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CLIQUE-TO-VERTEX DETOUR DISTANCE IN GRAPHS

I. KEERTHI ASIR[1], S. ATHISAYANATHAN[2]

ABSTRACT. Let C be a clique and v a vertex in a connected graph G. A clique-to-vertex $C-v$ path P is a $u-v$ path, where u is a vertex in C such that P contains no vertices of C other than u. The clique-to-vertex distance, $d(C,v)$ is the length of a smallest $C-v$ path in G. A $C-v$ path of length $d(C,v)$ is called a $C-v$ geodesic. The clique-to-vertex eccentricity $e_2(C)$ of a clique C in G is the maximum clique-to-vertex distance from C to a vertex $v \in V$ in G. The clique-to-vertex radius r_2 of G is the minimum clique-to-vertex eccentricity among the cliques of G, while the clique-to-vertex diameter d_2 of G is the maximum clique-to-vertex eccentricity among the cliques of G. Also The clique-to-vertex detour distance, $D(C,v)$ is the length of a longest $C-v$ path in G. A $C-v$ path of length $D(C,v)$ is called a $C-v$ detour. The clique-to-vertex detour eccentricity $e_{D2}(C)$ of a clique C in G is the maximum clique-to-vertex detour distance from C to a vertex $v \in V$ in G. The clique-to-vertex detour radius R_2 of G is the minimum clique-to-vertex detour eccentricity among the cliques of G, while the clique-to-vertex detour diameter D_2 of G is the maximum clique-to-vertex detour eccentricity among the cliques of G. It is shown that $R_2 \leq D_2$ for every connected graph G and that every two positive integers a and b with $2 \leq a \leq b$ are realizable as the clique-to-vertex detour radius and the clique-to-vertex detour diameter respectively of some connected graph. Also it is shown that for any two positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $r_2 = a$, $R_2 = b$ and it is shown that for any two positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $d_2 = a$, $D_2 = b$.

Key words : clique-to-vertex detour distance, clique-to-vertex detour center, clique-to-vertex detour periphery.
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[1]Department of Mathematics, St. Xavier's College (Autonomous), Palayamkottai - 627 002, Tamil Nadu, India. Email: asirsxc@gmail.com
[2]Head, Department of Mathematics, St. Xavier's College (Autonomous), Palayamkottai - 627 002, Tamil Nadu, India. Email: athisxc@gmail.com.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected simple graph. For basic graph theoretic terminologies, we refer to Buckley and Harary [2] and Chartrand [6]. If $X \subseteq V$, then $\langle X \rangle$ is the subgraph induced by X. A vertex v is called a simplicial vertex if the subgraph induced by its neighbors is complete. A clique C of a graph G is a maximal complete subgraph and we denoted it by its vertices. A $u - v$ path P beginning with u and ending with v in a graph G is a sequence of distinct vertices such that consecutive vertices in the sequence are adjacent in G. For a graph G, the length of a path is the number of edges on the path. In [2] distance in graphs is defined in a natural way. For any two vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic in G. For a vertex v in a connected graph G, the eccentricity of v is defined by $e(v) = \max\{d(u, v) : u \in V\}$. A vertex u of G such that $d(u, v) = e(v)$ is called a eccentric vertex of v. The radius of G is defined by $r = rad(G) = \min\{e(v) : v \in V\}$ and the diameter of G is defined by $d = diam(G) = \max\{e(v) : v \in V\}$. A vertex v in a graph G is called a central vertex if $e(v) = r$ and the center of G is defined by $C = Cen(G) = \langle\{v \in V : e(v) = r\}\rangle$. A vertex v in a graph G is called a peripheral vertex if $e(v) = d$ and the periphery of G is defined by $P = Per(G) = \langle\{v \in V : e(v) = d\}\rangle$.

Santhakumaran and Arumugam [1] investigated in detail the facility location problems namely vertex-serves-structure and structure-serves-vertex, where the structure is a clique. Correspondingly they defined as follows: For a vertex u and a clique C in a connected graph G, the vertex-to-clique distance is defined by $d(u, C) = \min\{d(u, v) : v \in C\}$. For our convenience a $u - C$ path of length $d(u, C)$ is called a vertex-to-clique $u - C$ geodesic or simply $u - C$ geodesic. The vertex-to-clique eccentricity of u is defined by $e_1(u) = \max\{d(u, C) : C \in \zeta\}$, where ζ is the set of all cliques in G. A clique C of G such that $e_1(u) = d(u, C)$ is called a vertex-to-clique eccentric vertex of u. The vertex-to-clique radius r_1 and vertex-to-clique diameter d_1 of G are defined by $r_1 = \min\{e_1(v) : v \in V\}$ and $d_1 = \max\{e_1(v) : v \in V\}$ respectively. A vertex v in a graph G is called a vertex-to-clique central vertex if $e_1(v) = r_1$ and the vertex-to-clique center of G is defined by $Z_1(G) = \langle\{v \in V : e_1(v) = r_1\}\rangle$. For our convenience we denote $C_1(G) = Z_1(G)$. A vertex v in a graph G is called a vertex-to-clique peripheral vertex if $e_1(v) = d_1$ and the vertex-to-clique periphery of G is defined by $P_1 = \langle\{v \in V : e_1(v) = d_1\}\rangle$. Also the clique-to-vertex distance in graph is defined as as follows: For a clique C and a vertex v in a connected graph G, the clique-to-vertex distance is defined by $d(C, v) = \min\{d(u, v) : u \in C\}$. For our convenience a $C - v$

path of length $d(C, v)$ is called a clique-to-vertex $C - v$ geodesic or simply $C - v$ geodesic. The clique-to-vertex eccentricity of C is defined by $e_2(C) = \max\{d(C, v) : v \in V\}$. A vertex v of G such that $e_2(C) = d(C, v)$ is called a clique-to-vertex eccentric vertex of C. The clique-to-vertex radius r_2 and clique-to-vertex diameter d_2 of G are defined by $r_2 = \min\{e_2(C) : C \in \zeta\}$ and $d_2 = \max\{e_2(C) : C \in \zeta\}$ respectively. A clique C in a graph G is called a clique-to-vertex central clique if $e_2(C) = r_2$ and the clique-to-vertex center of G is defined by $Z_2(G) = \langle\{C \in \zeta : e_2(C) = r_2\}\rangle$. For our convenience we denote $C_2(G) = Z_2(G)$. A clique C in a graph G is called a clique-to-vertex peripheral clique if $e_2(C) = d_2$ and the clique-to-vertex periphery of G is defined by $P_2 = \langle\{C \in \zeta : e_2(C) = d_2\}\rangle$.

Chartrand et.al. [4, 5] introduced and studied the concepts of detour distance in graphs as follows: For any two vertices u and v in a connected graph G, the detour distance $D(u, v)$ is the length of a longest $u - v$ path in G. A $u - v$ path of length $D(u, v)$ is called a $u - v$ detour in G. For a vertex v in a connected graph G, the detour eccentricity of a vertex v is defined by $e_D(v) = \max\{D(u, v) : u \in V\}$. A vertex u of G such that $D(u, v) = e_D(v)$ is called a detour eccentric vertex of v. The detour radius R and detour diameter D of G are defined by $R = rad_D G = \min\{e_D(v) : v \in V\}$ and $D = diam_D(G) = \max\{e(v) : v \in V\}$ respectively. A vertex v in a graph G is called a detour central vertex if $e_D(v) = R$ and the detour center of G is defined by $C_D = Cen_D(G) = \langle\{v \in V : e_D(v) = R\}\rangle$. A vertex v in a graph G is called a detour peripheral vertex if $e_D(v) = D$ and the detour periphery of G is defined by $P_D = Per_D(G) = \langle\{v \in V : e_D(v) = D\}\rangle$.

Keerthi Asir and Athisayanathan [7] introduced and studied the concepts of vertex-to-clique detour distance in graph as follows: Let C be a clique and v a vertex in a connected graph G. A vertex-to-clique $u - C$ path P is a $u - v$ path, where v is a vertex in C such that P contains no vertices of C other than v and the vertex-to-clique detour distance $D(u, C)$ is the length of a longest $u - C$ path. A $u - C$ path of length $D(u, C)$ is called a $u - C$ vertex to clique detour. The vertex-to-clique detour eccentricity, $e_{D1}(u)$ of a vertex u in G is defined as $e_{D1}(u) = \max\{D(u, C) : C \in \zeta\}$, where ζ is the set of all cliques in G. A clique C for which $e_{D1}(u) = D(u, C)$ is called a vertex-to-clique detour eccentric clique of u. The vertex-to-clique detour radius of G is defined as, $R_1 = rad_{D1}(G) = \min\{e_{D1}(v) : v \in V\}$ and the vertex-to-clique detour diameter of G is defined as, $D_1 = diam_{D1}(G) = \max\{e_{D1}(v) : v \in V\}$. A vertex v in a graph G is called a vertex-to-clique detour central vertex if $e_{D1}(v) = R_1$ and the vertex-to-clique detour center of G is defined as, $C_{D1}(G) = Cen_{D1}(G) = \langle\{v \in V : e_{D1}(v) = R_1\}\rangle$. A vertex v in a graph G is called a vertex-to-clique detour peripheral vertex if $e_{D1}(v) = D_1$ and the vertex-to-clique detour periphery of G is defined as, $P_{D1}(G) = Per_{D1}(G) = \langle\{v \in V : e_{D1}(v) = D_1\}\rangle$.

In this paper, we introduce a new distance called clique-to-vertex detour distance in a connected graph G and investigate certain results related to clique-to-vertex detour distance and other distances in G. Throughout this paper, G denotes a connected graph with atleast two vertices.

2. Clique-to-Vertex Detour Distance

Definition 1. *Let C be a clique and v a vertex in a connected graph G. A clique-to-vertex $C - v$ path P is a $u - v$ path, where u is a vertex in C such that P contains no vertices of C other than u.*

Definition 2. *The clique-to-vertex detour distance, $D(C, u)$ between a clique C and a vertex v in a graph G is the length of a longest $C - v$ path. A $C - v$ path of length $D(C, v)$ is called a clique-to-vertex $C - v$ detour or simply $C - v$ detour.*

Example 1. *Consider the graph G given in Fig 2.1. For the vertex v and the clique $C = \{x, y, z\}$ in G, the paths $P_1 : x, s, t, u, w, v$; $P_2 : z, r, v$ and $P_3 : z, u, w, v$ are $C - v$ paths, while the paths $Q_1 : x, y, z, u, w, v$ and $Q_2 : x, s, t, u, z, r, v$ are not $C - v$ paths. Now the clique-to-vertex distance $d(C, v) = 2$ and the clique-to-vertex detour distance $D(C, v) = 5$. Also P_1 is a $C - v$ detour and P_2 is a $C - v$ geodesic. Note that the $C - x$, $C - y$ and $C - z$ paths are the trivial paths of length 0 and any non-trivial $C - v$ path does not contain a simplicial vertex of C.*

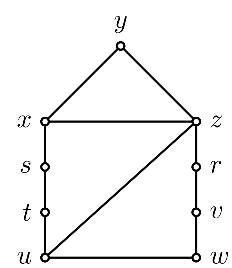

Fig 2.1: G

Since the length of a $C - v$ path between a clique C and a vertex v in a graph G of order n is atmost $n - 2$, we have the following observation.

Observation 1. *For any clique C and a vertex v in a non-trivial connected graph G of order n, $0 \leq d(C, v) \leq D(C, v) \leq n - 2$. The bounds are sharp. If G is a path $P : u_1, u_2, ..., u_{n-1}, u_n = v$ of order n, then $d(C, v) = D(C, v) = n - 2$, where $C = \{u_1, u_2\}$ and if G is a complete graph of order n, then $d(C, v) = D(C, v) = 0$ for every vertex v in G. Also we note that if G is a* *tree, then $d(C,v) = D(C,v)$ and if C is a clique and $v \notin C$ is a vertex in an even cycle, then $d(C,v) < D(C,v)$.*

Since a vertex of degree $n-1$ in a graph G of order n, belongs to every clique C in G, we have the following observation.

Observation 2. *Let G be a connected graph of order n and C a clique in G. If v is a vertex of degree $n-1$, then $D(C,v) = 0$.*

But the converse is not true. Consider the graph G given in Fig. 2.1., $D(C,v) = 0$, where $C = \{v, w\}$, but $deg(v) \neq n-1$.

Observation 3. *Let $K_{n,m}(n < m)$ be a complete bipartite graph with the partition V_1, V_2 of $V(K_{n,m})$ such that $|V_1| = n$ and $|V_2| = m$. Let C be a clique and v a vertex such that $v \notin C$ in $K_{n,m}$, then*

$$D(C,v) = \begin{cases} 2n-2, & \text{if } v \in V_1 \\ 2n-1 & \text{if } v \in V_2 \end{cases}$$

Observation 4. *Let v be a vertex and C a clique in a complete bipartite graph $K_{n,n}$ such that $v \notin C$, then $D(C,v) = 2n-2$.*

Since every tree has unique $C-v$ path between a clique C and a vertex v, we have the following observation.

Observation 5. *If G is a tree, then $d(C,v) = D(C,v)$ for every vertex v and a clique C in G.*

But the converse is not true. For the graph G obtained from a complete bipartite graph $K_{2,n}(n \geq 2)$ by joining the vertices of degree n by an edge. In such a graph every clique C is isomorphic to K_3 and for every vertex v with $v \notin C$, $d(C,v) = D(C,v) = 1$, but G is not tree.

3. Clique-to-Vertex Central Concepts

Definition 3. *Let G be a connected graph and ζ be the set of all cliques in G. The clique-to-vertex detour eccentricity, $e_{D2}(C)$ of a clique C in G is defined as $e_{D2}(C) = max\,\{D(C,v) : v \in V\}$. A vertex v for which $e_{D2}(C) = D(C,v)$ is called a clique-to-vertex detour eccentric vertex of C. The clique-to-vertex detour radius of G is defined as, $R_2 = rad_{D2}(G) = min\,\{e_{D2}(C) : C \in \zeta\}$ and the clique-to-vertex detour diameter of G is defined as, $D_2 = diam_{D2}(G) = max\,\{e_{D2}(C) : C \in \zeta\}$. A clique C in a graph G is called a clique-to-vertex detour central clique if $e_{D2}(C) = R_2$ and the clique-to-vertex detour center of G is defined as, $C_{D2}(G) = Cen_{D2}(G) = \langle C \in \zeta : e_{D2}(C) = R_2 \rangle$. A clique C in a graph G is called a clique-to-vertex detour peripheral clique if $e_{D2}(C) = D_2$ and the clique-to-vertex detour periphery of G is defined as, $P_{D2}(G) = Per_{D2}(G) = \langle C \in \zeta : e_{D2}(C) = D_2 \rangle$. If every clique of a graph G is a clique-to-vertex detour central clique, then G is called a clique-to-vertex detour self centered graph.*

Example 2. *For the connected graph G given in Fig. 3.1, the set of all cliques in G are given by, $\zeta = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}\}$ where $C_1 = \{v_1, v_2, v_3\}$, $C_2 = \{v_3, v_4\}$, $C_3 = \{v_4, v_5\}$, $C_4 = \{v_5, v_6\}$, $C_5 = \{v_6, v_7\}$, $C_6 = \{v_7, v_8\}$, $C_7 = \{v_8, v_{10}\}$, $C_8 = \{v_9, v_{10}\}$, $C_9 = \{v_4, v_9\}$ and $C_{10} = \{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$. The clique-to-vertex eccentricity $e_2(C)$, clique-to-vertex detour eccentricity $e_{D2}(C)$ of all the cliques of G are given in Table 1.*

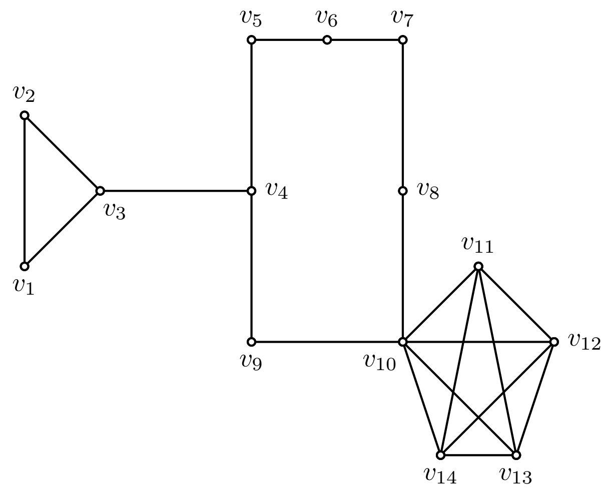

Fig. 3.1: G

C	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
$e_2(C)$	4	3	3	4	4	5	4	3	3	4
$e_{D2}(C)$	10	9	8	8	8	9	7	8	9	8

Table 1

For the graph G given in Fig. 3.1., clique-to-vertex detour eccentric vertex of all the cliques of G are given in Table 2.

Cliques C	Clique-to-Vertex Detour Eccentric Vertices
$C_1, C_2, C_3, C_5, C_{6,C_9}$	$v_{11}, v_{12}, v_{13}, v_{14}$
C_4, C_7, C_8, C_{10}	v_1, v_2

Table 2

For the graph G given in Fig. 3.1., the clique-to-vertex radius $r_2 = 3$, clique-to-vertex diameter $d_2 = 5$, clique-to-vertex detour radius $R_2 = 7$ and clique-to-vertex detour diameter $D_2 = 10$. Also the clique-to-vertex center $C_2(G)$, clique-to-vertex periphery $P_2(G)$, clique-to-vertex detour center $C_{D2}(G)$ and clique-to-vertex detour periphery $P_{D2}(G)$ are given in Table 3.

$C_2(G)$	$\langle\{C_2, C_3, C_8, C_9\}\rangle$	$P_2(G)$	$\langle\{C_6\}\rangle$
$C_{D2}(G)$	$\langle\{C_7\}\rangle$	$P_{D2}(G)$	$\langle\{C_1\}\rangle$

Table 3

Remark 1. *In a connected graph G, $C_2(G)$, $C_{D2}(G)$ and $P_2(G)$, $P_{D2}(G)$ need not be same. For the the graph G given in Fig 3.1, it is shown in Table 3, that $C_2(G)$, $C_{D2}(G)$ and $P_2(G)$, $P_{D2}(G)$ are distinct.*

Example 3. *The complete graph K_n, the cycle C_n, the wheel W_n and the complete bipartitate graph $K_{n,m}$ are clique-to-vertex detour self centered graphs.*

Remark 2. *A clique-to-vertex self-centered graph need not be a clique-to-vertex detour self centered graph. For the graph G given in Fig 3.2, $C_2(G) = \langle V(G)\rangle$ and $C_{D2}(G) = \langle\{v_3, v_4, v_5, v_6\}\rangle$.*

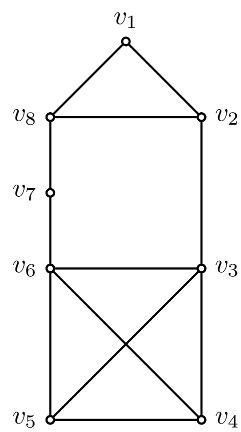

Fig. 3.2: G

Since the clique-to-vertex eccentricity is the maximum clique-to-vertex distance and the clique-to-vertex detour eccentricity is the maximum clique-to-vertex detour distance, the following theorem is a consequence of Observation 1.

Theorem 1. *For every clique C in G of order n, $0 \leq e_2(C) \leq e_{D2}(C) \leq n-2$.*

The bounds are sharp. If G is a path $P : u_1, u_2, ..., u_{n-1}, u_n = v$ of order n, then $e_2(C) = e_{D2}(C) = n - 2$, where $C = \{u_1, u_2\}$ and if G is a complete graph of order n, then $e_2(C) = e_{D2}(C) = 0$ for every clique C in G. Also we note that if G is a tree, then $e_2(C) = e_{D2}(C)$ and if G is an even cycle with $v \notin C$, then $e_2(C) < e_{D2}(C)$ for every clique C in G.

Since the clique-to-vertex radius (diameter) is the minimum (maximum) clique-to-vertex eccentricity and the clique-to-vertex detour radius (diameter) is the minimum (maximum) clique-to-vertex detour eccentricity, the following Corollary is a consequence of Theorem 1.

Corollary 2. *Let G be a connected graph. Then*
(i) $e_2(C) \leq e_{D2}(C)$ for every clique C in ζ.
(ii) $r_2 \leq R_2$.
(iii) $d_2 \leq D_2$.

Chartrand et. al. [6] showed that in a connected graph, the radius and diameter are related by $r \leq d \leq 2r$ and the detour radius and detour diameter are related by $R \leq D \leq 2R$. Also Santhakumaran et. al. [1] showed that the vertex-to-clique radius and vertex-to-clique diameter are related by $r_1 \leq d_1 \leq 2r_1 + 1$ and the clique-to-vertex radius and clique-to-vertex diameter are related by $r_2 \leq d_2 \leq 2r_2 + 1$. Keerthi Asir et.al. [7] showed that the similar inequality does not hold for the vertex-to-clique detour distance. Also this similar inequality does not hold for the clique-to-vertex detour distance.

Remark 3. *For the graph G_1 given in Fig. 3.3(a), $D_2 > 2R_2 + 1$ and the graph G_2 given in Fig. 3.3(b), $D_2 > 2R_2$ and $D_2 = 2R_2 + 1$. For the cycle $C_n (n \geq 3), D_2 < 2R_2$ and $D_2 < 2R_2 + 1$. Also for the path $P_{2n}, D_2 = 2R_2$.*

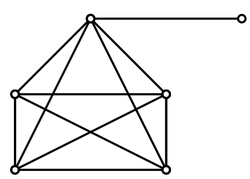

Fig. 3.3(a): G_1

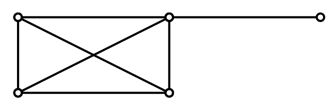

Fig. 3.3(b): G_2

Remark 4. *The clique-to-vertex radius r_2, the clique-to-vertex diameter d_2, clique-to-vertex detour radius R_2 and the clique-to-vertex detour diameter D_2 of some standard graphs are given in Table 4.*

Graph G	r_2	d_2	R_2	D_2
$K_n; n \geq 1$	0	0	0	0
$P_n; n \geq 2$	$\lceil n/2 \rceil$-1	$n-2$	$\lceil n/2 \rceil$-1	$n-2$
$C_n; n \geq 4$	$\lfloor (n-1)/2 \rfloor$	$\lfloor (n-1)/2 \rfloor$	$n-2$	$n-2$
$W_n; n \geq 5$	1	1	$n-3$	$n-3$
$K_{1,n}; n \geq 2$	0	1	0	1
$K_{n,n}; n \geq 2$	1	1	$2(n-1)$	$2(n-1)$
$K_{n,m}; 2 \leq n < m$	1	1	$2n-1$	$2n-1$

Table 4

Ostrand [8] showed that every two positive positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter respectively of some connected graph and Chartrand et. al. [4] showed that every two positive positive integers a and b with $a \leq b \leq 2a$ are realizable as the detour radius and detour diameter respectively of some connected graph. Also Santhakumaran et. al. [1] showed that every two positive integers a and b with $a \leq b \leq 2a+1$ are realizable as the vertex-to-clique radius and vertex-to-clique diameter respectively of some connected graph and every two positive integers a and b with $a \leq b \leq 2a+1$ are realizable as the clique-to-vertex radius and clique-to-vertex diameter respectively of some connected graph. Keerthi Asir et.al. [7] showed that every two positive integers a and b with $2 \leq a \leq b$ are realizable as the vertex-to-clique detour radius and vertex-to-clique detour diameter respectively of some connected graph. Now, we have a similar realization theorem for the clique-to-vertex detour radius and the clique-to-vertex detour diameter.

Theorem 3. *For each pair a, b of positive integers with $2 \leq a \leq b$, there exists a connected graph G with $R_2 = a$ and $D_2 = b$.*

Proof. **Case 1.** $a = b$. Let $C_{a+2} : u_1, u_2, ..., u_{a+2}, u_1$ be a cycle of order $a+2$. If $C = \{u_i, u_{i+1}\}$ then $e_{D2}(C) = a$ for $1 \leq i \leq a+2$. It is easy to verify that every clique S in G with $e_{D2}(S) = a$. Thus $R_2 = a$ and $D_2 = b$ as $a = b$..
Case 2. $2 \leq a < b \leq 2a$. Let $C_{a+2} : u_1, u_2, ..., u_{a+2}, u_1$ be a cycle of order $a+2$ and $P : v_1, v_2, ..., v_{b-a+1}$ be a path of order $b-a+1$. We construct the graph G as shown in the Fig. 3.4 by identifying the vertex u_1 of C_{a+2} and v_1 of P.

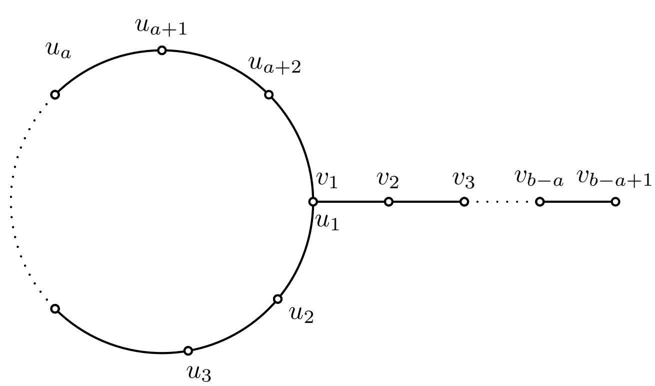

Fig. 3.4: G

If $C = \{u_1, u_2\} = \{u_1, u_{a+2}\}$, then $e_{D2}(C) = a$. Also if $C = \{u_i, u_{i+1}\}$, then $e_{D2}(C) = b - i + 2$ for $2 \leq i \leq \lceil \frac{a+2}{2} \rceil$ and $e_{D2}(C) = b - a + i - 1$ for $\lceil \frac{a+2}{2} \rceil < i \leq a+1$. Also if $C = \{v_i, v_{i+1}\}$ then $e_{D2}(C) = a + i$ for

$1 \leq i \leq b-a$. In particular, if $C = \{u_2, u_3\} = \{u_{a+1}, u_{a+2}\} = \{v_{b-a}, v_{b-a+1}\}$ then $e_{D2}(C) = b$. It is easy to verify that there is no clique S in G with $e_{D2}(S) < a$ and there is no clique S' in G with $e_{D2}(S') > b$. Thus $R_2 = a$ and $D_2 = b$ as $a < b$.

Case 3. $b > 2a(a \geq 2)$. Let $P_{a+1} : v_1, v_2, ..., v_{a+1}$ be a path of order $a+1$ and $K_{b-a+2} : u_1, u_2, ..., u_{b-a+2}$ be a complete graph of order $b-a+2$. We construct the graph G as shown in the Fig. 3.5 by identifying the vertex u_1 of K_{b-a+2} and v_1 of P_{a+1}. If $C = K_{b-a+2}$ then $e_{D2}(C) = a$ and if $C = \{v_i, v_{i+1}\}$ then $e_{D2}(C) = b-a+i$ for $1 \leq i \leq a$. In particular, if $C = \{v_a, v_{a+1}\}$ then $e_{D2}(C) = b$. It is easy to verify that there is no clique S in G with $e_{D2}(S) < a$ and there is no clique S' in G with $e_{D2}(S') > b$. Thus $R_2 = a$ and $D_2 = b$ as $b > 2a$.

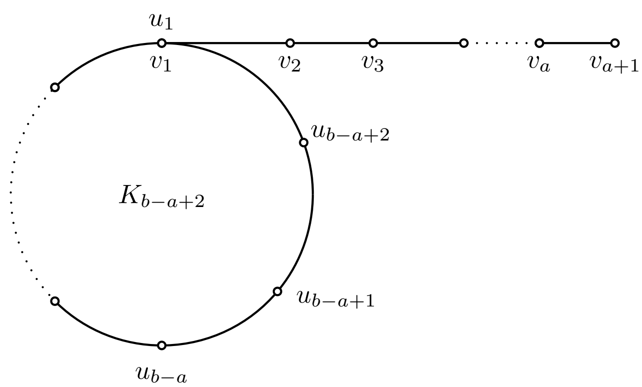

Fig. 3.5: G

□

Keerthi Asir et.al. [7] showed that for every two positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G with vertex-to-clique radius $r_1 = a$ and vertex-to-clique detour radius $R_1 = b$. Now, we have a similar realization theorem for the clique-to-vertex radius and the clique-to-vertex detour radius.

Theorem 4. *For any two positive integers a, b with $2 \leq a \leq b$, there exists a connected graph G such that $r_2 = a$ and $R_2 = b$.*

Proof. **Case 1.** $a = b$. Let $P_1 : u_1, u_2, ..., , u_a, u_{a+1}$ and $P_2 : v_1, v_2, ..., v_a, v_{a+1}$ be two copies of the path P_{a+1} of order $a+1$. We construct the graph G by joining u_1 in P_1 and v_1 in P_2 by an edge. If $C = \{u_1, v_1\}$ then $e_2(C) = e_{D2}(C) = a$ and if $C = \{u_i, u_{i+1}\} = \{v_i, v_{i+1}\}$ then $e_2(C) = a+i$ for $1 \leq i \leq a$. It is easy to verify that there is no clique S in G with $e_2(S) = e_{D2}(S) < a$. Thus $r_2 = a$ and $R_2 = b$ as $a = b$.

Case 2. $2 \leq a < b$. Let $P_1 : u_1, u_2, ..., , u_a, u_{a+1}$ and $Q_1 : v_1, v_2, ..., v_a, v_{a+1}$ be two copies of the path P_{a+1} of order $a + 1$. Let $P_2 : w_1, w_2, ..., w_{b-a+2}$ and $Q_2 : z_1, z_2, ..., z_{b-a+2}$ be two copies of the path P_{b-a+2} of order $b - a + 2$. We construct the graph G as follows: (i) identify the vertices u_1 in P_1 with w_1 in P_2 and also identify the vertices v_1 in Q_1 with z_1 in Q_2 (ii) identify the vertices u_3 in P_1 with w_{b-a+2} in P_2 and also identify the vertices z_{b-a+2} in Q_2 with v_3 in Q_1 (iii) join each vertex $w_i(2 \leq i \leq b-a+1)$ in P_2 with u_2 in P_1 and join each vertex $z_i(2 \leq i \leq b-a+1)$ in Q_2 with v_2 in Q_1 (iv) join u_1 in P_1 with v_1 in Q_1. The resulting graph G is shown in Fig. 3.6. If $C = \{u_1, v_1\}$ then $e_2(C) = a$ and $e_{D2}(C) = b$. Also if $C = \{u_i, u_{i+1}\} = \{v_i, v_{i+1}\}$ then $e_2(C) = a + i$ and $e_{D2}(C) = 2b - a + i$ for $3 \leq i \leq a$. Also if $C = \{u_2, w_i, w_{i+1}\} = \{v_2, z_i, z_{i+1}\}$ then $e_2(C) = a + 1$ for $i = 1$ and $e_2(C) = a + 2$ for $2 \leq i \leq b - a + 1$. However $e_{D2}(C) = b + i$ for $1 \leq i \leq b - a + 1$. It is easy to verify that there is no clique S in G with $e_2(S) < a$ and $e_{D2}(S) < b$. Thus $r_2 = a$ and $R_2 = b$ as $a < b$. □

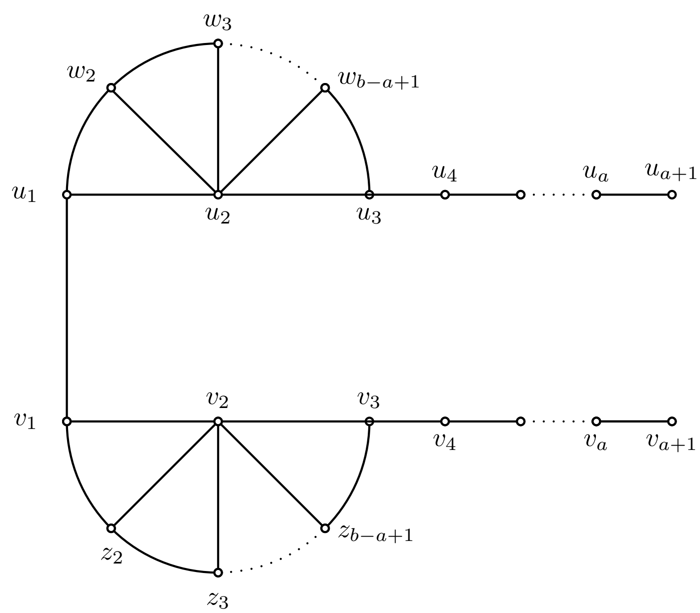

Fig. 3.6: G

Keerthi Asir et.al. [7] showed that for every two positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G with vertex-to-clique diameter $d_1 = a$ and vertex-to-clique detour diameter $D_1 = b$. Now, we have a similar realization theorem for the clique-to-vertex diameter and the clique-to-vertex detour diameter.

Theorem 5. *For any two positive integers a, b with $2 \leq a \leq b$, there exists a connected graph G such that $d_2 = a$ and $D_2 = b$.*

Proof. **Case 1.** $a = b$. Let $P_{a+2} : u_1, u_2, ..., , u_a, u_{a+1}, u_{a+2}$ be a path of order $a+2$. If $C = \{u_i, u_{i+1}\}$ then $e_2(C) = e_{D2}(C) = a-i+1$ for $1 \leq i \leq \lceil \frac{a+1}{2} \rceil$ and $e_2(C) = e_{D2}(C) = i-1$ for $\lceil \frac{a+1}{2} \rceil < i \leq a+1$. In particular if $C = \{u_1, u_2\} = \{u_{a+1}, u_{a+2}\}$ then $e_2(C) = e_{D2}(C) = a$. It is easy to verify that there is no clique S in G with $e_2(S) = e_{D2}(S) > a$. Thus $d_2 = a$ and $D_2 = b$ as $a = b$.
Case 2. $2 \leq a < b$. Let $P_1 : u_1, u_2, ..., u_a, u_{a+1}$ be a path of order $a + 1$. Let $P_2 : w_1, w_2, ..., w_{b-a+2}$ be a path of order $b - a + 2$. Let $P_3 : x_1, x_2$ be a path of order 2. We construct the graph G as follows: (i) identify the vertices u_1 in P_1, w_1 in P_2 with x_1 in P_3 and identify the vertices u_3 in P_1 with w_{b-a+2} in P_2 (ii) join each vertex $w_i (2 \leq i \leq b - a + 1)$ in P_2 with u_2 in P_1. The resulting graph G is shown in Fig. 3.7.

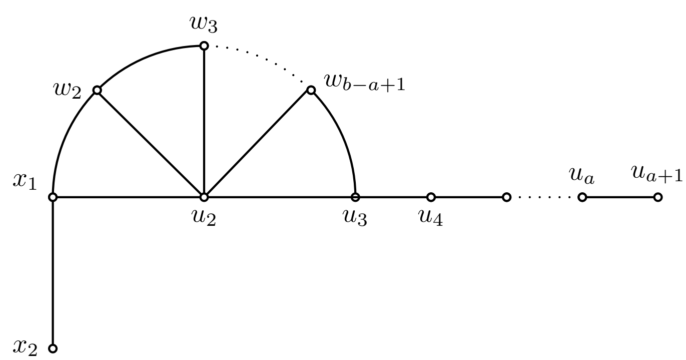

Fig. 3.7: G

If $C = \{x_1, x_2\}$ then $e_2(C) = a$ and $e_{D2}(C) = b$. Also if $C = \{u_i, u_{i+1}\}$ then $e_2(C) = i$ and $e_{D2}(C) = b - a + i$ for $3 \leq i \leq a$. If $C = \{u_2, w_i, w_{i+1}\}$ then $e_{D2}(C) = b - i - 1$ for $1 \leq i \leq \lceil \frac{b-a+1}{2} \rceil$ and $e_{D2}(C) = i$ for $\lceil \frac{b-a+1}{2} \rceil < i \leq b - a + 1$. Also we have to find $e_2(C)$ by the following subcases:

Subcase 1 of Case 2. When $a = 2$

$e_2(C) = 1$ for $i = 1$

$e_2(C) = 2$ for $2 \leq i \leq b - a + 1$

Subcase 2 of Case 2. When $a = 3$

$e_2(C) = 2$ for $1 \leq i \leq b - a + 1$

Subcase 3 of Case 2. When $a > 3$

$e_2(C) = a - 1$ for $1 \leq i \leq b - a$

$e_2(C) = a - 2$ for $i = b - a + 1$

It is easy to verify that there is no clique S in G with $e_2(S) > a$ and $e_{D2}(S) > b$. Thus $d_2 = a$ and $D_2 = b$ as $a < b$. □

Remark 5. *Harary and Norman* [3] *showed that the center of every connected graph G lies in a single block of G and Chartrand et. al.* [4] *showed that the detour center of every connected graph G lies in a single block of G. Also Santhakumaran et. al.* [1] *showed that the vertex-to-clique center of every connected graph G lies in a single block of G and Keerthi Asir et.al.* [7] *showed that the vertex-to-clique detour center of every connected graph G lies in a single block of G. However it is not true for the clique-to-vertex detour center of a graph. For the Path P_{2n+1}, the clique-to-vertex detour center is always P_3, which does not lie in a single block.*

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