{if} \quad x = d, \\ 0.20, & \text{if} \quad x = e. \end{cases}$$

Then, by routine calculation it can be checked that $A =< x, \mu_A, \gamma_A >$ is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S.

Theorem 3. *If $A =< x, \mu_A(x), \gamma_A(x) >$ is an intuitionistic fuzzy subset of S, then A is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S if and only if for all $x, y, a \in S$ the following conditions hold:*

(C_3) $\max\{\mu_A(xy), \frac{1-k}{2}\} \geq \min\{\mu_A(x), \mu_A(y)\}$ and
$\min\{\gamma_A(xy), \frac{1-k}{2}\} \leq \max\{\gamma_A(x), \gamma_A(y)\}$,
(C_4) $\max\{\mu_A(xay), \frac{1-k}{2}\} \geq \mu_A(a)$ and
$\min\{\gamma_A(xay), \frac{1-k}{2}\} \leq \gamma_A(a)$.

Proof. (C_1)$\Rightarrow$(C_3) On contrary, if

$$\begin{aligned} \max\{\mu_A(ab), \frac{1-k}{2}\} &< \min\{\mu_A(a), \mu_A(b)\} = t \\ &\text{and} \\ \min\{\gamma_A(ab), \frac{1-k}{2}\} &> \max\{\gamma_A(a), \gamma_A(b)\} = s \end{aligned}$$

for some $a, b \in S$ and $t \in (\frac{1-k}{2}, 1]$, $s \in [0, \frac{1-k}{2})$. In which it follows that $\mu_A(ab) < t$ and $\gamma_A(ab) > s$ and hence $[ab; (t, s)] \overline{\in} A$. On the other hand $\mu_A(a) = t$ and $\gamma_A(a) = s$ or $\mu_A(b) = t$ and $\gamma_A(b) = s$ i.e., $[a; (t, s)] \in A$ or $[b; (t, s)] \in A$ but $[a; (t, s)] \in A$, $[b; (t, s)] \in A$ also $\mu_A(a) + t = t + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$ and $\gamma_A(a) + s = 2s < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$ or $\mu_A(b) + t = t + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$ and $\gamma_A(b) + s = 2s < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$ and therefore $[a; (t, s)]\, q_k A$ or $[b; (t, s)]\, q_k A$. This contradicts by (C_1) and hence (C_3) is true.

$(C_2)\Rightarrow(C_4)$ Again let us suppose on contrary that condition (C_4) is not true that is there exist $x, y, a \in S$ such that

$$\begin{aligned}\max\{\mu_A(xay), \frac{1-k}{2}\} &< \mu_A(a) \\ &\text{and} \\ \min\{\gamma_A(xay), \frac{1-k}{2}\} &> \gamma_A(a).\end{aligned}$$

Then there exist $t \in (\frac{1-k}{2}, 1]$ and $s \in [0, \frac{1-k}{2})$ such that

$$\begin{aligned}\max\{\mu_A(xay), \frac{1-k}{2}\} &< t \leq \mu_A(a) \\ &\text{and} \\ \min\{\gamma_A(xay), \frac{1-k}{2}\} &> s \geq \gamma_A(a),\end{aligned}$$

this shows that $[xay; (t, s)]\,\overline{\in} A$ but $[a; (t, s)] \in A$ and $[a; (t, s)]\, q_k A$ contradicting Condition (C_2). Therefore, we accept that for all $x, y, a \in S$

$$\begin{aligned}\max\{\mu_A(xay), \frac{1-k}{2}\} &\geq \mu_A(a) \\ &\text{and} \\ \min\{\gamma_A(xay), \frac{1-k}{2}\} &\leq \gamma_A(a).\end{aligned}$$

Conversely, $(C_3)\Rightarrow(C_1)$ If $[ab; \min\{t_1, t_2\}, \max\{s_1, s_2\}]\overline{\in} A$ for some $a, b \in S$, then $\mu_A(ab) < \min\{t_1, t_2\}$ and $\gamma_A(ab) > \max\{s_1, s_2\}$. We consider the following two cases:

Case I: If $\mu_A(ab) \geq \min\{\mu_A(a), \mu_A(b)\}$ and $\gamma_A(ab) \leq \max\{\gamma_A(a), \gamma_A(b)\}$, then $\min\{\mu_A(a), \mu_A(b)\} < \min\{t_1, t_2\}$ and $\max\{\gamma_A(a), \gamma_A(b)\} > \max\{s_1, s_2\}$. In which it follows that $\mu_A(a) < t_1$ and $\gamma_A(a) > s_1$ or $\mu_A(b) < t_2$ and $\gamma_A(b) > s_2$ that is $[a; (t_1, s_1)]\,\overline{\in} A$ or $[b; (t_2, s_2)]\,\overline{\in} A$.

Case II: If $\mu_A(ab) < \min\{\mu_A(a), \mu_A(b)\}$ and $\gamma_A(ab) > \max\{\gamma_A(a), \gamma_A(b)\}$, then by (C_3) $\frac{1-k}{2} \geq \min\{\mu_A(a), \mu_A(b)\}$ and $\frac{1-k}{2} \leq \max\{\gamma_A(a), \gamma_A(b)\}$. If $[a; (t_1, s_1)] \in A$ or $[b; (t_2, s_2)] \in A$, then $\frac{1-k}{2} \geq \mu_A(a) \geq t_1$ and $\frac{1-k}{2} \leq \gamma_A(a) \leq s_1$ or $\frac{1-k}{2} \geq \mu_A(b) \geq t_2$ and $\frac{1-k}{2} \leq \gamma_A(b) \leq s_2$, then $\mu_A(a) + t_1 \leq \frac{1-k}{2} + t_1 \leq \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$ or $\gamma_A(a) + t_1 \geq \frac{1-k}{2} + s_1 \geq \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$. It follows that $[a; (t_1, s_1)]\,\overline{q}_k A$ or $[b; (t_2, s_2)]\,\overline{q}_k A$. Hence, $[a; (t_1, s_1)]\,\overline{\in} \vee \overline{q}_k A$ or $[b; (t_2, s_2)]\,\overline{\in} \vee \overline{q}_k A$.

$(C_4)\Rightarrow(C_2)$ If for some $x, y, a \in S$ we have $[xay; (t, s)]\overline{\in} A$, then $\mu_A(xay) < t$ and $\gamma_A(xay) > s$. We consider the following two cases:

Case I: If $\mu_A(xay) \geq \mu_A(a)$ and $\gamma_A(xay) \leq \gamma_A(a)$, then $\mu_A(a) < t$ and $\gamma_A(a) > s$, it follows that $[a; (t, s)]\,\overline{\in} A$.

Case II: If $\mu_A(xay) < \mu_A(a)$ and $\gamma_A(xay) > \gamma_A(a)$, then $\frac{1-k}{2} \geq \mu_A(a)$, $\frac{1-k}{2} \leq \gamma_A(a)$ (by (C_4)). If $[a; (t, s)] \in A$, then $\frac{1-k}{2} \geq \mu_A(a) \geq t$ and $\frac{1-k}{2} \leq$

$\gamma_A(a) \leq s$ i.e., $[a;(t,s)]\,\overline{q}_k A$. Thus in both case we see that $[a;(t,s)]\,\overline{\in} \vee \overline{q}_k A$. □

Theorem 4. *Let $A =< x, \mu_A(x), \gamma_A(x) >$ be an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S and $\overline{\mu}_A(x) = 1 - \mu_A(x)$, then $\Box A =< x, \mu_A(x), \overline{\mu}_A(x) >$ is also is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S if $\overline{\mu}_A(x) \leq \frac{1-k}{2}$ for all $x \in S$.*

Proof. To prove that $\Box A =< x, \mu_A(x), \overline{\mu}_A(x) >$ is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S, it is enough to show that

$$\begin{aligned} \min\{\overline{\mu}_A(xy), \frac{1-k}{2}\} &\leq \max\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} \\ &\text{and} \\ \min\{\overline{\mu}_A(xay), \frac{1-k}{2}\} &\leq \overline{\mu}_A(a). \end{aligned}$$

If $a, x, y \in S$, then

$$\begin{aligned} \min\{\overline{\mu}_A(xy), \frac{1-k}{2}\} &= \min\{1 - \mu_A(xy), \frac{1-k}{2}\} \\ &= 1 - \mu_A(xy) \text{ (as } 1 - \mu_A(xy) \leq \frac{1-k}{2}) \\ &\leq 1 - \min\{\mu_A(x), \mu_A(y)\} \\ &= \max\{1 - \mu_A(x), 1 - \mu_A(y)\} \\ &= \max\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}, \end{aligned}$$

and

$$\begin{aligned} \min\{\overline{\mu}_A(xay), \frac{1-k}{2}\} &= \min\{1 - \mu_A(xay), \frac{1-k}{2}\} \\ &= 1 - \mu_A(xay) \text{ (as } 1 - \mu_A(xay) \leq \frac{1-k}{2}) \\ &\leq 1 - \mu_A(a) \\ &= \overline{\mu}_A(a). \end{aligned}$$

Follows that $\Box A$ is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S. □

Definition 10. *Consider a fuzzy set f in S. Then, f is called an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy interior ideal of S if the following hold $x, y, a \in S$, $t_1, t_2 \in (0, 1]$ and $k \in [0, 1)$:*

(C_5) $[xy; \min\{t_1, t_2\}]\overline{\in} f \rightarrow [x; t_1]\overline{\in} \vee \overline{q}_k f$ or $[y; t_2]\overline{\in} \vee \overline{q}_k f$,
(C_6) $[xay; t]\overline{\in} f \rightarrow [a; t]\overline{\in} \vee \overline{q}_k f$.

Example 2. *Consider a semigroup $S = \{a, b, c\}$ with the multiplication given below in Table 2:*

Table: 2

.	a	b	c
a	a	d	a
b	a	b	a
c	a	d	c

Define a fuzzy set $f : S \to [0,\ 1]$ by

$$f(x) = \begin{cases} 0.9, & \text{if} \quad x = a, \\ 0.7, & \text{if} \quad x = b, \\ 0.4, & \text{if} \quad x = c, \end{cases}$$

Then, it is easy to check that f is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy interior ideal of S.

Theorem 5. *A fuzzy subset f of S is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy interior ideal of S if and only if for all $x, y, a \in S$ the following conditions hold:*

(C_7) $\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}$,
(C_8) $\max\{f(xay), \frac{1-k}{2}\} \geq f(a)$.

Proof. Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy interior ideal of S and consider (C_7) does not hold for some $a, b \in S$ that is

$$\max\{f(ab), \frac{1-k}{2}\} < \min\{f(a), f(b)\},$$

then there exists $t \in (\frac{1-k}{2}, 1]$ such that

$$\max\{f(ab), \frac{1-k}{2}\} < t \leq \min\{f(a), f(b)\}.$$

In which it follows that $[ab; t]\,\overline{\in} f$ but $[a; t] \in f$, $[b; t] \in f$ also $[a; t]\, q_k f$, $[b; t]\, q_k f$, a contradiction by (C_5). Hence, (C_7) holds for all $x, y \in S$.

If $\max\{f(xay), \frac{1-k}{2}\} < f(a)$, then

$$\max\{f(xay), \frac{1-k}{2}\} < t_1 \leq f(a).$$

This implies $[xay; t_1]\,\overline{\in} f$ but $[a; t_1] \in f$ and $[a; t_1]\, q_k f$, again contradiction by (C_6). So that

$$\max\{f(xay), \frac{1-k}{2}\} \geq f(a)$$

for all $x, y, a \in S$.

Conversely, let both Condition (C_7) and (C_8) hold for all $x, y, a \in S$. If $[xy; \min\{t_1, t_2\}]\overline{\in} f$ for $x, y \in S$, $t_1, t_2 \in (0, 1]$, then $f(xy) < \min\{t_1, t_2\}$. The following two conditions are considered:

Case I: If $\min\{f(x), f(y)\} \leq f(xy)$, then

$$\min\{\mu_A(x), \mu_A(y)\} \leq f(xy) < \min\{t_1, t_2\}.$$

In which it follows that $\mu_A(x) < t_1$ or $\mu_A(y) < t_2$ that is $[x; t_1]\overline{\in} f$ or $[y; t_2]\overline{\in} f$.

Case II: If $f(xy) < \min\{f(x), f(y)\}$, then by (C_7) $\frac{1-k}{2} \geq \min\{f(x), f(y)\}$. If $[x; t_1] \in f$ or $[x; t_2] \in f$, then $\frac{1-k}{2} \geq f(x) \geq t_1$ or $\frac{1-k}{2} \geq f(y) \geq t_2$. It follows that $[x; t_1]\,\overline{q}_k f$ or $[y; t_2]\,\overline{q}_k f$. Thus in both case we see that $[x; t_1]\,\overline{\in} \vee \overline{q}_k f$ or $[y; t_2]\,\overline{\in} \vee \overline{q}_k f$.

Finally, consider $[xay; t]\overline{\in} f$, then $f(xay) < t$. We consider the following two cases:

Case I: If $f(xay) \geq f(a)$, then $f(a) < t$ and hence $[a; t]\,\overline{\in} f$.

Case II: If $f(xay) < f(a)$, then by (C_8) we have $\frac{1-k}{2} \geq f(a)$. If $[a; t] \in f$, then $\frac{1-k}{2} \geq f(a) \geq t$ and therefore $[a; t]\,\overline{q}_k f$. Thus in both case we see that $[a; t]\,\overline{\in} \vee \overline{q}_k f$. Eventually, f is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy interior ideal of S. □

The following result establish a relationship between intuitionistic fuzzy interior ideals and fuzzy interior ideals of type $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$.

Theorem 6. *An intuitionistic fuzzy set $A =< x, \mu_A(x), \gamma_A(x) >$ is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S if and only if and $\mu_A(x)$ and $\overline{\gamma}_A(x)$ are $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy interior ideal of S, where $\overline{\gamma}_A(x) = 1 - \gamma_A(x)$ and $\mu_A(x) \geq \frac{1-k}{2}$, $\gamma_A(x) \leq \frac{1-k}{2}$ for all $x \in S$.*

Proof. Let $A =< x, \mu_A(x), \gamma_A(x) >$ is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S. Then clearly $\mu_A(x)$ is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S. To prove that $\overline{\gamma}_A(x)$ is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S, it is enough to show that

$$\max\{\overline{\gamma}_A(xy), \frac{1-k}{2}\} \geq \min\{\overline{\gamma}_A(x), \overline{\gamma}_A(y)\}.$$

Let $x, y, a \in S$, then

$$\begin{aligned} \max\{\overline{\gamma}_A(xy), \frac{1-k}{2}\} &= \overline{\gamma}_A(xy) \\ &= 1 - \gamma_A(xy) \\ &= 1 - \min\{\gamma_A(xy), \frac{1-k}{2}\} \\ &\geq 1 - \max\{\gamma_A(x), \gamma_A(y)\} \\ &= \min\{1 - \gamma_A(x), 1 - \gamma_A(y)\}. \end{aligned}$$

and

$$\begin{aligned}\max\{\overline{\gamma}_A(xay), \frac{1-k}{2}\} &= \overline{\gamma}_A(xay)\\ &= 1-\gamma_A(xay)\\ &= 1-\min\{\gamma_A(xay), \frac{1-k}{2}\}\\ &\geq 1-\gamma_A(a)\\ &= \overline{\gamma}_A(a).\end{aligned}$$

Hence $\overline{\gamma}_A(x)$ is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy interior ideal of S. □

Theorem 7. *If $\{A_i : i \in \Lambda\}$ is non-empty family of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideals of S and $\mu_{A_i}(x) \geq \frac{1-k}{2}$, $\gamma_{A_i}(x) \leq \frac{1-k}{2}$ for all $x \in S$. Then $\cap A_i = (\wedge\mu_{A_i}, \vee\gamma_{A_i})$ is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S, where*

$$\begin{aligned}\wedge\mu_{A_i}(x) &= \inf\{\mu_{A_i}(x) : i \in \Lambda\},\\ &\quad \textit{and}\\ \vee\gamma_{A_i}(x) &= \sup\{\gamma_{A_i}(x) : i \in \Lambda\}.\end{aligned}$$

Proof. Let $x, y, a \in S$ and consider

$$\begin{aligned}\max\{\wedge\mu_{A_i}(xy), \frac{1-k}{2}\} &= \wedge\mu_{A_i}(xy),\\ &= \wedge\max\{\mu_{A_i}(xy), \frac{1-k}{2}\},\\ &\quad \text{by } (C_3)\\ &\geq \wedge\min\{\mu_{A_i}(x), \mu_{A_i}(y)\},\\ &= \min\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\}\},\\ &= \min\{\min\{\mu_{A_i}(x)\}, \min\{\mu_{A_i}(y)\}\},\\ &= \min\{\wedge\mu_{A_i}(x), \wedge\mu_{A_i}(y)\},\end{aligned}$$

and

$$\begin{aligned}\min\{\vee\gamma_{A_i}(xy), \frac{1-k}{2}\} &= \vee\gamma_{A_i}(xy),\\ &= \vee\min\{\gamma_{A_i}(xy), \frac{1-k}{2}\},\\ &\quad \text{by } (C_3)\\ &\leq \vee\max\{\gamma_{A_i}(x), \gamma_{A_i}(y)\},\\ &= \max\{\max\{\gamma_{A_i}(x), \gamma_{A_i}(y)\}\},\\ &= \max\{\max\{\gamma_{A_i}(x)\}, \max\{\gamma_{A_i}(y)\}\},\\ &= \max\{\vee\gamma_{A_i}(x), \vee\gamma_{A_i}(y)\}.\end{aligned}$$

Next consider

$$\begin{aligned}\max\{\wedge\mu_{A_i}(xay), \frac{1-k}{2}\} &= \wedge\mu_{A_i}(xay), \\ &= \wedge\max\{\mu_{A_i}(xay), \frac{1-k}{2}\}, \\ &\quad \text{by } (C_4) \\ &\geq \wedge\mu_{A_i}(a),\end{aligned}$$

and

$$\begin{aligned}\min\{\vee\gamma_{A_i}(xay), \frac{1-k}{2}\} &= \vee\gamma_{A_i}(xay), \\ &= \vee\min\{\gamma_{A_i}(xay), \frac{1-k}{2}\}, \\ &\quad \text{by } (C_4) \\ &\leq \vee\gamma_{A_i}(a).\end{aligned}$$

Follows that $\cap A_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S. □

Proposition 8. *If $\{f_i : i \in \Lambda\}$ be a non-empty family of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy interior ideals of S and $\mu_{A_i}(x) \geq \frac{1-k}{2}$ for all $x \in S$. Then $\cap f_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy interior ideal of S.*

Proof. The proof follows from *Theorem 7.* □

Definition 11. *An intuitionistic fuzzy subset $A =< x, \mu_A(x), \gamma_A(x) >$ of S is called an $(\overline{\in}, \overline{\in})$-intuitionistic fuzzy interior ideal of S, if for all $x, y, a \in S$, $t, t_1, t_2 \in (0, 1]$ and $s, s_1, s_2 \in [0, 1)$, satisfies the following conditions:*

(C_{11}) $[xy; \min\{t_1, t_2\}, \max\{s_1, s_2\}]\overline{\in}A \rightarrow [x; (t_1, s_1)]\overline{\in}A$ or $[y; (t_2, s_2)]\overline{\in}A$,
(C_{12}) $[xay; (t, s)]\overline{\in}A \rightarrow [a; (t, s)]\overline{\in}A$.

Example 3. *Consider a semigroup $S = \{0, a, b, c\}$ with the multiplication given below in Table 3:*

Table: 3

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Define an IFS $A =< x, \mu_A, \gamma_A >$ by

$$\mu_A \quad : \quad S \rightarrow [0,1] \mid \mu_A(x) = \begin{cases} 0.30, & \text{if} \quad x = 0, \\ 0.20, & \text{if} \quad x = a, \\ 0.40, & \text{if} \quad x = b, \\ 0.20, & \text{if} \quad x = c, \end{cases}$$

and

$$\gamma_A \quad : \quad S \rightarrow [0,1] \mid \gamma_A(x) = \begin{cases} 0.50, & \text{if} \quad x = 0, \\ 0.60, & \text{if} \quad x = a, \\ 0.30, & \text{if} \quad x = b, \\ 0.40, & \text{if} \quad x = c. \end{cases}$$

Then $A =< x, \mu_A, \gamma_A >$ is an $(\overline{\in}, \overline{\in})$-intuitionistic fuzzy interior ideals of S.

Proposition 9. *Let $A =< x, \mu_A(x), \gamma_A(x) >$ be an intuitionistic fuzzy subset of S. Then A is an intuitionistic fuzzy interior ideal of S if A is an $(\overline{\in}, \overline{\in})$-intuitionistic fuzzy interior ideal of S.*

Proof. Let A is an intuitionistic fuzzy interior ideal of S and $[xy; \min\{t_1, t_2\}, \max\{s_1, s_2\}]\overline{\in}A$, then $\mu_A(xy) < \min\{t_1, t_2\}$, $\gamma_A(xy) > \max\{s_1, s_2\}$. Then by definition of intuitionistic fuzzy interior ideal

$$\begin{aligned} \min\{\mu_A(x), \mu_A(y)\} &\leq \mu_A(xy) < \min\{t_1, t_2\} \\ &\text{and} \\ \max\{\gamma_A(x), \gamma_A(y)\} &\geq \gamma_A(xy) > \max\{s_1, s_2\}. \end{aligned}$$

In which it follows that $[x; \min\{t_1, t_2\}, \max\{s_1, s_2\}]\overline{\in}A$ or $[y; \min\{t_1, t_2\}, \max\{s_1, s_2\}]\overline{\in}A$.

Finally, if $[xay; (t, s)]\overline{\in}A$, then $\mu_A(xay) < t$ and $\gamma_A(xay) > s$ and by definition of intuitionistic fuzzy interior ideal

$$\mu_A(a) \leq \mu_A(xay) < t,$$

and

$$\gamma_A(a) \geq \gamma_A(xay) > s.$$

This shows that $[a; (t, s)]\overline{\in}A$. Hence, A is an $(\overline{\in}, \overline{\in})$-intuitionistic fuzzy interior ideal of S. □

From the above theorem it is clear that every intuitionistic fuzzy interior ideal is an $(\overline{\in}, \overline{\in})$-intuitionistic fuzzy interior ideal and it is obvious that every $(\overline{\in}, \overline{\in})$-intuitionistic fuzzy interior ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S.

In the next theorem we give a condition for an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S to be an $(\overline{\in}, \overline{\in})$-intuitionistic fuzzy interior ideal of S.

Theorem 10. *If $(\forall x \in S)$ $\mu_A(x) > \frac{1-k}{2}$, $\gamma_A(x) < \frac{1-k}{2}$, then an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S is an $(\overline{\in}, \overline{\in})$-intuitionistic fuzzy interior ideal of S.*

Proof. Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S and $\mu_A(x) > \frac{1-k}{2}$ and $\gamma_A(x) < \frac{1-k}{2}$ for all $x \in S$. Let $t_1, t_2 \in (0, 1]$, $s_1, s_2 \in [0, 1)$ and consider $[xy; \min\{t_1, t_2\}, \max\{s_1, s_2\}] \overline{\in} A$ then $\mu_A(xy) < \min\{t_1, t_2\}$, $\gamma_A(xy) > \max\{s_1, s_2\}$. By (C_3) we have;

$$\begin{aligned} \min\{\mu_A(x), \mu_A(y)\} &\leq \max\{\mu_A(xy), \frac{1-k}{2}\} < \max\{\min\{t_1, t_2\}, \frac{1-k}{2}\}, \\ &= \begin{cases} \min\{t_1, t_2\}, & \text{if } \min\{t_1, t_2\} \geq \frac{1-k}{2}, \\ \frac{1-k}{2}, & \text{if } \min\{t_1, t_2\} < \frac{1-k}{2}, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \max\{\gamma_A(x), \gamma_A(y)\} &\geq \min\{\gamma_A(xy), \frac{1-k}{2}\} > \min\left\{\max\{s_1, s_2\}, \frac{1-k}{2}\right\}, \\ &= \begin{cases} \frac{1-k}{2}, & \text{if } \max\{s_1, s_2\} \geq \frac{1-k}{2}, \\ \max\{s_1, s_2\}, & \text{if } \max\{s_1, s_2\} < \frac{1-k}{2}. \end{cases} \end{aligned}$$

This implies $\mu_A(x) < \min\{t_1, t_2\}$, $\gamma_A(x) > \max\{s_1, s_2\}$ or $\mu_A(y) < \min\{t_1, t_2\}$, $\gamma_A(y) > \max\{s_1, s_2\}$ i.e.

$$[x; (\min\{t_1, t_2\}, \max\{s_1, s_2\})] \overline{\in} A$$

or

$$[y; (\min\{t_1, t_2\}, \max\{s_1, s_2\}] \overline{\in} A.$$

Let $x, y, a \in S$ with $[xay; (t, s)] \overline{\in} A$, then $\mu_A(xay) < t$, $\gamma_A(xay) > s$ and from (C_4) we see that;

$$\begin{aligned} \mu_A(a) &\leq \max\{\mu_A(xay), \frac{1-k}{2}\} < \max\{t, \frac{1-k}{2}\}, \\ &= \begin{cases} t, & \text{if } t \geq \frac{1-k}{2}, \\ \frac{1-k}{2}, & \text{if } t < \frac{1-k}{2}, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(a) &\geq \min\{\gamma_A(xay), \frac{1-k}{2}\} > \min\{s, \frac{1-k}{2}\}, \\ &= \begin{cases} s, & \text{if } s \leq \frac{1-k}{2}, \\ \frac{1-k}{2}, & \text{if } s > \frac{1-k}{2}. \end{cases} \end{aligned}$$

Follows that $[a; (t, s)] \overline{\in} A$. Consequently, $A = \langle x, \mu_A, \gamma_A \rangle$ is an $(\overline{\in}, \overline{\in})$-intuitionistic fuzzy interior ideal of S. □

Theorem 11. *For an intuitionistic fuzzy subset $A = \langle x, \mu_A, \gamma_A \rangle$ of S, the following are equivalent for all $t \in (\frac{1-k}{2}, 1]$ and $s \in [0, \frac{1-k}{2})$:*

(C_{13}) A is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S.
(C_{14}) $C_{(t,s)}(A)\ (\neq \phi)$ is an interior ideal of S.

Proof. Assume that A is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S and $C_{(t,s)}(A) \neq \phi$. Let $x, y \in C_{(t,s)}(A)$, then $\mu_A(x) \geq t$, $\gamma_A(x) \leq s$ and $\mu_A(y) \geq t$, $\gamma_A(y) \leq s$. (C_5), implies that

$$\max\{\mu_A(xy), \frac{1-k}{2}\} \geq \min\{\mu_A(x), \mu_A(y)\} \geq \min\{t, t\} = t,$$

and

$$\min\{\gamma_A(xy), \frac{1-k}{2}\} \leq \max\{\gamma_A(x), \gamma_A(y)\} \leq \max\{s, s\} = s.$$

Thus $xy \in C_{(t,s)}(A)$.

Next we suppose $x, y, a \in S$ with $a \in C_{(t,s)}(A)$, then $\mu_A(a) \geq t$, $\gamma_A(a) \leq s$. From (C_6), we have that

$$\max\{\mu_A(xay), \frac{1-k}{2}\} \geq \mu_A(a) \geq t,$$

and

$$\min\{\gamma_A(xay), \frac{1-k}{2}\} \leq \gamma_A(a) \leq s.$$

Thus $xay \in C_{(t,s)}(A)$. Consequently $C_{(t,s)}(A)$ is an interior ideal of S.

Conversely, let $C_{(t,s)}(A)$ is an interior ideal of S. Assume that there exist $a, b \in S$ such that

$$\begin{aligned} \max\{\mu_A(ab), \frac{1-k}{2}\} &< \min\{\mu_A(a), \mu_A(b)\}, \\ &\text{and} \\ \min\{\gamma_A(ab), \frac{1-k}{2}\} &> \max\{\gamma_A(a), \gamma_A(b)\}, \end{aligned}$$

then for some $t_0 \in (\frac{1-k}{2}, 1]$ and $s_0 \in [0, \frac{1-k}{2})$

$$\begin{aligned} \max\{\mu_A(ab), \frac{1-k}{2}\} &< t_0 \leq \min\{\mu_A(a), \mu_A(b)\}, \\ &\text{and} \\ \min\{\gamma_A(ab), \frac{1-k}{2}\} &> s_0 \geq \max\{\gamma_A(a), \gamma_A(b)\}, \end{aligned}$$

follows that $a \in C_{(t_0,s_0)}(A)$ and $b \in C_{(t_0,s_0)}(A)$ but $ab \overline{\in} C_{(t_0,s_0)}(A)$, a contradiction. Therefore

$$\begin{aligned} \max\{\mu_A(ab), \frac{1-k}{2}\} &\geq \min\{\mu_A(a), \mu_A(b)\}, \\ &\text{and} \\ \min\{\gamma_A(ab), \frac{1-k}{2}\} &\leq \max\{\gamma_A(a), \gamma_A(b)\}, \end{aligned}$$

for all $x, y \in S$.

If

$$\max\{\mu_A(xay), \frac{1-k}{2}\} < \mu_A(a) \text{ and } \min\{\gamma_A(xay), \frac{1-k}{2}\} > \gamma_A(a),$$

for some $x, y, a \in S$, then there exist $t_1 \in (\frac{1-k}{2}, 1]$ and $s_1 \in s_0 \in [0, \frac{1-k}{2})$ such that

$$\begin{aligned} \max\{\mu_A(xay), \frac{1-k}{2}\} &< t_1 \leq \mu_A(a), \\ &\text{and} \\ \min\{\gamma_A(xay), \frac{1-k}{2}\} &> s_1 \geq \gamma_A(a), \end{aligned}$$

then $a \in C_{(t_0,s_0)}(A)$ but $xay \overline{\in} C_{(t_0,s_0)}(A)$, a contradiction. Therefore,

$$\begin{aligned} \max\{\mu_A(xay), \frac{1-k}{2}\} &\geq \mu_A(a), \\ &\text{and} \\ \min\{\gamma_A(xay), \frac{1-k}{2}\} &\leq \gamma_A(a), \end{aligned}$$

for all $x, y, a \in S$. Consequently, A is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S. □

Proposition 12. *Let $A = \langle x, \mu_A, \gamma_A \rangle$ is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S, then $Q^k_{(t,s)}(A) (\neq \phi)$ is an interior ideal of S for all $t \in (\frac{1-k}{2}, 1]$, $s \in [0, \frac{1-k}{2})$.*

Proof. Consider $A = \langle x, \mu_A, \gamma_A \rangle$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-intuitionistic fuzzy interior ideal of S. Let $t \in (\frac{1-k}{2}, 1]$ and $s \in [0, \frac{1-k}{2})$ such that $Q^k_{(t,s)}(A) \neq \phi$. Let $x, y \in Q^k_{(t,s)}(A)$ then $\mu_A(x) + t > 1$, $\gamma_A(x) + s < 1$ and $\mu_A(y) + t > 1$, $\gamma_A(y) + s < 1$. Then by (C_3)

$$\max\{\mu_A(xy), \frac{1-k}{2}\} \geq \min\{\mu_A(x), \mu_A(y)\} > \min\{1-t, 1-t\} = 1-t,$$

and

$$\min\{\gamma_A(xy), \frac{1-k}{2}\} \leq \max\{\gamma_A(x), \gamma_A(y)\} < \max\{1-s, 1-s\} = 1-s.$$

Thus $xy \in Q^k_{(t,s)}(A)$.

Next we suppose $x, y, a \in S$ such that $a \in Q^k_{(t,s)}(A)$, then $\mu_A(a) + t > 1$, $\gamma_A(a) + s < 1$. Then (C_4) implies

$$\begin{aligned} \max\{\mu_A(xay), \frac{1-k}{2}\} &\geq \mu_A(a) > 1-t \\ &\text{and} \\ \min\{\gamma_A(xay), \frac{1-k}{2}\} &\leq \gamma_A(a) < 1-s. \end{aligned}$$

Follows that $Q^k_{(t,s)}(A)$ is an interior ideal of S.

□

Conclusion: In the world of contemporary mathematics the use of algebraic structures in computer science, control theory and fuzzy automata theory always gain the interest of researchers. Algebraic structures particularly semigroups play a key role in such applied branches. Further, the fuzzification of several subsystems of semigroups are used in various models involving uncertainties. In this paper we introduced new types of subsystems of semigroup called $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subsystems and characterized semigroups in terms of $(\overline{\in}, \overline{\in} \vee \overline{q})$-intuitionistic fuzzy interior ideals and $(\overline{\in}, \overline{\in})$-intuitionistic fuzzy interior ideals. Finally, ordinary intuitionistic interior ideals and intuitionistic fuzzy interior ideal of type $(\overline{\in}, \overline{\in} \vee \overline{q})$ are connected by intuionistic fuzzy level subset.

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