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ZAGREB INDICES AND COINDICES OF PRODUCT GRAPHS

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Abstract. For a (molecular) graph, the first Zagreb index M_1 is equal to the sum of squares of the degrees of vertices, and the second Zagreb index M_2 is equal to the sum of the products of the degrees of pairs of adjacent vertices. Similarly, the first and second Zagreb coindices are defined as $\overline{M}_1(G) = \sum\limits_{uv \notin E(G)} (d_G(u) + d_G(v))$ and $\overline{M}_2(G) = \sum\limits_{uv \notin E(G)} d_G(u)d_G(v)$. In this paper, we compute the Zagreb indices and coindices of strong, tensor and edge corona product of two connected graphs. We apply some of our results to compute the Zagreb indices and coindices of open and closed fence graphs.

Key words : Zagreb index, Zagreb coindex, strong product, tensor product, edge corona product.
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1. Introduction

All the graphs considered in this paper are connected and simple. For vertex $u \in V(G)$, the degree of the vertex u in G, denoted by $d_G(u)$, is the number of edges incident to u in G. The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) : u \in V(G), v \in V(H)\}$ and $(u, x)(v, y)$ is an edge whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$, or (iii) $uv \in E(G)$ and $xy \in E(H)$. For two simple graphs G and H their *tensor product,* denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever g_1g_2 is an edge in G and h_1h_2 is an edge in H, see Fig.1. Note that if G and H are connected graphs, then $G \times H$ is connected only if at least one of the graph is nonbipartite.

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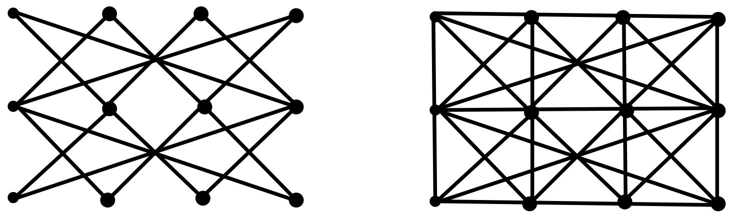

Fig.1. The tensor and strong product of P_3 and C_4

Let G and H be two graphs on disjoint sets of n and m vertices, p and q edges, respectively. The *edge corona product* $G \bullet H$ of G and H is defined as the graph obtained by taking one copy of G and p copies of H, and then joining two end vertices of the i^{th} edge of G to every vertex in the i^{th} copy of H.

A *topological index* of a graph is a parameter related to the graph; it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [7]. Several types of such indices exist, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index. Two of these topological indices are known under various names, the most commonly used ones are the first and second Zagreb indices.

The Zagreb indices have been introduced more than thirty years ago by Gutman and Trinajestic [8]. They are defined as $M_1(G) = \sum\limits_{u \in V(G)} d_G(u)^2$ and $M_2(G) = \sum\limits_{uv \in E(G)} d_G(u)d_G(v)$. Note that the first Zagreb index may also written as $M_1(G) = \sum\limits_{uv \in E(G)} (d_G(u) + d_G(v))$. The Zagreb indices are found to have appilications in QSPR and QSAR studies as well, see [4].

Noticing that contribution of nonadjacent vertex pair should be taken into account when computing the weighted wiener polynomials of certain composite graphs, see [5], Ashrafi et al. [1, 2] defined the first Zagreb coindex and second Zagerb coindex as $\overline{M}_1(G) = \sum\limits_{uv \notin E(G)} (d_G(u) + d_G(v))$ and $\overline{M}_2(G) = \sum\limits_{uv \notin E(G)} d_G(u)d_G(v)$, respectively. Notation and definitions which are not given here can be found in [3] or [11].

For the survey on theory and application of Zagreb indices, see [9]. Feng et al.[12] have given a sharp bounds for the Zagreb indices of graphs with a given matching number. Khalifeh et al. [6] have obtained the Zagreb indices of the Cartesian product, composition, join, disjunction and symmetric difference of graphs. Ashrafi et al. [2] determined the extremal values of Zagreb coindices over some special class of graphs. Hua and Zhang [10] have given some relations between Zagreb coindices and some other topolodical indices. Ashrafi et al. [1] have obtained the Zagreb indices of the Cartesian product, composition, join, disjunction and symmetric difference of graphs. In this paper, we compute the Zagreb indices and coindices of strong, tensor and edge corona product of two connected graphs. We apply some of our results to compute the Zagreb indices and coindices of open and closed fence graphs.

2. The first and second Zagreb indices of $G \boxtimes H$.

In this section, we compute the first and second Zagreb indices of the strong product of graphs. The following lemma is easily follows from the structure of $G \boxtimes H$.

Lemma 1. *Let G and H be two graphs. Then*
(i) $|V(G \boxtimes H)| = |V(G)|\,|V(H)|$.
(ii) $|E(G \boxtimes H)| = |V(G)|\,|E(H)| + |E(G)|\,|V(H)| + 2\,|E(G)|\,|E(H)|$.
(iii) The degree of the vertex (u_r, v_i) of $V(G \boxtimes H)$ is $d_G(u_r) + d_H(v_i) + d_G(u_r)d_H(v_i)$, that is, $d_{G\boxtimes H}((u_r, v_i)) = d_G(u_r) + d_H(v_i) + d_G(u_r)d_H(v_i)$.

Theorem 2. *Suppose G and H are graphs with $|V(G)| = n$, $|V(H)| = m$, $|E(G)| = p$ and $|E(H)| = q$. Then $M_1(G \boxtimes H) = (m + 4q)M_1(G) + (n + 4p)M_1(H) + M_1(G)M_1(H) + 8pq$.*

Proof. By the definition of first Zagreb index

$$M_1(G \boxtimes H) = \sum_{(u_i,v_j)\in V(G\boxtimes H)} d_{G\boxtimes H}((u_i, v_j))^2$$

$$\begin{aligned}
&= \sum_{u_i\in V(G)} \sum_{v_j\in V(H)} \Big(d_G(u_i) + d_H(v_j) + d_G(u_i)d_H(v_j)\Big)^2, \text{ by Lemma 1} \\
&= \sum_{u_i\in V(G)} \sum_{v_j\in V(H)} d_G(u_i)^2 + \sum_{u_i\in V(G)} \sum_{v_j\in V(H)} d_H(v_j)^2 \\
&\quad + \sum_{u_i\in V(G)} \sum_{v_j\in V(H)} d_G(u_i)^2 d_H(v_j)^2 + 2 \sum_{u_i\in V(G)} \sum_{v_j\in V(H)} d_G(u_i)d_H(v_j) \\
&\quad +2 \sum_{u_i\in V(G)} \sum_{v_j\in V(H)} d_G(u_i)^2 d_H(v_j) + 2 \sum_{u_i\in V(G)} \sum_{v_j\in V(H)} d_G(u_i)d_H(v_j)^2 \\
&= (m + 4q)M_1(G) + (n + 4p)M_1(H) + M_1(G)M_1(H) + 8pq.
\end{aligned}$$

In $G \boxtimes H$, define, $E_1 = \{(u, v)(x, y) \in E(G \boxtimes H) \,|\, ux \in E(G) \text{ and } v = y\}$, $E_2 = \{(u, v)(x, y) \in E(G \boxtimes H) \,|\, ux \in E(G) \text{ and } vy \in E(H)\}$ and $E_3 = \{(u, v)(x, y) \in E(G \boxtimes H) \,|\, u = x \text{ and } vy \in E(H)\}$. Clearly, $E_1 \cup E_2 \cup E_3 = E(G \boxtimes H)$. Also, $|E_1| = |E(G)|\,|V(H)|$, $|E_2| = 2\,|E(G)|\,|E(H)|$ and $|E_3| = |V(G)|\,|E(H)|$.

Theorem 3. *Let G and H be two connected graphs with n and m vertices, p and q edges, respectively. Then* $M_2(G \boxtimes H) = mM_2(G) + nM_2(H) + 2M_2(G)M_2(H) + 3M_1(G)\Big(q + M_1(H) + M_2(H)\Big) + 3M_1(H)\Big(p + M_2(G)\Big) + 6\Big(qM_2(G) + pM_2(H)\Big)$.

Proof. From the above partition of the edge set in $G \boxtimes H$, we have

$$\begin{aligned}
M_2(G \boxtimes H) &= \sum_{(u_i,v_j)(u_k,v_r)\in E(G\boxtimes H)} d_{G\boxtimes H}((u_i,v_j))d_{G\boxtimes H}((u_k,v_r)) \\
&= \sum_{(u_i,v_j)(u_k,v_j)\in E_1} d_{G\boxtimes H}((u_i,v_j))d_{G\boxtimes H}((u_k,v_j)) \\
&\quad + \sum_{(u_i,v_j)(u_k,v_r)\in E_2} d_{G\boxtimes H}((u_i,v_j))d_{G\boxtimes H}((u_k,v_r)) \\
&\quad + \sum_{(u_i,v_j)(u_i,v_r)\in E_3} d_{G\boxtimes H}((u_i,v_j))d_{G\boxtimes H}((u_i,v_r)). \qquad (1)
\end{aligned}$$

We shall obtain the above sums separately.

$$\begin{aligned}
& \sum_{(u_i,v_j)(u_k,v_j)\in E_1} d_{G\boxtimes H}((u_i,v_j))d_{G\boxtimes H}((u_k,v_j)) \\
= & \sum_{u_iu_k\in E(G)}\sum_{v_j\in V(H)} \Big(d_G(u_i) + d_H(v_j) + d_G(u_i)d_H(v_j)\Big) \\
& \Big(d_G(u_k) + d_H(v_j) + d_G(u_k)d_H(v_j)\Big), \text{by Lemma 1} \\
= & \sum_{u_iu_k\in E(G)}\sum_{v_j\in V(H)} d_G(u_i)d_G(u_k) + \sum_{u_iu_k\in E(G)}\sum_{v_j\in V(H)} (d_G(u_i) + d_G(u_k))d_H(v_j) \\
+ & \sum_{u_iu_k\in E(G)}\sum_{v_j\in V(H)} d_H(v_j)^2 + 2\sum_{u_iu_k\in E(G)}\sum_{v_j\in V(H)} d_G(u_i)d_G(u_k)d_H(v_j) \\
+ & \sum_{u_iu_k\in E(G)}\sum_{v_j\in V(H)} (d_G(u_i) + d_G(u_k))d_H(v_j)^2 \\
+ & \sum_{u_iu_k\in E(G)}\sum_{v_j\in V(H)} d_G(u_i)d_G(u_k)d_H(v_j)^2 \\
= & \; mM_2(G) + 2qM_1(G) + pM_1(H) + 4qM_2(G) + M_1(G)M_1(H) \\
& + M_2(G)M_1(H). \qquad (2)
\end{aligned}$$

$$
\begin{aligned}
& \sum_{(u_i,v_j)(u_k,v_r)\in E_2} d_{G\boxtimes H}((u_i,v_j))d_{G\boxtimes H}((u_k,v_r)) \\
= \; & 2\sum_{u_iu_k\in E(G)}\sum_{v_jv_r\in E(H)} \Big(d_G(u_i)+d_H(v_j)+d_G(u_i)d_H(v_j)\Big) \\
& \Big(d_G(u_k)+d_H(v_r)+d_G(u_k)d_H(v_r)\Big), \text{by Lemma 1} \\
= \; & 2\sum_{u_iu_k\in E(G)}\sum_{v_jv_r\in E(H)} d_G(u_i)d_G(u_k) + 2\sum_{u_iu_k\in E(G)}\sum_{v_jv_r\in E(H)} d_G(u_i)d_H(v_r) \\
& +2\sum_{u_iu_k\in E(G)}\sum_{v_jv_r\in E(H)} d_G(u_i)d_G(u_k)d_H(v_r) + 2\sum_{u_iu_k\in E(G)}\sum_{v_jv_r\in E(H)} d_H(v_j)d_G(u_k) \\
& +2\sum_{u_iu_k\in E(G)}\sum_{v_jv_r\in E(H)} d_H(v_j)d_H(v_r) + 2\sum_{u_iu_k\in E(G)}\sum_{v_jv_r\in E(H)} d_G(u_k)d_H(v_j)d_H(v_r) \\
& +2\sum_{u_iu_k\in E(G)}\sum_{v_jv_r\in E(H)} d_G(u_i)d_G(u_k)d_H(v_j) + 2\sum_{u_iu_k\in E(G)}\sum_{v_jv_r\in E(H)} d_G(u_i)d_H(v_j)d_H(v_r) \\
& +2\sum_{u_iu_k\in E(G)}\sum_{v_jv_r\in E(H)} d_G(u_i)d_H(v_j)d_G(u_k)d_H(v_r) \\
= \; & 2qM_2(G)+2pM_2(H)+2M_2(G)M_2(H)+2M_2(G)M_1(H) \\
& +2M_1(G)M_2(H)+M_1(G)M_1(H). \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{(u_i,v_j)(u_i,v_r)\in E_3} d_{G\boxtimes H}((u_i,v_j))d_{G\boxtimes H}((u_i,v_r)) \\
= \; & \sum_{u_i\in V(G)}\sum_{v_jv_r\in E(H)} \Big(d_G(u_i)+d_H(v_j)+d_G(u_i)d_H(v_j)\Big) \\
& \Big(d_G(u_i)+d_H(v_r)+d_G(u_i)d_H(v_r)\Big), \text{by Lemma 1} \\
= \; & \sum_{u_i\in V(G)}\sum_{v_jv_r\in E(H)} d(u_i)^2 + \sum_{u_i\in V(G)}\sum_{v_jv_r\in E(H)} (d_G(u_i)+d_G(u_k))d_H(v_j) \\
& + \sum_{u_iu_k\in E(G)}\sum_{v_j\in V(H)} d_H(v_j)^2 + 2\sum_{u_iu_k\in E(G)}\sum_{v_j\in V(H)} d_G(u_i)d_G(u_k)d_H(v_j) \\
& + \sum_{u_iu_k\in E(G)}\sum_{v_j\in V(H)} (d_G(u_i)+d_G(u_k))d_H(v_j)^2 \\
& + \sum_{u_iu_k\in E(G)}\sum_{v_j\in V(H)} d_G(u_i)d_G(u_k)d_H(v_j)^2 \\
= \; & qM_1(G)+2pM_1(H)+nM_2(H)+4pM_2(H)+M_1(G)M_1(H) \\
& +M_1(G)M_2(H). \qquad (4)
\end{aligned}
$$

Using (1) and the sums in (2), (3) and (4) respectively, we have

$M_2(G \boxtimes H) = mM_2(G) + nM_2(H) + 2M_2(G)M_2(H) + 3M_1(G)\Big(q + M_1(H) + M_2(H)\Big) + 3M_1(H)\Big(p + M_2(G)\Big) + 6\Big(qM_2(G) + pM_2(H)\Big).$

One can easily see that $M_1(C_n) = 4n,\ n \geq 3,\ M_1(P_1) = 0,\ M_1(P_n) = 4n - 6,\ n > 1$ and $M_1(K_n) = n(n-1)^2$. Similarly, $M_2(C_n) = 4n,\ n \geq 3$ and $M_2(P_n) = 4(n-2),\ n > 2$. Moreover $M_2(P_1) = 0$ and $M_2(P_2) = 1$. Also, $M_2(K_n) = \frac{n(n-1)^3}{2}$.

As an application we present formulae for Zagreb indices of open and closed fences, $P_n \boxtimes K_2$ and $C_n \boxtimes K_2$, see Fig. 2.

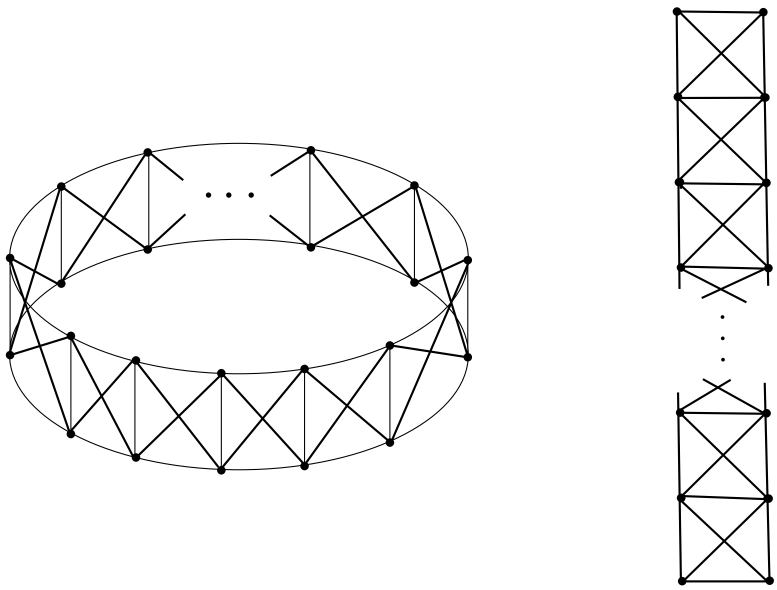

Fig.2. Closed and open fence graphs

Example 1. (*i*) $M_1(P_n \boxtimes K_2) = 2(25n - 32)$.
(*ii*) $M_1(C_n \boxtimes K_2) = 50n$.
(*iii*) $M_2(P_n \boxtimes K_2) = 125n - 212$.
(*iv*) $M_2(C_n \boxtimes K_2) = 125n$.

3. Zagreb indices of $G \times H$.

In this section, we compute the first and second Zagreb indices of the tensor product of graphs. The following lemma is follows from the structure of $G \times H$.

Lemma 4. *Let* G *and* H *be two graphs. Then*
(*i*) $|V(G \times H)| = |V(G)|\,|V(H)|$.
(*ii*) $|E(G \times H)| = 2\,|E(G)|\,|E(H)|$.

(iii) The degree of a vertex (u_r, v_i) of $G \times H$ is given by $d_{G\times H}((u_r, v_i)) = d_G(u_r)d_H(v_i)$.

For a positive integer, we let $N_a(G) = \sum\limits_{u\in V(G)} (d_G(u))^a$. One can easily see that $N_0(G) = |V(G)|$, $N_1(G) = 2\,|E(G)|$ and $N_2(G) = M_1(G)$.

Theorem 5. *Let G and H be two connected graphs. Then $N_a(G \times H) = N_a(G)N_a(H)$.*

Proof.

$$\begin{aligned} N_a(G \times H) &= \sum_{(u_i,v_j)\in V(G\times H)} (d((u_i, v_j)))^a \\ &= \sum_{u_i\in V(G)} \sum_{v_j\in V(H)} d_G(u_i)^a d_H(v_j)^a, \text{ by Lemma 4} \\ &= \Big(\sum_{u_i\in V(G)} d_G(u_i)^a\Big)\Big(\sum_{v_j\in V(H)} d_H(v_j)^a\Big) \\ &= N_a(G)N_a(H). \end{aligned}$$

In the above theorem, if we set $a = 2$, we obtain the following corollary.

Corollary 6. *Let G and H be two connected graphs. Then $M_1(G \times H) = M_1(G)M_1(H)$.*

Theorem 7. *Let G and H be two connected graphs. Then $M_2(G \times H) = 2M_2(G)M_2(H)$.*

Proof. By the definition of second Zagreb index

$$\begin{aligned} M_2(G \times H) &= \sum_{(u_i,v_j)(u_k,v_r)\in E(G\times H)} d_{G\times H}((u_i, v_j))d_{G\times H}((u_k, v_r)) \\ &= 2 \sum_{u_iu_k\in E(G)} \sum_{v_jv_r\in E(H)} d_G(u_i)d_H(v_j))d_G(u_k)d_H(v_r), \text{ by Lemma 4} \\ &= 2\Big(\sum_{u_iu_k\in E(G)} d_G(u_i)d_G(u_k)\Big)\Big(\sum_{v_jv_r\in E(H)} d_H(v_r)d_H(v_j)\Big) \\ &= 2M_2(G)M_2(H). \end{aligned}$$

4. Zagreb coindices of $G \times H$ and $G \boxtimes H$

In this section, we compute the first and seconf Zagreb coindices of tensor and strong product of two graphs. One can easily check that the contribution of each vertex $u \in V(G)$ to $\overline{M}_1(G)$ is exactly $(|V(G)| - d_G(u) - 1)d_G(u)$. Thus we have $\overline{M}_1(G) = \sum\limits_{u\in V(G)} (|V(G)| - d_G(u) - 1)d_G(u)$.

Theorem 8. *Let G and H be two connected graphs with n, m vertices and p, q edges, respectively. Then $\overline{M}_1(G \times H) = 4pq\big(nm - 1\big) - M_1(G)M_1(H)$.*

Proof. By the definition of first Zagreb coindex

$$
\begin{aligned}
\overline{M}_1(G \times H) &= \sum_{(u_i,v_j)\in V(G\times H)} \Big(nm - d_{G\times H}((u_i,v_j)) - 1\Big) d_{G\times H}((u_i,v_j)) \\
&= \sum_{u_i\in V(G)} \sum_{v_j\in V(H)} \Big(nm - d_G(u_i)d_H(v_j) - 1\Big) d_G(u_i)d_H(v_j), \text{ by Lemma 4} \\
&= nm\Big(\sum_{u_i\in V(G)} d_G(u_i)\Big)\Big(\sum_{v_j\in V(H)} d_H(v_j)\Big) - \Big(\sum_{u_i\in V(G)} d_G(u_i)^2\Big)\Big(\sum_{v_j\in V(H)} d_H(v_j)^2\Big) \\
&\quad - \Big(\sum_{u_i\in V(G)} d_G(u_i)\Big)\Big(\sum_{v_j\in V(H)} d_H(v_j)\Big) \\
&= 4pq\big(nm - 1\big) - M_1(G)M_1(H).
\end{aligned}
$$

Lemma 9. [1] *Let G be a connected graphs with p edges. Then $\overline{M}_2(G) = 2p^2 - M_2(G) - \frac{M_1(G)}{2}$.*

The following theorem is follows from Lemma 9. Corollary 6 and Theorem 7.

Theorem 10. *Let G and H be two connected graphs with n, m vertices and p, q edges, respectively. Then $\overline{M}_1(G \times H) = 8p^2q^2 - 2M_2(G)M_2(H) - \frac{M_1(G)M_1(H)}{2}$.*

Theorem 11. *Let G and H be two connected graphs with n, m vertices and p, q edges, respectively. Then $\overline{M}_1(G \boxtimes H) = m\overline{M}_1(G) + n\overline{M}_1(H) - M_1(G)M_1(H) - 4\Big(qM_1(G) + pM_1(H)\Big) + 4(nm-3)pq + 2nm\Big((n-1)q + (m-1)p\Big)$.*

Proof.

$$\overline{M}_1(G\boxtimes H) = \sum_{(u_i,v_j)\in V(G\boxtimes H)} \Big(nm - d_{G\boxtimes H}((u_i,v_j)) - 1\Big) d_{G\boxtimes H}((u_i,v_j))$$

$$= \sum_{u_i\in V(G)}\sum_{v_j\in V(H)} \Big(nm - (d_G(u_i)+d_H(v_j)+d_G(u_i)d_H(v_j)) - 1\Big)$$

$$\Big(d_G(u_i)+d_H(v_j)+d_G(u_i)d_H(v_j)\Big), \text{ by Lemma 1}$$

$$= \sum_{u_i\in V(G)}\sum_{v_j\in V(H)} \Big(nm - (d_G(u_i)+d_H(v_j)+d_G(u_i)d_H(v_j)) - 1\Big) d_G(u_i)$$

$$+ \sum_{u_i\in V(G)}\sum_{v_j\in V(H)} \Big(nm - (d_G(u_i)+d_H(v_j)+d_G(u_i)d_H(v_j)) - 1\Big) d_H(v_j)$$

$$+ \sum_{u_i\in V(G)}\sum_{v_j\in V(H)} \Big(nm - (d_G(u_i)+d_H(v_j)+d_G(u_i)d_H(v_j)) - 1\Big) d_G(u_i)d_H(v_j)$$

$$= \sum_{u_i\in V(G)}\sum_{v_j\in V(H)} \Big(n - d_G(u_i) - 1\Big) d_G(u_i)$$

$$+ \sum_{u_i\in V(G)}\sum_{v_j\in V(H)} \Big(nm - n - d_H(v_j) - d_G(u_i)d_H(v_j)\Big) d_G(u_i)$$

$$+ \sum_{u_i\in V(G)}\sum_{v_j\in V(H)} \Big(m - d_H(v_j) - 1\Big) d_H(v_j)$$

$$+ \sum_{u_i\in V(G)}\sum_{v_j\in V(H)} \Big(nm - m - d_G(u_i) - d_G(u_i)d_H(v_j)\Big) d_H(v_j)$$

$$+ \sum_{u_i\in V(G)}\sum_{v_j\in V(H)} \Big(nm - (d_G(u_i)+d_H(v_j)+d_G(u_i)d_H(v_j)) - 1\Big) d_G(u_i)d_H(v_j)$$

$$= m\overline{M}_1(G) + n\overline{M}_1(H) - M_1(G)M_1(H) - 4\Big(qM_1(G) + pM_1(H)\Big)$$

$$+4(nm-3)pq + 2nm\Big((n-1)q + (m-1)p\Big).$$

The following theorem is follows from Lemma 9, Theorems 2 and 3.

Theorem 12. *Let G and H be two connected graphs with n, m vertices and p, q edges, respectively. Then* $\overline{M}_2(G\boxtimes H) = 2(mp+nq)^2 - 2M_2(G)M_2(H) - M_1(G)\Big(3M_2(H) + q - \frac{m}{2}\Big) - M_1(H)\Big(3M_2(G) + p - \frac{n}{2}\Big) - M_2(G)(6q+m) - M_2(H)(6p+n) - \frac{7}{2}M_1(G)M_1(H)$.

It can be verified that $\overline{M}_1(K_n) = 0$, $\overline{M}_1(P_n) = 2(n-2)^2$ and $\overline{M}_1(C_n) = 2n(n-3)$. Using $\overline{M}_1(P_n)$, $\overline{M}_1(C_n)$ and Theorems 11 and 12, we compute the formulae for first and second Zagreb coindices of open and closed fence graphs.

Example 2. (i) $\overline{M}_1(P_n \boxtimes K_2) = 4(5n^2 - 19n + 18)$.
(ii) $\overline{M}_1(C_n \boxtimes K_2) = 20n(n-3)$.
(iii) $\overline{M}_2(P_n \boxtimes K_2) = 50n^2 - 192n + 200$.
(iv) $\overline{M}_2(C_n \boxtimes K_2) = 18n^2 - 108n$.

5. Zagreb indices and coindices of $G \bullet H$.

In this section, we compute the Zagreb indices and coindices of edge corona product of two graphs.

Lemma 13. *Let G and H be two graphs. Then*
$(i)|V(G \bullet H)| = |V(G)| + |E(G)|\,|V(H)|$.
$(ii)|E(G \bullet H)| = |E(G)| + |E(G)|\,|E(H)| + 2\,|E(G)|\,|V(H)|$.
(iii) *For each vertex $x \in V(G)$, we have $d_{G\bullet H}(x) = d_G(x)(|V(H)| + 1)$ and for each vertex $y \in V(H_i)$, $d_{G\bullet H}(y) = d_H(y) + 2$.*

Theorem 14. *Let G and H be two connected graphs with n and m vertices, p and q edges, respectively. Then $(i)M_1(G \bullet H) = (m+1)^2 M_1(G) + pM_1(H) + 4mp + 8pq$.*
$(ii)M_2(G \bullet H) = (m+1)^2 M_2(G) + pM_2(H) + 2pM_1(H) + 4pq + 2(m+1)(m+q)M_1(G)$.
$(iii)\overline{M}_1(G \bullet H) = (m+1)\Big(\overline{M}_1(G) + 2mp^2 - mM_1(G)\Big) + p\Big(2\big(n + mp - 3\big)(m+q) - M_1(H) - 4q\Big)$.
(iv) $\overline{M}_2(G \bullet H) = 2p^2(2m+q+1)^2 - (m+1)^2 M_2(G) - pM_2(H) - (m+1)M_2(G)\Big(\frac{m+1}{2} + 2(m+q)\Big) + \frac{3}{2}pM_1(H) - 2mp - 8pq$.

Proof.

$$\begin{aligned}
(i)\ M_1(G \bullet H) &= \sum_{x \in V(G\bullet H)} d_{G\bullet H}(x)^2 \\
&= \sum_{x \in V(G)} d_G(x)^2 (m+1)^2 + \sum_{i=1}^{p} \sum_{x \in V(H_i)} (d_H(x) + 2)^2, \\
&\quad \text{by Lemma 13} \\
&= (m+1)^2 M_1(G) + pM_1(H) + 4mp + 8pq.
\end{aligned}$$

$$(ii) M_2(G \bullet H) = \sum_{xy \in E(G \bullet H)} d_{G \bullet H}(x) d_{G \bullet H}(y)$$

$$\begin{aligned}
&= \sum_{xy \in E(G)} (m+1)^2 d_G(x) d_G(y) + \sum_{i=1}^{p} \sum_{xy \in E(H)} (d_H(x)+2)(d_H(y)+2) \\
&\quad +2 \sum_{xy \in E(G)} \sum_{u \in V(H)} (m+1)(d_G(x)+d_G(y))(d_H(u)+2), \text{ by Lemma 13} \\
&= (m+1)^2 M_2(G) + pM_2(H) + 2pM_1(H) + 4pq + 2(m+1)(m+q)M_1(G).
\end{aligned}$$

$$(iii) \overline{M}_1(G \bullet H) = \sum_{x \in V(G \bullet H)} \Big(n + mp - d_{G \bullet H}(x) - 1\Big) d_{G \bullet H}(x)$$

$$\begin{aligned}
&= \sum_{x \in V(G)} \Big(n + mp - d_G(x)(m+1) - 1\Big) d_G(x)(m+1) \\
&\quad + \sum_{i=1}^{p} \sum_{x \in V(H_i)} \Big(n + mp - (d_H(x)+2) - 1\Big)(d_H(x)+2), \text{ by Lemma 13} \\
&= (m+1) \sum_{x \in V(G)} \Big(\big(n - d_G(x) - 1\big) d_G(x) + mp d_G(x) - m d_G^2(x)\Big) \\
&\quad + \sum_{i=1}^{p} \sum_{x \in V(H_i)} \Big(\big(n + mp - 3\big) d_H(x) - d_H^2(x) + 2\big(n + mp - 3\big) - 2 d_H(x)\Big) \\
&= (m+1)\Big(\overline{M}_1(G) + 2mp^2 - mM_1(G)\Big) \\
&\quad + p\Big(2\big(n + mp - 3\big)(m+q) - M_1(H) - 4q\Big).
\end{aligned}$$

The formula (iv) is follows from Lemma 9 and part (i) and (ii).

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