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TOPOLOGICAL STRUCTURE OF 2-NORMED SPACE AND SOME RESULTS IN LINEAR 2-NORMED SPACES ANALOGOUS TO BAIRE'S THEOREM AND BANACH STEINHAUS THEOREM

P RIYAS[1], K T RAVINDRAN[2]

Abstract. In this paper we construct the topological structure of linear 2-normed space. This enable us to define the concept of open sets in linear 2-normed space and derive an analogue of Baire's theorem and Banach Steinhaus theorem in linear 2-normed spaces..

Key words : linear 2-normed space, locally convex topological vector space, 2-Banach space, equi-continuity, locally bounded set, equi-bounded..
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1. Introduction

The concept of a linear 2-normed space was introduced as a natural 2-metric analogue of that of a normed space. In 1963, Siegfried Gähler, a German Mathematician introduced the notion of a 2-metric space, a real valued function of point-triples on a set X, whose abstract properties were suggested by the area function for a triangle determined by a triple in Euclidean space[5]. Many Mathematician have intensively studied this concept in the last three decades and obtained new applications of these notions in some abstract settings. Recently R Pilakkat and S Thirumangalath proved Baire's theorem for 2-Banach(K) Spaces in [8],[9]. However there is yet no complete proof for Baire's theorem for 2-Banach Spaces. In this paper, we prove an analogue of Baire's theorem and Banach Steinhaus theorem in linear 2-normed spaces X by constructing a locally convex topology for X. We now state some definitions before presenting our main results.

[1]Department of Mathematics, K M M Govt.Women's College, Kannur, India-670004, Email-riyasmankadavu@gmail.com
[2]P G Departmentof Mathematics, Payyannur College, Kannur, India-670627, Email: ktravindran@rediffmail.com.

Let X be a linear space of dimension greater than 1 over $\mathbb{R}$. Suppose $\| \ , \ \|$ is a real valued function on $X \times X$ satisfying the following conditions:

a). $\|x, y\| = 0$ if and only if x and y are linearly dependent.
b). $\|x, y\| = \|y, x\|$ for all $x \ , y \in X$.
c). $\|\alpha x, y\| = |\alpha|\|x, y\|$ for all $\lambda \in \mathbb{R}$ and all $x \ , y \in X$.
d). $\|x + y, z\| \leq \|x, z\| + \|y, z\|$, for all $x \ , y$ and $z \in X$.

Then $\| \ , \ \|$ is called a 2-norm on X and the pair $(X, \| \ , \ \|)$ is called a linear 2-normed space. Some basic properties of linear 2-normed space can be immediately obtained as follows:

- $\|x, y\| \geq 0$, for all $x \ , y \ \in X$
- $\|x, y + \alpha x\| = \|x, y\|$, for all $x, y \in X$ and for all $\alpha \in$ R

A standard example of a linear 2-normed space is $\mathbb{R}^2$ equipped with the 2-norm: $\|x, y\| =$ area of the parallelogram determined by the vector x and y as the adjacent sides. In any given 2-normed space, we can define a function p_e on X by

$$p_e(x) = \|x, e\|$$

for some $e \in X$. It is easy to see that this function satisfies the following conditions:

(1) $p_e(x + y) \leq p_e(x) + p_e(y)$
(2) $p_e(\alpha x) = |\alpha| p_e(x)$

Any function defined on X and satisfying the conditions (1) and (2) is called seminorm on X. Since X is of dimension ≥ 2, corresponding to each $x \neq 0$ there exist some $e \in X$ such that x and e are linearly independent and therefore $p_e(x) \neq 0$.Thus if X is a 2-normed space, the collection $\mathbb{P} = \{p_e : \ e \in X\}$ forms a separating family of seminorms on X.

2. Main Results

Baire's Theorem in Linear 2-normed space

In this section we investigate the structure of open sets in linear 2-normed space and using this structure we formulate an analogue of Baire's Theorem in linear 2-normed space.

Theorem 1. *Let X be a real linear 2-normed space. Then the subset $B_e(0, 1) = \{x \in X : \|x, e\| < 1\}$ of X is convex, symmetric, balanced and absorbing.*

Proof. For any $x, \ y \in B_e(0, 1)$ and $t \in [0, 1]$,

$$\begin{aligned} \|tx + (1-t)y, e\| &\leq \|tx, e\| + \|(1-t)y, e\| \\ &= |t|\|x, e\| + |(1-t)|\|y, e\| \\ &< t + (1-t) = 1 \end{aligned}$$

implying that $tx + (1-t)y \in B_e(0,1)$. Hence $B_e(0,1)$ is convex. Also for any $x \in X$, $\|-x, e\| = |-1|\|x, e\| = |x, e\|$ implies that $B_e(0,1) = -B_e(0,1)$. This shows that $B_e(0,1)$ is symmetric.
For all α with $|\alpha| \leq 1$ and $x \in B_e(0,1)$,

$$\begin{aligned} \|\alpha x, e\| &= |\alpha|\|x, e\| \\ &\leq \|x, e\| < 1 \\ \text{That is,} \quad \alpha x &\in B_e(0,1), \text{ for all } x \in B_e(0,1). \end{aligned}$$

Hence $B_e(0,1)$ is balanced.
We shall now show that $B_e(0,1)$ is absorbing. Let $x \in X$. If x and e are linearly dependent then $\|x, e\| = 0 < 1$ and so $x \in B_e(0,1) = tB_e(0,1)$ where $t = 1$. On the other hand, if we take $t = 2\|x, e\| > 0$, then $\|\frac{1}{t}x, e\| = \frac{1}{t}\|x, e\| = \frac{1}{2} < 1$. This shows that $x \in tB_e(0,1)$ for some $t > 0$. Hence $B_e(0,1)$ is absorbing. □

Theorem 2. *Let X be a linear 2-normed space and $\mathbb{P} = \{p_e : e \in X\}$ where $p_e(x) = \|x, e\|$. Associate to each $p_e \in \mathbb{P}$ and each positive integer n set $V(p_e, n) = B_e(0, \frac{1}{n})$. Let $\mathbb{B}$ be the collection of all finite intersection of the sets $V(p_e, n)$. Then $\mathbb{B}$ is a convex balanced local base for a topology $\mathbb{T}$ on X which turns X into a locally convex space such that*

1) *Every $p_e \in \mathbb{P}$ is continuous.*
2) *A set $E \subseteq X$ is bounded if and only if every $p_e \in \mathbb{P}$ is bounded on E.*

Proof. Define a family $\mathbb{T}$ of subsets of X by $A \in \mathbb{T}$ if and only if A is a (possibly empty) union of translates of members of $\mathbb{B}$. For any $x \in X$, $\|x, e\| < n_x$ implies that $x \in n_x B_e(0,1) = n_x V(p_e, 1)$ and so $X = \bigcup_{n_x} n_x V(p_e, 1) \in \mathbb{T}$. clearly $\emptyset \in \mathbb{T}$ and closed under arbitrary union and finite intersection. This shows that $\mathbb{T}$ is a translation invariant topology on X. Since $\mathbb{B}$ is the family of finite intersection of convex and balanced subset $V(p_e, n)$ of X, each member of $\mathbb{B}$ is convex and balanced, and $\mathbb{B}$ forms a local base for $\mathbb{T}$. Next we shall prove that X is a locally convex topological vector space. Let $0 \neq x \in X$. The family $\mathbb{P}$ being separating, there exist $p_e \in \mathbb{P}$ such that $p_e(x) > 0$. Note that x is not in $V(p_e, n)$ if $np_e(x) = n\|x, e\| > 1$. This shows that 0 is not in the neighbourhood $x - V(p_e, n) = x - B_e(0, \frac{1}{n}) = B_e(x, \frac{1}{n})$ of x and so x is not in the closure of $\{0\}$. Since $\mathbb{T}$ is translation invariant, every singleton set $\{x\} = x + \{0\}$ is a closed set.

We now show that addition and scalar multiplication are continuous. Let U be a neighbourhood of 0. Then as $\mathbb{B}$ is a local base, there exist $p_{e_1}, p_{e_2}, \dots, p_{e_m}$ in $\mathbb{P}$ and some positive integers $n_1, n_2, \dots, n_m$ such that

$$V(p_{e_1}, n_1) \cap V(p_{e_2}, n_2) \cap \dots \cap V(p_{e_m}, n_m) \subseteq U.$$

Put $V = V(p_{e_1}, 2n_1) \cap V(p_{e_2}, 2n_2) \cap ... \cap V(p_{e_m}, 2n_m)$

For any $z = x + y \in V + V$,

$$\|z, e_i\| = \|x + y, e_i\| \leq \|x, e_i\| + \|y, e_i\| < \frac{1}{2n_i} + \frac{1}{2n_i} = \frac{1}{n_i}, \text{ for all } i,$$

implying that $z = x + y \in V(p_{e_i}, n_i)$, for all i and so $z \in U$. Therefore $V + V \subseteq U$. This shows that vector addition is continuous. Suppose that $x \in X$, α is any scalar and U and V are as above. Then $x \in sV$ for some $s > 0$. If we take $t = \dfrac{s}{1 + |\alpha|s}$ and $|\beta - \alpha| < \dfrac{1}{s}$, then

$$\begin{aligned} |\beta|t &= |(\beta - \alpha) + \alpha|t \leq (|\beta - \alpha| + |\alpha|) \frac{s}{1 + |\alpha|s} \\ &< \left(\frac{1}{s} + |\alpha|\right) \frac{s}{1 + |\alpha|s} = 1. \end{aligned}$$

Therefore if $y \in x + tV$ and $|\beta - \alpha| < \dfrac{1}{s}$, then as V is balanced,

$$\beta y - \alpha x = \beta(y - x) + (\beta - \alpha)x \in |\beta|tV + |\beta - \alpha|sV \subseteq V + V \subseteq U$$

Thus for any neighbourhood $\alpha x + U$ of αx, there exist a neighbourhood $W = x + tV$ of x such that $\beta W \subseteq \alpha x + U$ for all β with $|\beta - \alpha| < \dfrac{1}{s}$. This proves that scalar multiplication is continuous. Hence X is a locally convex topological vector space. If $U = (-\epsilon, \epsilon)$ is any neighbourhood of $p_e(0) = 0$ in $\mathbb{R}$ then we can find a neighbourhood $V = V(p_e, \dfrac{1}{\epsilon})$ of 0 in X such that $p_e(V) \subseteq U$. This shows that p_e is continuous at 0. Now let U be any neighbourhood of $p_e(x)$. Then $p_e(x) - U$ is a neighbourhood of 0 and therefore there exist some neighbourhood V of 0 in X such that $p_e(x) - p_e(V) \subseteq U$. Since V is balanced and p_e is a seminorm, it follows that $p_e(x + V) \subseteq U$. Hence p_e is continuous on X.

Now suppose that E is bounded and let $p_e \in \mathbb{P}$. Then corresponding to the neighbourhood $V(p_e, 1)$ of 0, there exist some $k > 0$ such that $E \subseteq kV(p_e, 1)$. Thus for any $x \in E$, $p_e(x) < k$. It follows that every $p_e \in \mathbb{P}$ is bounded on E.

Conversely suppose that every $p_e \in \mathbb{P}$ is bounded on E and let U be a neighbourhood of 0 in X. Then as B is a local base, there exist $p_{e_1}, p_{e_2}, \ldots, p_{e_m}$ in $\mathbb{P}$ and some positive integers $n_1, n_2, ..., n_m$ such that

$$V(p_{e_1}, n_1) \cap V(p_{e_2}, n_2) \cap ... \cap V(p_{e_m}, n_m) \subseteq U.$$

By our assumption, corresponding to each p_{e_i} there exist numbers M_i such that $P_{e_i}(x) < M_i$, for all $x \in E$ and $1 \leq i \leq m$. For any $x \in E$, $p_{e_i}(x) < M_i < \dfrac{n}{n_i}$, if $n > M_i n_i$. Then,

$$\begin{aligned} p_{e_i}\left(\frac{1}{n}x\right) &< \frac{1}{n_i} \text{ for all } i. \\ x &\in nV(p_{e_i}, n_i) \text{ for all } i. \\ \text{That is } x &\in nU \text{ and so } E \subseteq nU. \end{aligned}$$

Hence E is bounded. □

Definition 1. *Let A be a convex and absorbing set in a topological vector space X. The Minkowski's functional μ_A of A is defined by*

$$\mu_A(x) = \inf\{t > 0 \ : \ t^{-1}x \in A\} \ \ for \ \ x \in X.$$

Theorem 3. *Let X be a linear 2-normed space and let $\mathbb{B}$ be the collection of all finite intersection of the sets of the form $V(p_e, n) = B_e(0, \frac{1}{n})$. Then $V = \{x \in X \ : \ \mu_V(x) < 1\}$ for all $V \in \mathbb{B}$, where μ_V is the Minkowski's functional on X.*

Proof. For any $V \in \mathbb{B}$, we can take it as

$$V = \bigcap_{i=1}^{m} V(p_{e_i}, n_i) \tag{1}$$

Then for any $x \in V$, $\|x, e_i\| < \frac{1}{n_i}$ for $1 \leq i \leq m$. Choose t such that $n_i\|x, e_i\| < t < 1$ for all i. But then $\|\frac{x}{t}, e_i\| < \frac{1}{t}\frac{t}{n_i} = \frac{1}{n_i}$, for all i implies that $\frac{x}{t} \in V$. Thus if $x \in V$ then $\frac{x}{t} \in V$, for some $t < 1$ and so $\mu_V(x) < 1$.

Conversely if $x \notin V$, then $\frac{x}{t} \in V$ would imply that $\|x, e_i\| < \frac{t}{n_i}$ for all i. Also from (1) , if $x \notin V$ then $\|x, e_i\| \geq \frac{1}{n_i}$ for some i and so $t > n_i\|x, e_i\| \geq 1$. It follows that $\mu_V(x) \geq 1$. Equivalently if $\mu_V(x) < 1$ then $x \in V$.
Hence $V = \{x \in X \ : \ \mu_V(x) < 1\}$. □

Many authors have described an open set in a linear 2-normed space in different ways. Here by using theorem (2), we define open and closed sets in a linear 2-normed space as follows :

Definition 2. *A subset A of a linear 2-normed space X is said to be open if for any $x \in A$ then there exist $e_1, e_2, ..., e_n$ in X and $r_1, r_2, ...r_n > 0$ such that*

$$x+V(p_{e_1}, r_1)\cap V(p_{e_2}, r_2)\cap...\cap V(p_{e_n}, r_n) = B_{e_1}(x, r_1)\cap B_{e_2}(x, r_2)\cap...\cap B_{e_n}(x, r_n) \subseteq A$$

where $B_{e_i}(x, r_i) = \{z \in X \ : \ \|x - z, e_i\| < r_i\}$.

A subset B of a linear 2-normed space X is said to be closed if its complement is open in X.

Theorem 4. *Let X be a linear 2-normed space. Then the ball $B_e(0,r) = \{x : \|x,e\| < r\}$ is open in X*

Proof. Let $x \in B_e(0,1)$. Choose $e_m = me$ and $r_m = m(1 - \|x,e\|)$ for $m = 1,2,3,...,n$. If $y \in \cap_{m=1}^{n} B_{e_m}(x, r_m)$ then $\|y-x, e_m\| < r_m,$ for all m and

$$\begin{aligned} \|y,e\| &\leq \|y-x,e\| + \|x,e\| \\ &= \|y-x, \frac{e_m}{m}\| + \|x,e\| \\ &= \frac{1}{m}\|y-x,e_m\| + \|x,e\| \\ &< \frac{1}{m} r_m + \|x,e\| \\ &= \frac{1}{m}\left[m(1-\|x,e\|)\right] + \|x,e\| = 1. \end{aligned}$$

This shows that $\quad y \in B_e(0,1).$

Hence $B_e(0,1)$ is open in X. □

Corollary 5. *The ball $B_e(a,r) = \{x : \|x-a,e\| < r\}$ is open in a linear 2-normed space X for all $a,\ e \in X$ and $r > 0$.*

Proof. Let $x = a + ry \in a + rB_e(0,1) = B_e(a,r)$. Since $B_e(0,1)$ is open in X, there exist $e_1, e_2, ..., e_n$ in X and $r_1, r_2, ..., r_n > 0$ such that

$$\cap_{m=1}^{n} B_{e_m}(y, r_m) \subseteq B_e(0,1).$$

This implies that $a + r \cap_{m=1}^{n} B_{e_m}(y, r_m) \subseteq a + rB_e(0,1) = B_e(a,r).$

That is, $\cap_{m=1}^{n} B_{e_m}(x, R) = \cap_{m=1}^{n} B_{e_m}(a + ry, R) \subseteq B_e(a,r).$

Hence $B_e(a,r)$ is open in X □

Example 1. *Let $X = \mathbb{R}^2$ be a linear 2-normed space with 2-norm defined by $\|x,y\| = |x_1y_2 - x_2y_1|$, where $x = (x_1,x_2)$ and $y = (y_1,y_2)$ and let $e = (e_1,e_2)$. Then,*

$$\begin{aligned} B_e(0,1) &= \{(x_1,x_2) : \|x,e\| < 1\} \\ &= \{(x_1,x_2) : |x_1 - x_2| < 1\} \\ &= \{(x_1,x_2) : x_1 - 1 < x_2 < x_1 + 1\} \text{ is open in } X. \end{aligned}$$

Definition 3. *A sequence $\{x_n\} \longrightarrow x$ in a linear 2-normed space X if for any open set V containing 0 there exist a positive integer N such that $x_n - x \in V$ for all $n \geq N$.*

Theorem 6. *A sequence $\{x_n\} \longrightarrow x$ in a 2-normed space X if and only $\lim_{n\to\infty} \|x_n - x, e\| = 0$, for all $e \in X$.*

Proof. Consider the open set $V = B_e\left(0, \frac{1}{n}\right)$ containing 0 and for any $e \in X$. If the sequence $\{x_n\}$ converges to x then we can find some positive integer N such that $x_n - x \in B_e\left(0, \frac{1}{n}\right)$ for all $n \geq N$. Then

$$\|x_n - x, e\| < \frac{1}{n}, \text{for all } \; n \geq N \;\; and \;\; e \in X.$$

Letting $N \to \infty$, we get $\lim_{n\to\infty} \|x_n - x, e\| = 0$, for all $e \in X$.

Conversely, suppose that $\lim_{n\to\infty} \|x_n - x, e\| = 0$, for all $e \in X$ and V is any open set containing 0. Then there exist $e_1, e_2 \ldots e_n$ in X and $r_1, r_2, \ldots r_n$ such that

$$B_{e_1}(0, r_1) \cap B_{e_2}(0, r_2) \cap \ldots \cap B_{e_n}(0, r_n) \subseteq V.$$

But then by our assumption corresponding to each $r_i > 0$, there exist a positive integer N_i such that

$$\|x_n - x, e\| < r_i, \text{ for all } n \geq N_i \;\; and \; for \; all \; e = e_i.$$

In other words, $x_n - x \in B_{e_i}(0, r_i)$, for all $n \geq N = \max_i(N_i)$ *and* for all i . It follows that $x_n - x \in V$, for all $n \geq N$. Hence $\{x_n\} \longrightarrow x$. □

Definition 4. *A sequence $\{x_n\}$ in a linear 2-normed space X is said to be Cauchy sequence if there exist two linearly independent elements y and z such that $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$ and $\lim_{m,n\to\infty} \|x_n - x_m, z\| = 0$.*

Definition 5. *Let X be a linear 2-normed space and $A \subseteq X$. Then a point $x \in X$ is called limit point of A in X if for any open set U containing x, $A \cap (U - \{x\}) \neq \emptyset$.*

Theorem 7. *Let X be a linear 2-normed space and $A \subseteq X$. If x is a limit point of A then corresponding to each $e \in X$ there exist a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} \|x_n - x, e\| = 0$.*

Proof. If x is a limit point of A then corresponding to each open ball $B_e\left(x, \frac{1}{n}\right)$ we can choose $x_n \in A \cap \left(B_e\left(x, \frac{1}{n}\right) - \{x\}\right)$. Then $x_n \in A$ and $\|x_n - x, e\| < \frac{1}{n}$ for all n. Thus for any $e \in X$, there exist $\{x_n\}$ in A such that $\lim_{n\to\infty} \|x_n - x, e\| = 0$. □

Definition 6. *Let X be a linear 2-normed space and $A \subseteq X$. A point $x \in X$ is called a closure point of A if every open set containing x intersects A. The set of all closure points of A, denoted by $\overline{A}$ is called closure of A.*

Definition 7. *Let X be a linear 2-normed space and $A \subseteq X$. Then A is said to be dense in X if every open set U in X intersects A.*

Definition 8. *A linear 2-normed space in which every Cauchy sequence is convergent, is called a 2-Banach space or a complete space.*

We now prove the main objective of this section :

Theorem 8 (Analogue of Baire's Theorem in Linear 2-normed space)**.** *Let X be a 2-Banach space. Then intersection of a countable number of dense open subsets of X is dense in X.*

Proof. Let $V_1, V_2, ...$ be dense open subsets of X. For $x_0 \in X$, consider an arbitrary non-empty open subset B_0 of X containing x_0. Then there exist $f_1, f_2, ..., f_i$ in X and $p_1, p_2, ..., p_i > 0$ such that

$$B_{f_1}(x_0, p_1) \cap B_{f_2}(x_0, p_2) \cap ... \cap B_{f_i}(x_0, p_i) \subseteq B_0.$$

Since V_1 is dense in X and $\cap_{k=1}^{i} B_{f_k}(x_0, p_k)$ is open, $V_1 \cap \big(\cap_{k=1}^{i} B_{f_k}(x_0, p_k)\big) \neq \emptyset$. Choose an element $x_1 \in V_1 \cap \big(\cap_{k=1}^{i} B_{f_k}(x_0, p_k)\big)$. Then as $V_1 \cap \big(\cap_{k=1}^{i} B_{f_k}(x_0, p_k)\big)$ is open in X, we can find an open set B_1 containing x_1 such that $\overline{B_1} \subseteq V_1 \cap \big(\cap_{k=1}^{i} B_{f_k}(x_0, p_k)\big) \subseteq V_1 \cap B_0$. B_1 being an open set containing x_1, there exist $g_1, g_2, ..., g_j$ in X and $q_1, q_2, ..., q_j > 0$ such that

$$B_{g_1}(x_1, q_1) \cap B_{g_2}(x_1, q_2) \cap ... \cap B_{g_j}(x_1, q_j) \subseteq B_1.$$

Note that V_2 is dense in X and $\cap_{k=1}^{j} B_{g_k}(x_1, q_k)$ is open in X. Consequently, $V_2 \cap \Big(\cap_{k=1}^{j} B_{g_k}(x_1, q_k)\Big) \neq \emptyset$. Let $x_2 \in V_2 \cap \Big(\cap_{k=1}^{j} B_{g_k}(x_1, q_k)\Big)$. Then as above, we can choose an open set B_2 containing x_2 such that $\overline{B_2} \subseteq V_2 \cap \Big(\cap_{k=1}^{j} B_{g_k}(x_1, q_k)\Big) \subseteq V_2 \cap B_1$. Thus proceeding inductively we can find a sequence $\{x_n\}$ such that $x_n \in V_{m+1} \cap \big(\cap_{i=1}^{k} B_{e_i}(x_m, r_i)\big)$ for all $n > m$ and a decreasing sequence $\{R_n\}$ of positive real numbers such that $R_n < \dfrac{1}{n}$. where $R_1 = \max\{p_1, p_2, ..., p_i\}$, $R_2 = \max\{q_1, q_2, ..., q_j\}$, ... , $R_m = \max\{r_1, r_2, ..., r_k\}$.

$$\begin{aligned} \|x_n - x_m, e_i\| &< R_m < \frac{1}{m}, \text{ for all } n > m \text{ and for all } i. \\ \|x_n - x_r, e_i\| &\leq \|x_n - x_m, e_i\| + \|x_m - x_r, e_i\| \\ &< \frac{1}{m} + \frac{1}{m} = \frac{2}{m}, \text{ for all } n, r > m \textit{ and } \text{ for all } i. \end{aligned}$$

If we let $m \to \infty$, we obtain $\lim\limits_{n,r \to \infty} \|x_n - x_r, e\| = 0$, for all $e \in \text{span}\{e_1, e_2, ..e_k\}$. This shows that $\{x_n\}$ is a Cauchy sequence in a 2-Banach space X and hence there exist some $x \in X$ such that $x_n \to x$ in X. Since $x_n \in B_m$, for all $n \geq m$, it follows that $x \in \overline{B_m}$ and as $\overline{B_m} \subseteq V_m \cap B_0$ for $m = 1, 2, 3...$ we see that

$x \in (\cap_{m=1}^{\infty} V_m) \cap B_0$. Hence B_0 intersects $\cap_{m=1}^{\infty} V_m$ and therefore dense in X. □

3. Banach Steinhaus Theorem in Linear 2-normed space

In this section, we will consider linear operators defined on a linear 2-normed space into a linear 2-normed space. We will formulate Banach Steinhaus Theorem for a family of continuous linear operators.

Definition 9. *Let X and Y be linear 2-normed spaces over $\mathbb{R}$. Then a linear map $T : X \longrightarrow Y$ is continuous at x if for any open ball $B_d\left(T(x), R\right)$ in Y there exist an open ball $B_e(x, r)$ in X such that $T\left(B_e(x, r)\right) \subseteq B_d\left(T(x), R\right)$. In other words for any $d \in Y$ and $R > 0$, there exist some $e \in X$ and $r > 0$ such that $\|T(y) - T(x), d\| < R$ whenever $\|y - x, e\| < r$ and for all $y,\ x \in X$*

Theorem 9. *Let X and Y be linear 2-normed spaces over $\mathbb{R}$. If a linear operator $T : X \longrightarrow Y$ is continuous at 0 then it is continuous on X.*

Proof. Assume that the linear operator $T : X \longrightarrow Y$ is continuous at 0. For any open ball $B_d(0, R)$ in Y , we can find an open ball $B_e(0, r)$ such that

$$T\left(B_e(0, r)\right) \subseteq B_d(0, R)$$

Then by linearity, $T(y) - T(x) \in B_d(0, R)$ whenever $y - x \in B_e(0, r)$. Thus if $y \in x + B_e(0, r) = B_e(x, r)$ then $T(y) \in T(x) + B_d(0, R) = B_d(T(x), R)$. Hence

$$T\left(B_e(x, r)\right) \subseteq B_d(T(x), R)$$

implying that T is continuous on X. □

Definition 10. *Let X and Y be linear 2-normed spaces over $\mathbb{R}$ and $T : X \longrightarrow Y$ be a linear operator. The operator T is said to be sequentially continuous at $x \in X$ if for any sequence $\{x_n\}$ of X converging to x we have $T(x_n) \longrightarrow T(x)$.*

Theorem 10. *Every continuous linear map T from a linear 2-normed space X into a linear 2- normed space Y is sequentially continuous on X.*

Proof. Let $T : X \longrightarrow Y$ be continuous at $x \in X$. If $B_d\left(T(x), R\right)$ is any open ball in Y, then by the continuity of T, there exist some open ball $B_e(x, r)$ in X. Such that

$$T\left(B_e(x, r)\right) \subseteq B_d(T(x), R) \tag{2}$$

Let $\{x_n\}$ be any sequence in X such that $x_n \to x$ in X. Then corresponding to the open ball $B_e(x, r)$, there exist some $K > 0$ such that

$$x_n \in B_e(x, r), \text{ for all } n \geq K \tag{3}$$

$$\|x_n - x, e\| < r, \text{ for all } n \geq K$$

(2) and (3) shows that $T(x_n) \in B_d(T(x), R),$ for all $n \geq K$

$$\|T(x_n) - T(x), d\| < R, \text{ for all } n \geq K$$

Since $B_d(T(x), R)$ is arbitrary, it follows that $T(x_n) \longrightarrow T(x)$. Hence T is sequentially continuous on X. $\square$

Theorem 11. *Let X and Y be linear 2-normed spaces over $\mathbb{R}$. If X is finite dimensional, then every linear map from X into Y is sequentially continuous.*

Proof. Let X be finite dimensional and $T : X \longrightarrow Y$ be linear. If $X = \{0\}$ then there is nothing to prove. Let now $X \neq \{0\}$ and $\{e_1, e_2, ..., e_m\}$ be a basis for X. For a sequence $\{x_n\}$ in X, let $x_n = a_{n,1}e_1 + a_{n,2}e_2 + ... + a_{n,m}e_m$ where $a_{nj} \in \mathbb{R}$. If $x_n \longrightarrow x = a_1e_1 + a_2e_2 + ... + a_me_m$ in X, then

$$\begin{aligned}
\|x_n - x, e_j\| &= \|(a_{n,1} - a_1)e_1 + (a_{n,2} - a_2)e_2 + ... + (a_{n,m} - a_m)e_m, e_j\| \\
&= \|(a_{n,j} - a_j)e_j + y^j, e_j\| \\
&\quad \textit{where } y^j \in Y_j = span\{e_i \ : \ i = 1, 2, ...m \textit{ and } i \neq j\} \\
&= |a_{n,j} - a_j|\|e_j + y_j, e_j\|, \ \textit{where} \ \ y_j = \frac{1}{|a_{n,j} - a_j|}y^j \\
&\geq |a_{n,j} - a_j|dist(e_j, Y_j), \\
&\quad \textit{where } dist(e_j, Y_j) \ = \ \inf\{\|y, e_j\| \ : \ y \in Y_j\} \\
|a_{n,j} - a_j| &\leq \frac{\|x_n - x, e_j\|}{dist(e_j, Y_j)} \longrightarrow 0 \textit{ as } n \to \infty \text{ and for all } j. \\
\text{That is,} \quad & a_{n,j} \longrightarrow a_j \ \text{ for all } j.
\end{aligned}$$

By the linearity of T, it then follows that

$$\begin{aligned}
T(x_n) &= T(a_{n,1}e_1 + a_{n,2}e_2 + ... + a_{n,m}e_m) \\
&= a_{n,1}T(e_1) + a_{n,2}T(e_2) + ... + a_{n,m}T(e_m) \\
&\longrightarrow a_1T(e_1) + a_2T(e_2) + ... + a_mT(e_m) \\
&= T(a_1e_1 + a_2e_2 + ... + a_me_m) \\
&= T(x)
\end{aligned}$$

Thus every linear map T from X to Y is sequentially continuous. $\square$

Definition 11. *Let X and Y be two real linear 2-normed spaces, $\{T_\lambda\}_{\lambda\in\Lambda}$ a family of linear operator from X to Y. We say that $\{T_\lambda\}_{\lambda\in\Lambda}$ is equi-continuous if for any neighbourhood $B_d(0, R)$ in Y there exist some $B_e(0, r)$ in X such that*

$$T_\lambda\left(B_e(0, r)\right) \subseteq B_d(0, R), \ \ \textit{for all } \lambda \in \Lambda.$$

In other words if for any $d \in Y$ and $R > 0$, there exist $e \in X$ and $r > 0$ such that $\|T(x), d\| < R$, whenever $\|x, e\| < r$ and for all $\lambda \in \Lambda$.

Definition 12. *A subset E of a linear 2-normed space X is said to be locally bounded if there exist some $e \in X - \{0\}$ and $r > 0$ such that $E \subseteq B_e(0, r)$.*

A subset E of a linear 2-normed space X is bounded if for any open ball $B_e(0, r)$ there exist some $t > 0$ such that $E \subseteq tB_e(0, r) \subset B_e(0, R)$.

A linear map $T : X \to Y$ is bounded if and only if it maps bounded set into bounded set.

Theorem 12. *Suppose X and Y are linear 2-normed spaces over $\mathbb{R}$. Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be an equi-continuous collection of linear mappings from X into Y and B be a bounded subset of X. Then $T_\lambda(B)$ is a bounded subset of Y for all $\lambda \in \Lambda$, that is, $\{T_\lambda\}_{\lambda\in\Lambda}$ is equi-bounded.*

Proof. Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be an equi-continuous collection of linear mappings from X into Y. For any open ball $B_d(0, R)$ in Y, we can find an open ball $B_e(0, r)$ in X such that $T_\lambda (B_e(0, r)) \subseteq B_d(0, R)$ for all $\lambda \in \Lambda$.

$$\text{That is } \|T_\lambda(x), d\| < R, \text{ whenever } \|x, e\| < r \text{ and for all } \lambda \in \Lambda \tag{4}$$

Since B is bounded, corresponding to the open ball $B_e(0, r)$ there exist some $t > 0$ such that

$$B \subseteq tB_e(0, r) \tag{5}$$

If $x \in B$ then $\|\frac{x}{t}, e\| < r$. But then from (4), we obtain

$$\|T_\lambda(x), d\| < tR = R^1, \text{ for all } \lambda \in \Lambda, \ d \in Y \textit{ and } x \in B.$$

This shows that $T_\lambda(B)$ is a bounded subset of Y for all $\lambda \in \Lambda$, that is, $\{T_\lambda\}_{\lambda\in\Lambda}$ is equi-bounded. □

Theorem 13 (Banach Steinhauss Theorem in Linear 2-normed space)**.** *Let X and Y be linear 2-normed spaces over $\mathbb{R}$. If X is a 2-Banach space and $\{T_\lambda\}_{\lambda\in\Lambda}$ is a family of continuous linear operator from X to Y such that for any $x \in X$, there exist $c_x > 0$ such that*

$$\|T_\lambda(x), y\| < c_x\|x, e\|, \textit{ for all } \lambda \in \Lambda, \ y \in Y \textit{ and } e \notin \textit{span}\{x\} \tag{6}$$

then the family $\{T_\lambda\}_{\lambda\in\Lambda}$ is equi-continuous.

Proof. Let $B_d(0, R)$ be any open ball in Y. Note that $B_d(0, R)$ is absorbing in Y. choose a positive real number r such that $\overline{B_d(0, r)} + \overline{B_d(0, r)} \subseteq B_d(0, R)$.

$$\begin{aligned} Define \quad A_n &= \left\{x \in X \ : \ T_\lambda(x) \in n\overline{B_d(0, r)}, \text{ for all } \lambda \in \Lambda\right\} \\ &= \left\{x \in X \ : \ \frac{x}{n} \in T_\lambda^{-1}\left(\overline{B_d(0, r)}\right), \text{ for all } \lambda \in \Lambda\right\} \\ &= \left\{x \in X \ : \ x \in nT_\lambda^{-1}\left(\overline{B_d(0, r)}\right), \text{ for all } \lambda \in \Lambda\right\} \\ &= \cap_{\lambda\in\Lambda} nT_\lambda^{-1}\left(\overline{B_d(0, r)}\right). \end{aligned}$$

Then A_n is closed for all n and by using the given condition (1), we obtain $X = \cup_{n\in\mathbb{N}}A_n$. Since X is a 2-Banach space, Baire's theorem shows that atleast one of A_n has non-empty interior. Let x_0 be an interior point of A_{n_0}. Then there exist an open ball $B_e(0,t)$ such that

$$\begin{aligned} x + B_e(0,t) &\subseteq B_e(x,t) \subseteq A_{n_0} \\ T_\lambda\left(B_e(0,t)\right) &\subseteq T_\lambda(A_{n_0}) - T_\lambda(x) \\ &\subseteq n_0\overline{B_d(0,r)} - n_0\overline{B_d(0,r)} \\ &= n_0\left(\overline{B_d(0,r)} + \overline{B_d(0,r)}\right) \\ &\subseteq n_0 B_d(0,R), \quad \text{for all } \lambda \in \Lambda. \\ T_\lambda\left(B_e(0,\frac{t}{n})\right) &\subseteq B_d(0,R), \quad \lambda \in \Lambda. \end{aligned}$$

This shows that $\{T_\lambda\}_{\lambda\in\Lambda}$ is equi-continuous and hence equi-bounded. □

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