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A NUMERICAL APPROACH FOR SOLVING HAMMERSTEIN INTEGRAL EQUATIONS IN BANACH SPACES

MOSTEFA NADIR

Abstract. In this work, we give a weaker conditions guarantee the boundedness of the Hammerstein integral equation in L^p spaces, also we study conditions of the convergence of the approximate solution to the exact one of the integral equation using the successive approximations method. Finally, we treat numerical examples compared with other papers in order to confirm the efficiency of our results.

Key words :Hammerstein integral equation, successive approximation, interpolation spaces.

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1. Introduction

Many problems which arise in mathematical physics, lead to mathematical models described by nonlinear integral equations. In particular, Hammerstein integral equations are of high applicability in different areas of applied mathematics, physics, potential theory and electostatics. Also, they are widely used in many applied areas, which include engineering, radiation of surface water wave, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomenal in biology, quantum mechanics, mathematical economics, and queuing theory. generally, we can say Hammerstein integral equation is one of the most practical ones. Many different methods have been used to approximate the solution of such integral equations. In [1, 4], an iterative scheme based on the homotopy analysis method has been used to solve this

Department of Mathematics University of Msila. Email: mostefanadir@yahoo.fr.

integral equations, in [10] an approach based on single-term Walsh series is proposed. We Note that, due to the Green's function we can transform any ordinary differential equation of the second order with boundary conditions into an Hammerstein integral equation of the general form

$$\psi(t_0) - \int_a^b k(t_0,t)l(t,\psi(t))dt = f(t_0), \tag{1}$$

or equivalently

$$\varphi(t_0) = \int_a^b k(t_0,t)l(t,\varphi(t))dt, \tag{2}$$

where $k(t_0,t)$ is a map from $[a,b]\times[a,b]$, into $\mathbb{R}$ and $l(t,\varphi(t))$ a nonlinear map from $[a,b]\times\mathbb{R}$, into $\mathbb{R}, \psi(t)$ and $f(t_0)$ are in $L^p([a,b])$. The goal of this paper is to give sufficient conditions for the existence and uniqueness of a solution $\varphi \in L^p([a,b])$ of the equation (1) under weaker hypotheses; where we shall assume that

1) the function $l(t,\varphi(t))$ is strongly measurable in t and continuous in φ.

2) $\|l(t,\varphi(t))\| \le a_0(t) + b_0\|\varphi\|$ for $t\in[a,b]$ and $\varphi(t)\in\mathbb{R}$, where $a_0 \in L^q([a,b])$ such that $\dfrac{1}{p}+\dfrac{1}{q}=1$ and $b_0\ge 0$. Let us recall that, another existence theorems for L^p-solutions of (2) with a kernel $k\in L^p$ were proved in the papers [7, 11]Obviously, in this paper the kernel k is not necessary L^p-integrable.

2. Main Results

Theorem 1. *Suppose that the functions $k(t_0,t)$, and $l(t,\varphi(t))$ satisfy the following conditions*

(A1) *The kernel $k(t_0,t)$ is measurable on $[a,b]\times[[a,b]$ and such that*

$$\left(\int_a^b |k(t_0,t)|^{\frac{p-\sigma}{q-1}}\,dt_0\right)^{\frac{q-1}{p-\sigma}} \le M_1, \text{ for all } t\in[a,b],$$

where $\sigma<p$ and $\sigma,p,q>1$.

(A2) *The kernel $k(t_0,t)$ is measurable on $[a,b]\times[[a,b]$ and such that*

$$\left(\int_a^b |k(t_0,t)|^{\frac{q-p+\sigma}{p+q}}\,dt\right)^{\frac{p+q}{q-p+\sigma}} \le M_2, \quad \text{for all } t_0\in[a,b].$$

(A3) *The function $l(t,\varphi(t))$ is a nonlinear map from $[a,b]\times\mathbb{R}$, into $\mathbb{R}$ satisfying the Carathéodory condition and such that*

$$|l(t,\varphi(t))| \le a_0(t) + b_0\,|\varphi(t)|^{\frac{p}{q}}$$

where $a_0(t)\in L^q([a,b],\mathbb{R})$, $b_0>0$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Under conditions ***(A1)***,***(A2)***,***(A3)****and* $q \leq p$ *the operator*

$$A\varphi(t_0) = \int_a^b k(t_0,t)l(t,\varphi(t))dt, \tag{3}$$

is a map from $L^p([a,b])$ *into* $L^p([a,b])$.

Proof. From the condition **(A3)**, we can write

$$|l(t,\varphi(t))|^q \leq \left(|a_0(t)| + b_0\,|\varphi(t)|^{\frac{p}{q}}\right)^q,$$

and therefore

$$\|l(t,\varphi(t))\|_q = \left(\int_a^b |l(t,\varphi(t))|^q\,dt\right)^{\frac{1}{q}} \leq \left(\int_a^b \left(|a_0(t)| + b_0\,|\varphi(t)|^{\frac{p}{q}}\right)^q dt\right)^{\frac{1}{q}}.$$

Using Minkovski's inequality, it comes

$$\begin{aligned}\|l(t,\varphi(t))\|_q &\leq c\left(\left(\int_a^b |a_0(t)|^q\right)^{\frac{1}{q}} + \left(\int_a^b b_0^q\,|\varphi(t)|^p\right)^{\frac{1}{q}}\right)\\ &\leq c\left(\|a_0(t)\|_q + b_0\,\|\varphi(t)\|_p^{\frac{p}{q}}\right).\end{aligned}$$

Hence the operator $l(t,\varphi(t))$ is a continuous element of $L^q([a,b])$ [12].
However, on the space $L^p([a,b])$ we consider,

$$A\varphi(t_0) = \int_a^b k(t_0,t)l(t,\varphi(t))dt, \tag{4}$$

where following [3], we have

$$\begin{aligned}|A\varphi(t_0)| &= \left|\int_a^b k(t_0,t)l(t,\varphi(t))dt\right|,\\ &\leq \int_a^b |k(t_0,t)l(t,\varphi(t))|\,dt,\\ &= \int_a^b \left(|k(t_0,t)|^{\frac{p-\sigma}{q-1}}\,|l(t,\varphi(t))|^q\right)^{\frac{1}{p}} |k(t_0,t)|^{\frac{q-p+\sigma}{q}}\,|l(t,\varphi(t))|^{1-\frac{q}{p}}\,dt,\\ &\leq \left(\int_a^b |k(t_0,t)|^{\frac{p-\sigma}{q-1}}\,|l(t,\varphi(t))|^q\,dt\right)^{\frac{1}{p}} \left(\int_a^b |k(t_0,t)|^{\frac{q-p+\sigma}{pq}}\,dt\right)^{\frac{1}{p}} \left(\int_a^b |l(t,\varphi(t))|^q\,dt\right)^{\frac{p-q}{pq}},\\ |A\varphi(t_0)| &\leq M_2^{\frac{q-p+\sigma}{p^2q}}\,\|l(t,\varphi(t))\|^{\frac{p-q}{p}} \left(\int_a^b |k(t_0,t)|^{\frac{p-\sigma}{q-1}}\,|l(t,\varphi(t))|^q\,dt\right)^{\frac{1}{p}},\end{aligned}$$

or again,

$$\begin{aligned}
|A\varphi(t_0)|^p &\leq \left(M_2^{\frac{q-p+\sigma}{p^2q}} \|l(t,\varphi(t))\|^{\frac{p-q}{p}} \left(\int_a^b |k(t_0,t)|^{\frac{p-\sigma}{q-1}} |l(t,\varphi(t))|^q \, dt \right)^{\frac{1}{p}} \right)^p \\
\left(\int_a^b |A\varphi(t_0)|^p \, dt_0 \right)^{\frac{1}{p}} &\leq M_2^{\frac{q-p+\sigma}{p^2q}} \|l(t,\varphi(t))\|^{\frac{p-q}{p}} \left(\int_a^b \left(\int_a^b |k(t_0,t)|^{\frac{p-\sigma}{q-1}} |l(t,\varphi(t))|^q \, dt \right) dt_0 \right)^{\frac{1}{p}} \\
&\leq M_2^{\frac{q-p+\sigma}{p^2q}} \|l(t,\varphi(t))\|^{\frac{p-q}{p}} \left(\int_a^b |k(t_0,t)|^{\frac{p-\sigma}{q-1}} \, dt_0 \right)^{\frac{1}{p}} \left(\int_a^b |l(t,\varphi(t))|^q \, dt \right)^{\frac{1}{p}} \\
&\leq M_2^{\frac{q-p+\sigma}{p^2q}} \|l(t,\varphi(t))\|^{1-\frac{q}{p}} M_1^{\frac{\sigma}{p}} \|l(t,\varphi(t))\|^{\frac{q}{p}} \\
\|A\varphi(t_0)\|_p &\leq M_2^{\frac{(p-\sigma)}{p}} M_1^{\frac{p-\sigma}{q}} \|l(t,\varphi(t))\|_q .
\end{aligned}$$

Hence, the operator $A\varphi(t_0)$ is well defined from $L^p([a,b])$ to $L^p([a,b])$. □

3. Existence and uniqueness of $L^p([a,b])$ solution

Consider the nonlinear integral equation

$$\varphi(t_0) = \int_a^b k(t_0,t) l(t,\varphi(t)) dt,$$

we would like to know what conditions one require on $k(t_0,t)$ and $l(t,\varphi(t))$ in order for this equation to have a solution $\varphi(t) \in L^p([a,b])$.

Theorem 2. *Suppose that the functions $k(t_0,t)$ and $l(t,\varphi(t))$ satisfy the following conditions*

(B1)*The kernel $k(t_0,t)$ belongs to the space L^p for all $t_0 \in [a,b]$*

$$\left(\int_a^b |k(t_0,t)|^p \, dt \right)^{\frac{1}{p}} \leq N_1(t_0), \quad \forall t_0 \in [[a,b].$$

(B2)*the function $l(t,\varphi(t))$ belongs to the space L^q for all $t \in [[a,b]$*

$$\left(\int_a^b |l(t,\varphi(t))|^q \, dt \right)^{\frac{1}{q}} \leq C,$$

and satisfying the Lipschitz condition

$$|l(t,\varphi_1(t)) - l(t,\varphi_2(t))| \leq L(t) \, |\varphi_1(t) - \varphi_2(t)| ,$$

with the function $L(t)$ belongs to the space $L^{\frac{pq}{p-q}}$ with $q \leq p$,

$$\left(\int_a^b |L(t)|^{\frac{pq}{p-q}}\, dt\right)^{\frac{p-q}{pq}} \leq N_2.$$

*Under assumptions **(B1)** and **(B2)**, the successive approximation*

$$\varphi_{n+1}(t_0) = \int_a^b k(t_0,t)l(t,\varphi_n(t))dt,$$

converges almost everywhere to the solution of the equation (2)*provided*

$$N_2^p \int_a^b N_1^p(t)dt = N^p < 1.$$

Proof. For this method we put $\varphi_0(t)$ as an identically null function and successively

$$\varphi_{n+1}(t_0) = \int_a^b k(t_0,t)l(t,\varphi_n(t))dt, \quad n = 0,1,2,...,n..,$$

and therefore, we obtain

$$\begin{aligned}
|\varphi_{n+1} - \varphi_n| &\leq \int_a^b |k(t_0,t)|\, |l(t,\varphi_n(t)) - l(t,\varphi_{n-1}(t))|\, dt,\\
|\varphi_{n+1} - \varphi_n| &\leq \int_a^b |k(t_0,t)|\, L(t)\, |\varphi_n - \varphi_{n-1}|\, dt,\\
&\leq \left(\int_a^b |k(t_0,t)|^p\, dt\right)^{\frac{1}{p}} \left(\int_a^b |L(t)|^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} \left(\int_a^b |\varphi_n - \varphi_{n-1}|^p\, dt\right)^{\frac{1}{p}}\\
|\varphi_{n+1} - \varphi_n|^p &\leq N_1^p(t_0)N_2^p \int_a^b |\varphi_n - \varphi_{n-1}|^p\, dt, \qquad (5)
\end{aligned}$$

using the condition $\varphi_0(t) = 0$,we get

$$|\varphi_1(t_0)|^p \leq N_1^p(t_0)\left(\int_a^b |l(t,0)|^q\, dt\right)^{\frac{p}{q}} = N_1^p(t_0)C^p,$$

and from (5), it comes

$$\begin{aligned}
|\varphi_2(t_0) - \varphi_1(t_0)|^p &\leq N_1^p(t_0)N_2^p \int_a^b N_1^p(t_0)C^p dt_0 = C^p N^p N_1^p(t_0),\\
|\varphi_3(t_0) - \varphi_2(t_0)|^p &\leq N_1^p(t_0)N_2^p \int_a^b C^p N_1^p(t_0) N^p dt_0 = C^p N^{2p} N_1^p(t_0),
\end{aligned}$$

more generally

$$|\varphi_{n+1}(t_0) - \varphi_n(t_0)|^p \leq C^p N^{2np} N_1^p(t_0),$$

or again after simplification

$$|\varphi_{n+1}(t_0) - \varphi_n(t_0)| \leq C N^{2n} N_1(t_0).$$

This expression gives that the sequence $\varphi_n(t_0)$ taken by the series

$$\varphi_1(t_0) + (\varphi_2(t_0) - \varphi_1(t_0)) + ... + (\varphi_p(t_0) - \varphi_{p-1}(t_0)) + ...,$$

has the majorant

$$CN_1(t_0)(1 + N + N^2 + ...N^{p-1} + ...$$

Naturally, this series converges. Hence the sequence $\varphi_n(t_0)$ converges to the solution of the equation (2). □

4. Numerical Experiments

In this section we describe some of the numerical experiments performed in solving the Hammerstein integral equation (1). In all cases, the interval is $[0, 1]$ and we chose the right hand side $f(t)$ in such way that we know the exact solution. This exact solution is used only to show that the numerical solution obtained with the method is correct [8, 9][8, 9]. In each table, φ represents the given exact solution of the Hammerstein equation and $\widetilde{\varphi}$ corresponds to the approximate solution of the equation produced by the iterative method

Example 1. *In the following Hammerstein integral equation* (1) *we seek the solution into L^p space for $p \geq 1$ and $q \leq p$ with* $\frac{1}{p} + \frac{1}{q} = 1$,

$$\varphi(t_0) - \int_0^1 2tt_0 \exp(-\varphi^2(t))dt = \frac{t_0}{\exp(1)},$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = t$$

Values of t	*Exact solution φ*	*Approx solution $\widetilde{\varphi}$*	*Error*	*Error* [5]
0.000000	*0.000000e+000*	*0.000000e+000*	*0.000000e+000*	*2.45505e-007*
0.200000	*2.000000e-001*	*2.000000e -001*	*2.252524e-009*	*2.45505e-007*
0.400000	*4.000000e-001*	*4.000000e -001*	*4.505049e-009*	*2.45505e-007*
0.600000	*6.000000e-001*	*6.000000e -001*	*6.757573e-009*	*2.45505e-007*
0.800000	*8.000000e-001*	*8.000000e -001*	*9.010097e-009*	*2.45505e-007*

Table 1. *The exact and approximate solutions of example 1 in some arbitrary points, and the error compared with the ones treated in* [5].

Example 2. *In the following Hammerstein integral equation* (1) *we seek the solution into* L^p *space for* $p \geq 1$ *and* $q \leq p$ *with* $\frac{1}{p} + \frac{1}{q} = 1$,

$$\varphi(t_0) - \int_0^1 tt_0 \varphi^2(t)dt = \cos(t_0) - \frac{1}{8}t_0(1 + \cos(2) + \sin(2)),$$

where the right hand function $f(t_0)$ *is chosen so that the solution* $\varphi(t)$ *is given by*

$$\varphi(t) = \cos(t)$$

Values of t	*Exact solution* φ	*Approx solution* $\widetilde{\varphi}$	*Error*	*Error* [2]
0.000000	*1.000000e+000*	*1.000000e+000*	*0.000000e+000*	*0.000000e+000*
0.200000	*9.800666e-001*	*9.800666e-001*	*4.135778e-009*	*6.05778e-005*
0.400000	*9.210610e-001*	*9.210610e-001*	*8.271556e-009*	*1.21999e-004*
0.600000	*8.253356e-001*	*8.253356e-001*	*1.240733e-008*	*1.83615e-004*
0.800000	*6.967067e-001*	*6.967067e-001*	*1.654311e-008*	*2.44709e-004*

Table 2. *The exact and approximate solutions of example 2 in some arbitrary points, and the error compared with the ones treated in* [2].

Example 3. *In the following Hammerstein integral equation* (1) *we seek the solution into* L^p *space for* $p \geq 1$ *and* $q \leq p$ *with* $\frac{1}{p} + \frac{1}{q} = 1$,

$$\varphi(t_0) - \int_0^1 \frac{4tt_0 + \pi t_0 \sin(\pi t)}{\varphi^2(t) + t^2 + 1} dt = \sin(\frac{\pi}{2}t_0) - 2t_0 \ln(3),$$

where the function $f(t_0)$ *is chosen so that the solution* $\varphi(t)$ *is given by*

$$\varphi(t) = \sin(\frac{\pi}{2}t)$$

Values of t	*Exact solution* φ	*Approx solution* $\widetilde{\varphi}$	*Error*	*Error* [6]
0.000000	*1.000000e+000*	*1.000000e+000*	*0.000000e+000*	*7.1385e-005*
0.200000	*3.090170e-001*	*3.090249e-001*	*7.857684e-006*	*7.1385e-005*
0.400000	*5.877853e-001*	*5.878010e-001*	*1.571537e-005*	*7.1385e-005*
0.600000	*8.090170e-001*	*8.090406e-001*	*2.357305e-005*	*7.1385e-005*
0.800000	*9.510565e-001*	*9.510879e-001*	*3.143074e-005*	*7.1385e-005*

Table 3. *The exact and approximate solutions of example 3 in some arbitrary points, and the error compared with the ones treated in* [6].

5. Conclusion

In this work we remark the convergence of the successive approximation method to the exact solution with a considerable accuracy for the Hammerstein integral equation under conditions of the theorems cited above, This numerical results show that the accuracy improves with increasing of the number of iterations. Finally, we confirm that, the theorems cited above lead us to the good approximation of the exact solution.

References

[1] F. Awawdeh, A. Adawi: A Numerical Method for Solving Nonlinear Integral Equations, in International Mathematical Forum, 4,2009,no. 17, 805 - 817

[2] R. Ezzati, K. Shakibi: On Approximation and Numerical Solution of Fredholm Hammerstein Integral Equations Using Multiquadric Quasi-interpolation, in Communications in Numerical Analysis 112(2012).

[3] L. Kantorovich, G. Akilov: Functional analysis Pergamon Press, University of Michigan 1982.

[4] S.J. Liao: On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147 (2004) 499-513.

[5] K. Maleknejad, H. Derili: The collocation method for Hammerstein equations by Daubechies wavelets in Applied Mathematics and Computation 172 (2006) 846-864.

[6] K. Maleknejad, M. Karami, N. Aghazadeh: Numerical solution of Hammerstein equations via an interpolation method, in Applied Mathematics and Computation 168(2005) 141-145

[7] K. Maleknejad, K. Nouri, M. Nosrati: Convergence of approximate solution of nonlinear Fredholm–Hammerstein integral equations in Commun Nonlinear Sci Numer Simulat 15 (2010)1432-1443.

[8] M. Nadir, B. Gagui: Two Points for the Adaptive Method for the Numerical Solution of Volterra Integral Equations , in International JournalMathematicalManuscripts (IJMM) 1, (2) 2007.

[9] M. Nadir, A. Rahmoune: Solving linear Fredholm integral equations of the second kind using Newton divided dixoerence interpolation polynomial, in International Journal of Mathematics and Computation (IJMC) 7, (10) pp 1-6 (2010).

[10] B. Sepehrian, M. Razzaghi: Solution of nonlinear Volttera-Hammerstein integral equations via single-term Walsh series method, in Mathematical Problems in engineering, vol. 5,(2005), pp 547-554.

[11] S. Szufla: On the Hammerstein integral equation in Banach spaces, Math. Nachr. 124 (1985), pp 7-14.

[12] F. G. Tricomi: Integral equations, University press, University of Cambridge 1957.