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A COMPARISON OF PERTURBATION TECHNIQUES FOR NONLINEAR PROBLEMS

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Abstract. The present paper presents the comparison of analytical techniques. We establish the existence of the phenomena of the *noise terms* in the perturbation series solution and find the exact solution of the nonlinear problems. If the *noise terms* exist, the *Homotopy perturbation method* gives the same series solution as in *Adomian Decomposition Method* and we get the exact solution using two iterations only.

Key words : Perturbation Methods, Noise terms phenomena, Homotopy Perturbation Method, Adomian Decomposition Method.
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1. Introduction

For the last few years, there has been a great deal in the research of handeling the nonlinear problems with perturbation methodss. *Adomian Decomposition Method* [?] is one of the most successful, reliable and effective tool among the other techniques for dealing with the nonlinear problems. This is a method for solving a variety of problems whose mathematical models yield equations or system of equations involving algebraic equations, system of algebraic equations, linear or strongly nonlinear, homogeneous and non homogeneous ordinary and partial differential equations, and system of such differential equations either with variable coefficients or constant coefficients, Fredholm or Volterra integral equations and system of such equations and

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integro-differential equations whether they contain or not small or large parameter(for example see [?], [?]-[?]).

The advantages of this method over other methods are that the series solution given by this method converges quickly. It avoids any restrictive assumptions such as linearization, smallness assumption of parameter, any kind of transformation, perturbation or discretization of variables (like in numerical methods which require large size of computational work and usually rounding off errors causes loss of accuracy). This work has been original extracted from the thesis [?] and [?].

2. Modified Adomian Decomposition Method

A. M. Wazwaz is the first one who modified the standard *Adomian decomposition method* to get the exact solution of the differential equations by separating the first component of the iterative solution into two parts [?]. He did not provided any idea to deal with this separation as according to Wazwaz [?], the separation of the first component of the recursive scheme depends only on the trial basis. Here, we shall critically comment on some idea about the separation of the first component into two parts. We shall see that if we use *standard Adomian method* instead of the *modified technique* developed by Wazwaz, we get the best results if some specific criteria is justified. It is pointed out by Wazwaz [?], [?] that the *modified decomposition method* accelerates the fast convergence of the approximate series solution and the exact solution is obtained with the couple of iterations. The complete details of the method can be found in [?] and [?].

Let us start with the differential equation in general form

$$Lu + Nu + Ru = g, \tag{1}$$

with given conditions. Here u is the unknown function, L is the linear invertible differential operator of the highest order, R is another linear differential operator of order less than that of L, Nu being the nonlinear term while g is the non-homogeneous term. The inverse operator L^{-1} applying on (??), together with given conditions, we yields,

$$u = f - L^{-1}(Ru) - L^{-1}(Nu), \tag{2}$$

where f is obtained by integrating g along with given conditions. The *standard Adomian method* suggests $u(\overrightarrow{r}, t)$ as a series solution,

$$u(\overrightarrow{r}, t) = \sum_{n=0}^{\infty} u_n(\overrightarrow{r}, t), \tag{3}$$

where the other terms u_0, u_1, ..., are found by using the following recursive relation

$$\begin{cases} u_0 = f, \\ u_{n+1} = -L^{-1}[Ru_n] - L^{-1}[Nu_n], \quad n \geq 1. \end{cases} \tag{4}$$

The *decomposition method* suggests that the zeroth component u_0 is usually identified by the function f and he components u_1, u_2 ,..., u_n are determined recursively. The closed form solution $u(\overrightarrow{r}, t)$, if it exists, can be found because of the fast convergence enhanced by this technique [?]-[?]. In the *modified method* the function f in (??) is split into two parts as f_0 and f_1 such that

$$f = f_0 + f_1. \tag{5}$$

According to Wazwaz, the proper selection of f_0 and f_1 is trial based. This brings a slight variation in the u_0 and u_1 since f_0 is assigned to the initial component u_0 whereas the other part f_1 is combined with the other terms in (??) for u_1 and the following recursive formula is established;

$$\begin{cases} u_0 = f_0, \\ u_1 = f_1 - L^{-1}[Ru_0] - L^{-1}[Nu_0], \\ u_{n+1}(x) = -L^{-1}[Ru_n] - L^{-1}[Nu_n], \quad n \geq 1. \end{cases} \tag{6}$$

In [?] it is stated that only two iterations are enough to find out the exact solution. According to Wazwaz, the success of *modified technique* is dependent upon the proper right choice of f_0 and f_1 which has no criterion to be judged.

Here, we only describe the method proposed by Wazwaz as well as a criteria as a guess to judge the form of f_0, but this criteria is not always helpful and we shall show that we have another criteria to judge the forms of f_0 and f_1. Perhaps, this is the only specific criteria to judge the forms of f_0 and f_1 which depends upon the *standard Adomian decomposition method* and its two components only, namely u_0 and u_1.

2.1. **Methods of Choosing the Initial Approximation.**

Here we will discuss only one method to choose the initial approximation for iteration. The other method will be discussed in detail in next sections.

First of all we consider the given initial conditions and observe that the minimum of the terms of f that is enough to satisfy the given initial conditions. However this method is not so exact to choose the first component but it gives us some direction and helps us for trials instead of calculating the so called *Adomian Polynomials* which require heavy calculations.

Like the restrictions on the *standard Adomian decomposition method*, the *modified decomposition method* has also some restrictions. In addition to the restriction in *standard Adomian decomposition method*, this technique is not applicable to the homogeneous problems at all (Why? The details for this will be given in in details in next sections). And moreover, this method will not always work for inhomogeneous problems. This depends mainly on the phenomena of the existence of so called "*noise terms*". To check the performance of the *modified decomposition method*, Wazwaz [?] has made a selection six examples of nonlinear PDEs, including *Klein-Gordon* equation, and the nonlinear *Lane-Emden* differential equations, which are studied to exhibit the performance of this technique.

2.2. Introduction to the Noise term phenomena.

More recently, *Adomian* and *Rach*, first time introduced the phenomena of the so called "*noise terms*" and they were defined as "*the identical terms with opposite signs that appear in the first two components of the series solution of u*". It was proved that if a term or terms in u_0 are canceled out by a term or terms in u_1, whether u_1 contains some further terms in it or not, then the remaining non canceled terms in u_0 provide the exact solution u. It was suggested that these terms may appear only for the case inhomogeneous problems and this discovery demonstrated a fast convergence. The fast convergence of the approximated series solution obtained by this method is presented in many papers (see for example [?]-[?]). Most recently, Wazwaz developed a necessary condition that is necessary to ensure the appearance of "*noise terms*" in the non-homogeneous problems which states that the "*noise terms*" may appear in the components u_0 and u_1 only if the exact solution u appears as part of u_0. Moreover, these non-canceled terms must satisfy the equation and the given conditions. Certainly, it is not always true that "*noise terms*" appear for all non-homogeneous equations as mentioned in [?]. Also see [?],[?], [?].

Although the *Modified Technique* may give the exact solution for nonlinear equations without any need of the so-called *Adomian polynomials* and this *Modified Technique* needs only a slight variation from the *Standard Adomian Method*, yet it depends upon the first two components of the *Standard Adomian method*. Here in this chapter we shall show that this *Modified Technique* can be only combined with the phenomena of self-canceling "*noise terms*". If these *noise terms* do not exist, then this *Modified Technique* will not work properly to get the exact solution. Here we select the same examples studied by wazwaz [?] for his modified technique, and we shall use the Standard Adomian Method to check whether the "noise terms" exist or not for these problems. Also we use the method discussed before to choose the initial approximation. Moreover, our objective here is to introduce the "noise terms" in homotopy Perturbation Method (HPM), if they appear in the series solution. With the help of this addition in HPM, we will be able to write the exact solution without any further calculation. We shall see that the calculation adopted in HPM will produce the same result as in the standard Adomian Method. In the next section we shall examine our ideas through examples.

2.3. **Examples.**

2.3.1. *Nonlinear partial differential equations.*

Example 1.

$$u_{tt} + u_x^2 + u - u^2 = te^{-x}, \tag{7}$$

along with the given conditions $u(x,0) = 0$ and $u_t(x,0) = e^{-x}$. Applying the linear operator $L = \frac{\partial^2}{\partial t^2}$, with prescribed conditions, the following result is obtained

$$u(x,t) = \frac{t^3}{6}e^{-x} + te^{-x} - \int_0^t \int_0^s (u_x^2 + u - u^2) dq ds. \tag{8}$$

Now considering the *standard Adomian method*,

$$f(x,t) = \frac{t^3}{6}e^{-x} + te^{-x}. \tag{9}$$

Wazwaz could split $f(x,t)$ into two parts, say f_0 and f_1 or $u_0(x,t)$ and $u_1(x,t)$ on trial basis. First of all we consider the given initial conditions and see that the minimum of the terms of f that is enough to satisfy the given initial conditions. We observe that the term te^{-x} is enough to satisfy the initial conditions. Therefore, we can choose it as $u_0 = f_0$ and remaining part of f is added to the definition of u_1, that is remaining part is selected as f_1, so that the following recursive relation is obtained,

$$u_1(x,t) = \frac{t^3}{6}e^{-x} - \int_0^t \int_0^s \left(u_{0x}^2 + u_0 - u_0^2\right) \quad dq ds, \tag{10}$$

and $u_{n+1}(x,t) = -L^{-1}[u_{nx}^2 + u_n - u_n^2]$, which gives $u_1(x,t) = \frac{t^3}{6}e^{-x} - \frac{t^3}{6}e^{-x} = 0$, so that $u_2 = 0$. Thus

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + \texttt{all other zero components} \tag{11}$$

Therefore, $u(x,t) = te^{-x}$. is the exact solution. We see that how a slight modification in the method leads us to the exact solution. But the difficulty here is the proper choice of the part f_0 which is the main task here to apply modified technique, and is tackled by using given initial conditions. But this method of choosing and splitting f into two parts does not work always. It is just a guess. And on trial basis we can find the exact solution.

Here we also show that trials are not only criteria to determine this part. In fact, this *modified technique* mainly depends upon two components of the *standard Adomian method*, say u_0 and u_1.

Now considering the *standard Adomian method*, we have

$$f(x,t) = \frac{t^3}{6}e^{-x} + te^{-x} = u_0(x,t), \tag{12}$$

$$u_1 = -\frac{t^3e^{-x}}{6} + \frac{t^5e^{-x}}{90}. \tag{13}$$

Now, we add the two components of the solution from equation (**??**) and (**??**) to observe the existence of the *noise term* in the solution, we can see that the *noise terms* exist as $-\frac{t^3}{6}e^{-x}$, in the first component and $\frac{t^5}{90}e^{-x}$, in the second component of the recursive relation, and the non cancelled term of u_0 is te^{-x}, which was actually taken as u_0 in the modified technique given by Wazwaz [**?**]. So we can see that existence of the *noise term* assures us the existence of the exact solution. If we see to the initial conditions, we observed that they are also satisfied by the non cancelled term of u_0 and also the governing equation also satisfied by it. The difficulty of splitting f into two parts was removed. But by the phenomena introduced by Adomian and Rack in [**?**], [**?**] that "*it was concluded that if a term or terms in the component u_0 are cancelled by a term or terms in u_1 , even though u_1 contains further terms, then the remaining non canceled terms in u_0 provide the exact solution u.*" If *noise term* exists and the remaining non cancelled terms of u_0 satisfy the given equation and prescribed conditions, then why we use the *modified algorithm*, given by Wazwaz [**?**]. So we can conclude that that the *modified algorithm* will be effective well if we combine it with the *noise term* phenomena, if exists. Otherwise if we have already applied *Standard Adomian Decomposition Method* and we find the existence of the *noise term*, then we must check whether the equation and the give conditions are satisfied or not. If they are satisfied by the non- cancelled term, then we should declare it the solution of the equation. So the exact solution is

$$u(x,t) = te^{-x}$$

Our next examples will focus on this issue.

Example 2.

$$u_{xx} + uu_x = x + \ln t, \tag{14}$$

along with given conditions $u(0,t) = \ln t$, and $u_x(0,t) = 1$. Let us define the linear operator as $L = \frac{\partial^2}{\partial x^2}$, to get

$$f(x,t) = \frac{x^3}{6} + x + \frac{x^2}{2}\ln t + \ln t - L^{-1}[uu_x]. \tag{15}$$

We now consider that by using initial conditions that minimum of terms of f that satisfies the given conditions is $x + \ln t$. We take it as f_0 and remaining part of f is added to the definitions of u_1. So taking $u_0(x,t) = x + \ln t$, and $u_1(x,t) = \frac{x^3}{6} + \frac{x^2}{2}\ln t - L^{-1}[u_0u_{0x}]$, so that

$$u_{n+1}(x,t) = -L^{-1}[u_nu_{nx}], \qquad n \geq 1. \tag{16}$$

Thus,

$$u_1(x,t) = \frac{x^3}{6} + \frac{x^2}{2}\ln t - \frac{x^3}{6} + \frac{x^2}{2}\ln t = 0. \tag{17}$$

Consequently from equation (**??**) we have $u_{n+1}(x,t) = 0, \quad \forall n \geq 1$, and therefore, the exact solution is

$$u(x,t) = x + \ln t. \tag{18}$$

It is clear from the given examples that the right choice of f_0 and f_1 can make the size of work smaller.
Now, using *Standard Adomian Method* we have,

$$u(x,t) = \frac{x^3}{6} + \frac{x^2}{2}\ln t + x + \ln t - L^{-1}[uu_x], \tag{19}$$

and selecting $u_0(x,t) = \frac{x^3}{6}+\frac{x^2}{2}\ln t+x+\ln t$, and $u_{n+1}(x,t) = -L^{-1}[u_n u_{nx}]$; $n \geq 0$, we get

$$u_1(x,t) = -[\frac{x^3}{6}+\frac{x^5}{30}+\frac{1}{6}x^4\ln t+\frac{1}{2}x^2\ln t+\frac{1}{72}x^6\ln t+(\ln t)^2\frac{x^3}{6}+\frac{1}{72.7}x^7]. \tag{20}$$

Now, we add the two components of the solution from equation (**??**) and (**??**) to observe the existence of the *noise term* in the solution. We see that the *noise terms* exist as $\frac{x^3}{6} + \frac{1}{2}x^2\ln t$, in the first component and $-\frac{x^3}{6} - \frac{1}{2}x^2\ln t$, in the second component of the *recursive relation* (**??**) and the non cancelled term of u_0 is $x + \ln t$, which was actually taken as f_0 in the *modified technique* given by Wazwaz in [**?**]. If we see to the initial conditions, we observed that they are also satisfied by the non cancelled term of u_0 and also the governing equation also satisfied by it.

2.3.2. *Nonlinear Klein Gordon Equations.* The most important nonlinear partial differential equation the nonlinear Klein Gordon Equation arising in the fields of nonlinear optics, ferromagnetic materials, charge density waves, in models for an electronic device called Josephson junction, and in quantum field theory for strongly interacting particles. In relativistic physics, it is used to describe dispersive wave phenomena. Physically, Klein Gordon Equation gives rise to an ideal model, such as for quasi-particle currents. The Klein-Gordon equation, as a relativistic version of the Schrdinger equation, was derived in 1928 which describe the free particles.
The standard form of nonlinear Klein-Gordon equationis given by

$$u_{tt} - u_{xx} + au + F = G, \tag{21}$$

subject to the initial conditions $u(x,0) = h(x)$, $u_t(x,0) = g(x)$, where $a \neq 0$ is a constant and G is the source term of space x and time t and F is a general nonlinear potential function of u which often yields oscillatory solution. For $G = exp(u)$, the equation (**??**) appears in the theory of constant curvature and with $G = cosu$, the equation (**??**) becomes the Sine-Gordon equation whereas the polynomial nonlinearity in G is used to model the field theory. The different forms of the nonlinear terms G in (**??**)have been investigated in [**?**]. We consider the following form of nonlinear Klein Gordon equation.

Example 1. Consider

$$u_{tt} - u_{xx} + u^2 = 6xt(x^2 - t^2) + x^6t^6, \tag{22}$$

along with given conditions $u(x,0) = 0$ and $u_t(x,0) = 0$. Applying L^{-1} on both sides of the equation (**??**), which, on being integrated, gives us

$$u(x,t) = x^3t^3 + \frac{3}{10}xt^5 + \frac{x^6t^8}{56} + L^{-1}[u_{nxx} - u_n^2] \tag{23}$$

Now, in $f(x,t) = x^3t^3 + \frac{3}{10}xt^5 + \frac{x^6t^8}{56}$, we see that the minimum of terms of f that satisfies these conditions is x^3t^3 only. We choose it as $u_0(x,t) = x^3t^3$, and $f_1 = -\frac{3}{10}xt^5 + \frac{x^6t^8}{56}$ so that using the recursive relation, $u_{n+1}(x,t) = L^{-1}[u_{nxx} - u_n^2]$, we get

$$u_1(x,t) = -\frac{3}{10}xt^5 + \frac{x^6t^8}{56} + \frac{3}{10}xt^5 - \frac{x^6t^8}{56} = 0, \tag{24}$$

The remaining components also vanish as $u_{n+1}(x,t) = 0, ; \forall n \geq 1$, so that $u(x,t) = x^3t^3$, is the exact solution. Using the Standard Adomian's Method we get $f(x,t) = x^3t^3 - \frac{3}{10}xt^5 + \frac{x^6t^8}{56}$, and we choose it as $u_0 = x^3t^3 - \frac{3}{10}xt^5 + \frac{x^6t^8}{56}$, and $u_{n+1}(x,t) = L^{-1}[u_{nxx} - u_n^2],$. Applying the iverse linear operator we get

$$\begin{aligned} u_1(x,t) &= \frac{3}{10}xt^5 + \frac{1}{174}x^4t^{10} - \frac{1}{56}x^6t^8 - \frac{3}{4400}x^2t^{12} - \frac{1}{(56)^2.306}x^{12}t^{18} \\ &+ \frac{1}{150}x^4t^{10} + \frac{1}{19600}x^7t^{15} - \frac{1}{28.12.13}x^9t^{13}. \end{aligned}$$

We can see that the *noise terms* exist as $-\frac{3}{10}xt^5 + \frac{x^6t^8}{56}$, in the first component and $\frac{3}{10}xt^5 - \frac{1}{56}x^6t^8$, in the second component of the *recursive relation* and the non cancelled term of u_0 is x^3t^3 which was actually taken as u_0 in the *modified technique.* clearly the initial conditions and the governing equation is also satisfied by the exact solution $u(x,t) = x^3t^3$.

Example 2.

$$u_{tt} - u_{xx} + u^2 = -x\cos t + x^2cos^2t, \tag{25}$$

with initial conditions $u(x,0) = x$, and $u_t(x,0) = 0$. Applying L^{-1} on both sides of the equation **??**, we get $f(x,t) = -x\cos t - \frac{x^2t^2}{4} + \frac{x^2}{8}\cos 2t - \frac{x^2}{8}$. Observe that the minimum of the terms of f that satisfy these conditions is $x\cos t$ only which is chosen as u_0 and

$$u_1(x,t) = \frac{x^2t^2}{4} - \frac{x^2}{8}\cos 2t + \frac{x^2}{8} - \frac{x^2t^2}{4} + \frac{x^2}{8}\cos 2t - \frac{x^2}{8} = 0. \tag{26}$$

The remaining components also vanish so that the exact solution is $u(x,t) = x\cos t$. Applying the Standard Adomian's Method, we have

$$u_0 = x\cos t - \frac{x^2t^2}{4} + \frac{x^2}{8}\cos 2t - \frac{x^2}{8}, \tag{27}$$

and

$$\begin{aligned} u_1(x,t) &= \frac{1}{24}t^4 - \frac{1}{8}t^2 + \frac{1}{16}\cos 2t - \frac{1}{16} - \frac{1}{4}x^2t^2 + \frac{1}{4}x^2\cos 2t - \frac{1}{4}x^2 \\ &+ x^4\frac{1}{2048}\cos 4t + \frac{1}{64}t\sin 2t - \frac{49}{2048}x^4 - x^4\frac{15}{256}t^2 + \frac{1}{192}t^4 \\ &+ \frac{1}{480}t^6 - x^3(\frac{5}{2})\cos t + 2t\sin t - \frac{1}{2}t^2\cos t + \frac{1}{18}\cos^3 t. \end{aligned}$$

Observe the existence of the *noise term* as $-\frac{x^2t^2}{4} + \frac{x^2}{8}\cos 2t - \frac{x^2}{8}$,in the first component and $+\frac{x^2t^2}{4} - \frac{x^2}{8}\cos 2t + \frac{x^2}{8}$, in the second component of the *recursive relation.* And the non cancelled term of u_0 is $x\cos t$, which was actually taken as u_0 in the *Modified Technique*

2.3.3. *Lane-Emden equations.* The Lane-Emden equation introduced by Lane in 1870 and studied in 1907 by Emden, has a fundamental importance in many problems of mathematical physics ranging from astrophysics to kinetic theory and quantum mechanics. The Lane-Emden equation of index m characterized by singular behaviours in its standard form is given by,

$$u'' + \frac{r}{x}u' + au^m(x) = h(x), \tag{28}$$

subject to the conditions $u'(0) = 0$, and $u(0) = u_0$. This equations was used to model the thermal behavior of a spherical cloud of gas subject to the classical laws of thermodynamics for different values of r. The other special forms of the nonlinear term u^m in (**??**) are used for modeling in the theory of stellar structure, the thermal behaviors of a spherical cloud of gas, isothermal gas spheres, and theory of thermionic currents. The index m in (**??**) has a physical significance only in the range $0 \leq m \leq 5$, which has some analytical solutions for $m = 0, 1, 5$ while the numerical solutions are still sought in case of other values of the index m, and the perturbation techniques are use to find a series solution of (**??**). However, these series solutions are often convergent in restricted regions.

The singularity behaviour of (**??**) at the point $x = 0$ as a main difficulty in its analysis was handled by wazwaz [**?**] by introducing a new differential operator L in terms of the two derivatives as,

$$L = x^{-n}\frac{d}{dx}\left[x^n\frac{d}{dx}\right]. \tag{29}$$

The *Adomian decomposition method* is applied in straightforward fashion with this new choice for the differential operator L [?].

Example 1. Consider the following inhomogeneous equation

$$u_{xx} + \frac{2}{x}u_x + u^3 = 6 + x^6, \tag{30}$$

along with the given conditions $u(0) = 0$, and $u_x(0) = 0$. Applying the inverse linear operator L^{-1} on the both side of the equation (ref26) and using the modified technique, we get

$$f(x) = x^2 + \frac{x^8}{72}. \tag{31}$$

We see that the minimum of the terms of f that satisfy these conditions is x^2 only. So $u_0(x) = x^2 = f_0$, $\quad u_1(x) = \frac{x^8}{72} - L^{-1}(u_0^3)$, and $u_{n+1}(x) = -L^{-1}(u_n^3); \forall n \geq 1$, so that $u_1(x) = 0$, with all other other vanishing components. Hence, the exact solution is $u(x) = x^2$.
On the other hand, using the *Standard Adomian's Method* we get $u_0(x) = f(x)$ and

$$u_1(x) = -\frac{1}{72}x^8 - \frac{1}{7!}x^{14} - \frac{5}{10!}x^{20} - \frac{1925}{11! \times 12636}x^{26}. \tag{32}$$

We can see that the *noise terms* exist as $+\frac{x^8}{72}$, in the first component (??) and $-\frac{x^8}{72}$, in the second component (??) of the *recursive relation*. And the non cancelled term of u_0 is x^2 which is the exact solution.

Example 2.

$$u_{xx} + \frac{4}{x}u_x + u^2 = 4 + 18x + 4x^3 + x^6, \tag{33}$$

with the initial conditions $u(0) = 2$, and $u_x(0) = 0$. Applying the inverse linear operator and using the Modified technique, we get,

$$u(x) = 2 + \frac{2x^2}{5} + x^3 + \frac{1}{10}x^5 + \frac{x^8}{88} - L^{-1}(u^2). \tag{34}$$

From equation (??) we can select

$$f(x) = 2 + \frac{2x^2}{5} + x^3 + \frac{1}{10}x^5 + \frac{x^8}{88}. \tag{35}$$

Note that the minimum of the terms of $f$that satisfy these conditions are $2 + x^3$ as well as 2. After making trials, we choose $u_0(x) = 2 + x^3 = f_0$, and remaining part of $f(x)$is added to the definition of u_1 so that

$$u_1(x) = \frac{2x^2}{5} + \frac{1}{10}x^5 + \frac{x^8}{88} - \frac{2x^2}{5} - \frac{1}{10}x^5 - \frac{x^8}{88} = 0. \tag{36}$$

So $u_{n+1}(x) = 0$, and the exact solution is $u(x) = 2 + x^3$.
On the other hand using *Standard Adomian's Method.* we have

$$u_0(x) = 2 + \frac{2x^2}{5} + x^3 + \frac{1}{10}x^5 + \frac{x^8}{88}, \tag{37}$$

$$\begin{aligned} u_1(x) &= -\frac{2}{5}x^2 + \frac{2}{35}x^4 + \frac{1}{10}x^5 + \frac{2}{225 \times 3}x^6 + \frac{4}{25 \times 7}x^7 + \frac{1}{11 \times 8}x^8 \\ &+ \frac{1}{75 \times 9}x^9 + \frac{49}{1430 \times 10}x^{10} + \frac{9}{2750 \times 12}x^{12} + \frac{1}{704 \times 13}x^{13} \\ &+ \frac{1}{3960 \times 15}x^{15} + \frac{1}{(88)^2 \times 21 \times 18}x^{18}. \end{aligned}$$

We can see that the *noise terms* exist as $+x^3 + \frac{1}{10}x^5 + \frac{x^8}{88}$, in the first component and $-x^3 - \frac{1}{10}x^5 - \frac{x^8}{88}$, in the second component of the *recursive relation.* And the non cancelled term of u_0 is $2 + x^3$, which is the exact solution.

3. The Homotopy Perturbation Method

Like the other available traditional perturbation methods, this technique does not require a *small parameter* in an equation at all. While ccording to the *homotopy technique*, a *homotopy* along with an *embedding parameter* $p \in [0, 1]$ is established, and the method is called the *Homotopy Perturbation Method*, which can take the many featured advantages of the traditional perturbation techniques as well as the *Homotopy techniques* [?][?]. Unfortunately, the available traditional perturbation techniques depend on the existence of a *small parameter* in that problem under discussion, which is a restriction in finding the applicability of wide applications, because not all nonlinear problems have such *small parameter* at all in them. Many other new techniques were developed in the recent years to eliminate the *"small parameter"* restriction, which inclde *the artificial parameter method* by Liu, the *Homotopy Analysis Method* proposed by Liao [?], [?], and *the variational iteration method* proposed by He. A review of recently developed nonlinear analysis methods can be found in detail in [Int. J. Nonlinear Sci. Numer. Simul. 1 (1 (2000) 51-75]. Recently much enthusiasm of *homotopy theory* among scientists was appeared, and *the homotopy theory* becomes a powerful tool, when coupling with the perturbation theory [?]-[?]. *The Homotopy Perturbation Method* (HPM), proposed first by Ji-Huan He in 1998, was further developed and improved by He, and was repeated by Mallil and his colleagues in 2000. We shall describe this method with examples as in previous technique, and compare our results with the previous results. Moreover, we shall show that if we choose the same linear operator as in *Adomian Decomposition Method*, we shall get the same results and this can be also combined with the *noise* *term* phenomena if it exists. So a new addition to the *Homotopy perturbation method* will be established.

3.1. **Iterated Homotopy perturbation method.**

To illustrate the *Iterated Homotopy Perturbation Method* (IHPM), He considered the following nonlinear differential equation

$$A(u) = f(\overrightarrow{r}), \qquad \overrightarrow{r} \in \Omega_j, \tag{38}$$

with boundary conditions $B(u, \partial u/\partial n) = 0, \overrightarrow{r} \in \overrightarrow{r_j}$, where A is a general differential operator which can be divided into linear part L and a nonlinear part N, $f(\overrightarrow{r})$ is a known analytic function, $\overrightarrow{r_j}$ is the boundary of the domain Ω_j and Then

$$L(u) + N(u) = f(\overrightarrow{r}).. \tag{39}$$

He constructed a Homotopy $v_j(\overrightarrow{r}, p) : \Omega_j \times [0, 1] \to R$, satisfying

$$H(v_j, p) = L(v_j) - L(y_{j,0}) + pL(y_{j,0}) + p[N(v_j) - f(\overrightarrow{r})] = 0, \tag{40}$$

where $p \in [0, 1]$ is an embedding parameter, $y_{j,0}$ is the initial approximation of (**??**). Hence, it is obvious that

$$\begin{cases} H(v_j, 0) = L(v_j) - L(y_{j,0}) = 0, \\ H(v_j, 1) = A(v_j) - f(\overrightarrow{r}) = 0. \end{cases} \tag{41}$$

Applying the perturbation technique, with $0 \leq p \leq 1$, the solution of **??** can be expressed as series as in p, as follows,

$$v_j = v_{j,0} + pv_{j,1} + p^2 v_{j,2} + p^3 v_{j,3} + ..., \tag{42}$$

When $p \to 1$, (**??**) corresponds to (**??**) and (**??**) becomes the approximate solution of (**??**), i.e.,

$$u = \lim_{p \to 1} v_j = v_{j,0} + v_{j,1} + v_{j,2} + v_{j,3} + ..., \tag{43}$$

The series (**??**) is convergent for most cases, and also the rate of convergence depends on $A(v_j)$. Now we consider the different examples that we considered in previous methods.

3.2. **Examples.**

3.2.1. *Nonlinear partial differential equations.*

Example 1. Consider the following nonlinear partial differential equation

$$u_{tt} + u_x^2 + u - u^2 = te^{-x}, \tag{44}$$

with initial conditions $u(x,0) = 0$, and $u_t(x,0) = e^{-x}$. Define a linear operator as $L = \frac{\partial^2}{\partial t^2} + 1$, and let $t \in \Omega_1$, and $x \in \Omega_2$, then $\Omega = \cup\Omega_j$. We construct a *homotopy* which satisfies

$$L[v(x,t;p) - u_0] - pL[v(x,t;p) - u_0] + p[Lv(x,t;p) + v_x^2 - v^2] = te^{-x}, \tag{45}$$

$$Lv(x,t;p) - Lu_0 + pLu_0 + p[v_x^2 - v^2] = te^{-x},$$

with initial approximation as $u_0(x,t) = u(x,0) = 0$. Suppose the solution of problem is of the form

$$v(x,t;p) = v_0(x,t) + pv_1(x,t) + p^2v_2(x,t) + ..., \tag{46}$$

such that $v^2 = v_0^2 + 2pv_1v_0 + p^2v_1^2 + ...$, and $v_x^2 = v_{0x}^2 + 2pv_{1x}v_{0x} + p^2v_{1x}^2 + ...$, Then from (**??**) we have,

$$\begin{aligned} L[v_0 + pv_1 + p^2v_2 + ...,] &- Lu_0 + pLu_0 + p[v_{0x}^2 + 2pv_{1x}v_{0x} + p^2v_{1x}^2 + ...,] \\ &- (v_0^2 + 2pv_1v_0 + p^2v^2 + ...,)] = te^{-x} \end{aligned}$$

with initial conditions

$$v(x,0;p) = v_0(x,0) + pv_1(x,0) + p^2v_2(x,0) + ..., = 0, \tag{47}$$

$$v_t(x,0;p) = v_{0t}(x,0) + pv_{1t}(x,0) + p^2v_{2t}(x,0) + ..., = e^{-x}, \tag{48}$$

so that the following set of equations are obtained

- (A) $\{Lv_0 = te^{-x}$ $\quad v_0(x,0) = 0$, $v_{0t}(x,0) = e^{-x}$,
- (B) $\{Lv_1 + v_{0x}^2 - v_0^2 = 0$ $\quad v_1(x,0) = 0, v_{1t}(x,0) = 0$,
- (C) $\{Lv_2 + 2[v_{0x}v_{1x} - v_0v_1] = 0,$ $\quad v_2(x,0) = 0, v_{2t}(x,0) = 0$

and so on. Here note that we are getting the same components of the series solution as we have calculated for the *Adomian Decomposition Method.* For example in set (B), the component v_{0x}^2 is the first component in *Adomian Decomposition Method.* for the nonlinear term u_x^2 appearing in the equation. Similarly, the component v_0^2 was calculated as u_0^2 for the nonlinear term u^2 .If we choose the same linear operator as in *Adomian Decomposition Method* to proceed, we get the same results as in *Adomian Decomposition Method.* Similarly in set of equation (C), the component $2[v_{0x}v_{1x} - v_0v_1]$ is the same as we calculated in previous method for nonlinear terms $u_x^2 - u^2$. When the embedding parameter p goes from 0 to 1, $v(x,t;p)$ goes from the initial approximation to the exact solution as

$$u(x,t) = \lim_{p\to 1} v(x,t;p) = v_0 + v_1 + v_2 + ..., \tag{49}$$

Now, consider the set of equations (A),which gives the solution, using initial conditions,

$$v_0(x,t) = te^{-x}. \tag{50}$$

Now, from set equations (B), we have

$$Lv_1 + v_{0x}^2 - v_0^2 = Lv_1 + t^2e^{-2x} - t^2e^{-2x} = 0, \tag{51}$$

so that $Lv_1 = 0$, with initial conditions $v_1(x,0) = 0$, and $v_{1t}(x,0) = 0$. This will give the solution, $v_1(x,t) = 0$. Now, consider the set of equations (C) $Lv_2 + 2[v_{0x}v_{1x} - v_0v_1] = 0$, with initial condition which will give the solution $v_2(x,t) = 0$. So that we get the solution from equation as

$$u(x,t) = \lim_{p\to 1} v(x,t;p) = v_0 + v_1 + v_2 + ..., = v_0 + 0 + 0 + ...,$$

So the exact solution is $u(x,t) = te^{-x}$. The solution is the same as we have calculated in *Adomian Decomposition Method* and *Modified Technique* even though the linear operator was different from the linear operators in the mentioned methods. We got the same result by this technique.

Example 2.

$$u_{xx} + uu_x = x + \ln t, \tag{52}$$

with initial conditions $u(0,t) = \ln t$, and $u_x(0,t) = 1$. Defining the linear operator as $L = \frac{\partial^2}{\partial x^2}$, we construct a *homotopy* which satisfies

$$Lv - Lu_0 + pLu_0 + p[vv_x] = x + \ln t, \tag{53}$$

with initial approximation $u_0(x,t) = u(0,t) = \ln t$. Suppose the solution of equation of the form (**??**) as,

$$v(x,t;p) = v_0(x,t) + pv_1(x,t) + p^2v_2(x,t) + ..., \tag{54}$$

so that $vv_x = v_0v_{0x} + p[v_0v_{1x} + v_1v_{0x}] + p^2[v_0v_{2x} + v_1v_{1x} + v_{0x}v_2] + ...,$ Then from (**??**), we have

$$\begin{aligned} & L(v_0 + pv_1 + p^2v_2) + ..., -Lu_0 + pLu_0 \\ + \quad & p[v_0v_{0x} + p[v_0v_{1x} + v_1v_{0x}] + p^2[v_0v_{2x} + v_1v_{1x} + v_{0x}v_2] + ...,] = x + \ln t, \end{aligned}$$

with initial conditions

$$v(0,t;p) = v_0(0,t) + pv_1(0,t) + p^2v_2(0,t) + ..., = \ln t, \tag{55}$$

$$v_x(0,t;p) = v_{0x}(0,t) + pv_{1x}(0,t) + p^2v_{2x}(0,t) + ..., = 1. \tag{56}$$

Now, equating the terms with equal powers of p we get the following set of equations;

- (A) $Lv_0 - Lu_0 = x + \ln t,$ $v_0(0,t) = \ln t,$ $v_{0x}(0,t) = 1$
- (B) $Lv_1 + v_0v_{0x} = 0,$ $v_1(0,t) = 0,$ $v_{1x}(0,t) = 0$
- (C) $Lv_2 + [v_0v_{1x} + v_1v_{0x}] = 0,$ $v_2(0,t) = 0,$ $v_{2x}(0,t) = 0$

and so on. Here note that we are getting the same components of the series solution as we have calculated for the *Adomian Decomposition Method.* Now, from (A), we have

$$v_0(x,t) = \ln t + x + L^{-1}[x + \ln t]. \tag{57}$$

The terms on the R.H.S., have already been calculated in *Adomian Decomposition Method* for $u_0(x,t)$ as equation (**??**). So we get from this equation,

$$v_0(x,t) = x + \ln t + \frac{x^3}{6} + \frac{x^2}{2}\ln t. \tag{58}$$

This term is similar to the term in *Adomian Decomposition Method* as $u_0(x,t)$. So $u_0 = v_0 = x + \ln t + \frac{x^3}{6} + \frac{x^2}{2}\ln t$. Now, we consider the set of equations (B), and get

$$v_1(x,t) = v_1(0,t) + xv_{1x}(0,t) - L^{-1}[v_0 v_{0x}],$$
$$v_1(x,t) = -L^{-1}[v_0 v_{0x}]. \tag{59}$$

This term also has been already calculated in *Adomian Decomposition Method* as equation (**??**) and so $v_1(x,t) = u_1(x,t)$, and we get,

$$v_1(x,t) = -[\frac{x^3}{6} + \frac{x^5}{30} + \frac{1}{6}x^4\ln t + \frac{1}{2}x^2\ln t + \frac{1}{72}x^6\ln t + (\ln t)^2\frac{x^3}{6} + \frac{1}{72\times 7}x^7]. \tag{60}$$

It is interesting to note that if we ad the two components of the series solution without the parameter p we get the following result.

$$v_0(x,t)+v_1(x,t) = [x+\ln t - \frac{x^5}{30} - \frac{1}{6}x^4\ln t] - [\frac{1}{72}x^6\ln t - (\ln t)^2\frac{x^3}{6} - \frac{1}{72\times 7}x^7] \tag{61}$$

which is the same result obtained in ADM obtained after the cancellation of terms in the first two components. So we can introduce the *noise term phenomena* here also as *"The identical terms with opposite signs that appear in the first two components of the series solution of* $v(x,t;p)$ *asp* $\to 1$*"*. So, it is concluded that if a term or terms in the component v_0 are cancelled by a term or terms in v_1 , even though v_1 contains further terms, then the remaining non canceled terms in v_0 provide the exact solution $v_1(x,t)$ as $p \to 1$. We can see that as $p \to 1$, and we consider only two components of the series solution, we have $u(x,t) = \lim_{p\to 1} v(x,t;p) = v_0 + v_1$. So we get the exact solution using the *noise term phenomena in HPM.* As in the next set of equation, we will have to face the same calculations as in ADM, so that the remaining *noise terms* may cancel as $p \to 1$. We can see that the *noise terms* exist as $\frac{x^3}{6} + \frac{1}{2}x^2\ln t$, in the first component and $-\frac{x^3}{6} - \frac{1}{2}x^2\ln t$, in the second component the series solution. And the non cancelled term of v_0 is $x + \ln t$. So the exact solution is $u(x,t) = x + \ln t$.

3.2.2. *Nonlinear Klein Gordon Equations.*

Example 1.

$$u_{tt} - u_{xx} + u^2 = 6xt(x^2 - t^2) + x^6t^6, \tag{62}$$

with initial conditions $u(x, 0) = 0$, and $u_t(x, 0) = 0$. We construct a homotopy which satisfies

$$Lv - Lu_0 + pLu_0 + p[v^2 - v_{xx}] = 6xt(x^2 - t^2) + x^6t^6, \tag{63}$$

with initial approximation $u_0(x, t) = u(x, 0) = \ln t$. Suppose the solution of equation of the form (**??**) so that $v^2 = v_0^2 + 2pv_1v_0 + p^2v_1^2 + ...,$ and $v_{xx}(x, t) = v_{0xx}(x, t) + pv_{1xx}(x, t) + p^2v_{2xx}(x, t) + ...,$. Then from equation (**??**) we get the following set of equations;

- $(A)\quad Lv_0 = 6xt(x^2 - t^2) + x^6t^6, \quad v_0(x, 0) = 0, \quad v_{0t}(x, 0) = 0$
- $(B)\quad Lv_1 + Lu_0 + [v_0^2 - v_{0xx}] = 0, \quad v_1(x, 0) = 0, \quad v_{1t}(x, 0) = 0$
- $(C)\quad Lv_2 + [2v_1v_0 - v_{1xx}] = 0, \quad v_2(x, 0) = 0, \quad v_{2t}(x, 0) = 0$

and so on. Here again note that we are getting the same components of the series solution as we have calculated in *Adomian Decomposition Method* . We choose the same linear operator as in *Adomian Decomposition Method* to proceed, to get the same results as in *Adomian Decomposition Method.* From set of equations (A), we have

$$v_0(x, t) = L^{-1}[6xt(x^2 - t^2) + x^6t^6], \tag{64}$$

The terms on the R.H.S., have already been calculated in *Adomian Decomposition Method* in equation (**??**) for $u_0(x, t)$. So, we get from this equation, $v_0(x, t) = x^3t^3 - \frac{3}{10}xt^5 + \frac{x^6t^8}{56}$. Similarly from (B) we have

$$\begin{aligned} v_1(x, t) &= \frac{3}{10}xt^5 + \frac{1}{174}x^4t^{10} - \frac{1}{56}x^6t^8 - \frac{3}{4400}x^2t^{12} - \frac{1}{(56)^2 \times 306}x^{12}t^{18} \\ &+ \frac{1}{150}x^4t^{10} + \frac{1}{19600}x^7t^{15} - \frac{1}{28 \times 12 \times 13}x^9t^{13} \end{aligned}$$

Again if we ad the two components of the series solution without the parameter p we observe that the *noise terms* exist as $-\frac{3}{10}xt^5 + \frac{x^6t^8}{56}$, in the first component and $\frac{3}{10}xt^5 - \frac{1}{56}x^6t^8$, in the second component. And the non cancelled term of v_0 is x^3t^3. So the exact solution is $u(x, t) = x^3t^3$.

Example 2.

$$u_{tt} - u_{xx} + u^2 = -x\cos t + x^2\cos^2 t, \tag{65}$$

with initial conditions $u(x, 0) = x$, and $u_t(x, 0) = 0$. We construct a Homotopy which satisfies

$$Lv - Lu_0 + pLu_0 + p[v^2 - v_{xx}] = -x\cos t + x^2\cos^2 t, \tag{66}$$

with initial approximation $u_0(x, t) = u(x, 0) = x$. Suppose the solution of equation of the form (**??**) to get the following set of equations;

- (A) $Lv_0 = -x\cos t + x^2\cos^2 t, \quad v_0(x,0) = x, \quad v_{0t}(x,0) = 0$
- (B) $Lv_1 + [v_0^2 - v_{0xx}] = 0, \quad v_1(x,0) = 0, \quad v_{1t}(x,0) = 0$
- (C) $Lv_2 + [2v_1v_0 - v_{1xx}] = 0, \quad v_2(x,0) = 0, \quad v_{2t}(x,0) = 0$

and so on. With the same arguments, from set of equations (A), we have

$$v_0(x,t) = x\cos t - \frac{x^2t^2}{4} + \frac{x^2}{8}\cos 2t - \frac{x^2}{8},$$

and from (B), $v_1(x,t) = u_1(x,t)$ and so So we get

$$\begin{aligned} v_1(x,t) &= \frac{1}{24}t^4 - \frac{1}{8}t^2 + \frac{1}{16}\cos 2t - \frac{1}{16} - \frac{1}{4}x^2t^2 + \frac{1}{4}x^2 cos2t - \frac{1}{4}x^2 \\ &+ x^4\frac{1}{2048}\cos 4t + \frac{1}{64}t\sin 2t - \frac{49}{2048}x^4 - x^4\frac{15}{256}t^2 + \frac{1}{192}t^4 + \frac{1}{480}t^6 \\ &- x^3\frac{5}{2}\cos t + 2t\sin t - \frac{1}{2}t^2\cos t + \frac{1}{18}\cos^3 t \end{aligned}$$

These two components of the series solution after the cancellation of terms using the *noise term phenomena in HPM*, give the exact solution

$$u(x,t) = x\cos t.$$

As we can see that the *noise terms* exist as $-\frac{x^2t^2}{4} + \frac{x^2}{8}\cos 2t - \frac{x^2}{8}$ in the first component and $+\frac{x^2t^2}{4} - \frac{x^2}{8}\cos 2t + \frac{x^2}{8}$, in the second component. And the non canceled term of v_0 is $x\cos t$.

3.2.3. *Lane-Emden equations.*

Example 1.

$$u_{xx} + \frac{2}{x}u_x + u^3 = 6 + x^6,$$

with the initial conditions $u(0) = 0$, and $u_x(0) = 0$. Let $x \in \Omega$, and we construct a homotopy as satisfying

$$Lv - Lu_0 + pLu_0 + p[v^3] = 6 + x^3, \tag{67}$$

with initial approximation $u_0 = u(0) = 0$. Suppose the solution of equation of the form (**??**), then from equation **??**, we get the following set of equations;

- (A) $Lv_0 = 6 + x^3, \quad v_0(0) = 0, \quad v_{0x}(0) = 0$
- (B) $Lv_1 + [v_0^3] = 0, \quad v_1(0) = 0, \quad v_{1x}(0) = 0$
- (C) $Lv_2 + [3v_1v_0^2] = 0, \quad v_2(0) = 0, \quad v_{2x}(0) = 0$

and so on. When the embedding parameter p goes from 0 to 1, $v(x,t;p)$ goes from the initial approximation to the exact solution. Thus from (A), we have $v_0 = u_0 = x^2 + \frac{x^8}{72}$, and from (B) $v_1(x) = u_1(x) = -\frac{1}{72}x^8 - \frac{1}{7!}x^{14} - \frac{5}{10!}x^{20} - \frac{1925}{11!\times 12636}x^{26}$. The existence of the *noise term* in the series solution as $+\frac{x^8}{72}$ in the first component and $-\frac{x^8}{72}$ in the second component ensures the exact solution as $u(x) = x^2$

Example 2.

$$u_{xx}+\frac{4}{x}u_x+u^2=4+18x+4x^3+x^6, \tag{68}$$

with the initial conditions $u(0)=2$, and $u_x(0)=0$. Constructing a homotopy which satisfies

$$Lv-Lu_0+pLu_0+p[v^2]=4+18x+4x^3+x^6, \tag{69}$$

with initial approximation $u_0=u(0)=2$. Suppose the solution of equation of the form (**??**), then from equation **??**, we have the following set of equations;

- (A) $Lv_0=4+18x+4x^3+x^6, \quad v_0(0)=2, \quad v_{0x}(0)=0$
- (B) $Lv_1+[v_0^2]=0, \quad v_1(0)=0, \quad v_{1x}(0)=0$
- (C) $Lv_2+[2v_1v_0]=0, \quad v_2(0)=0, \quad v_{2x}(0)=0$

and so on. Now, from set of equations (A), we have $v_0(x)=u_0(x)=2+\frac{2x^2}{5}+x^3+\frac{1}{10}x^5+\frac{x^8}{88}$, and from (B),

$$\begin{aligned} v_1(x) &= -[\frac{2}{5}x^2+\frac{2}{35}x^4+\frac{1}{10}x^5+\frac{2}{225.3}x^6+\frac{4}{25.7}x^7+\frac{1}{11.8}x^8+\frac{1}{75.9}x^9 \\ &+ \frac{49}{1430.10}x^{10}+\frac{9}{2750.12}x^{12}+\frac{1}{704.13}x^{13}+\frac{1}{3960.15}x^{15}+\frac{1}{(88)^2.21.18}x^{18}] \end{aligned}$$

We can see that the *noise terms* exist as $+x^3+\frac{1}{10}x^5+\frac{x^8}{88}$, in the first component and $-x^3-\frac{1}{10}x^5-\frac{x^8}{88}$, in the second component. And the non cancelled term of v_0 is $2+x^3$ which is the exact solution.

4. Conclusion: Comparison of the Analytical Methods

4.1. Comparison of Modified Adomian Decomposition Method and Adomian Decomposition Method.

In all examples we can see that Modified Decomposition Method is guess based method on its initial component. As far as the argument of Wazwaz is concerned, he showed that we can find the exact solution after using two iterations only. But did not provide any idea to choose the initial component. In Section 1.3, we have given an idea to choose the initial component, which seems not to be helpful always. If we want to find the exact solution, then either we should use *Adomian Decomposition Method* or any other technique for the non linear problem. Because we have shown in Section 1, that if there is some cancellation of the terms in the series solution then there may or may not be *noise terms* existence phenomena, this depends upon the necessary condition of the existence of the *noise terms*. And if such terms exist, then we should simply use the *noise terms* phenomena instead of the Modified Technique. Moreover *noise terms* do not exist always. In this case it was useless to apply this technique. And if we lessen the terms in the first component of the recursive relation in order to facilitate our work and to avoid the calculations of the Adomian polynomials, then the remaining terms are added to the second component. And if the exact solution does not exist then in the next iteration, we will have to calculate Adomian polynomials to get the next components of the series solution. Also this technique is not applicable to homogeneous equations as *Adomian method* has some restrictions on it if both initial conditions and the governing equations are homogeneous. So we give preference to the *Adomian Decomposition Method* instead of the Modified Technique because it depends upon some features of *Adomian Decomposition Method* which is a straight forward method.

4.2. **Comparison of Adomian Decomposition Method and Homotopy Perturbation Method (HPM).**

We have seen the phenomena of the existence of the *noise term* in *Homotopy Perturbation Method* also if we use the same linear operator as in *Adomian Decomposition Method*. In Example 1 of Section 2, we see that when we changed the linear operator, we obtained the exact solution after two iterations only. And in all other examples the HPM goes parallel to *Adomian Decomposition Method*. However both the methods are independent of the existence of the small or large parameter. In *Homotopy Perturbation Method*, we take the advantage of the Homotopy which is a fundamental concept of topology as well as the perturbation technique. However, both the techniques give the same series solution when we use the same linear operator. In some papers, comparison has been made between the two methods, (for example in [**?**]-[**?**] and [**?**]) and showed that Homotopy Perturbation is widely applicable where *Adomian Decomposition Method* has some restrictions.

5. Concluding Remarks

We have seen that the two perturbation techniques, Adomian Decomposition method and Homotopy Perturbation Method (HPM) working in parallel when treated with the nonlinear equations. We selected the same linear operators both techniques, because we have great freedom to choose the linear operator. Adomian Decomposition Method is a straightforward method which does not require an initial guess, while we select an initial guess in Homotopy Perturbation Method (HPM). Then the success of HPM depends upon the choice of initial guess, however, we found the exact solution. Both the techniques are free from the restriction of the existence of small or large parameter. We have suggested an idea to select the initial guess in Modified technique by Wazwaz [**?**] as well as related the success of this technique with the existence of noise terms. We have suggested that this Modified Technique can be only combined with the phenomena of self-canceling "noise terms". We also introduced the existence of the phenomena of noise terms in HPM, on the basis of which we are able to modify Homotopy Perturbation Method for future work. We see that all presented exact solutions to nonlinear equations are a kind of simple combinations of elementary functions. One may as the question; Can this procedure be used to solve other more complex initial value problems? There are many nonlinear problems available in literature which have been treated with Adomian Decomposition method and Homotopy Perturbation Method. As far as the existence of the exact solution is concerned, it depends upon the initial guess and the choice of linear operator appearing in the problem and most important, the existence of the phenomena of self-canceling "noise terms". Which is proved to be success of these techniques. We are also trying to introduce these ideal for other available perturbation techniques by applying it to some nonlinear models of diffusion [**?**]. However, there are still many improvements may be required to apply these techniques to more complex initial value problems.

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