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THE t-PEBBLING NUMBER OF SQUARES OF CYCLES

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ABSTRACT. Let C be a configuration of pebbles on a graph G. A pebbling move (step) consists of removing two pebbles from one vertex, throwing one pebble away, and moving the other pebble to an adjacent vertex. The t-pebbling number, $f_t(G)$, of a connected graph G, is the smallest positive integer such that from every configuration of $f_t(G)$ pebbles, t pebbles can be moved to any specified target vertex by a sequence of pebbling moves. In this paper, we determine the t-pebbling number for squares of cycles.

Key words : pebbling number, p^{th} power of a graph, cycle graph.
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1. INTRODUCTION

We begin by introducing relevant terminology and background on the subject. Here, the term graph refers to a simple graph. A *configuration* C of pebbles on a graph $G = (V, E)$ can be thought of as a function $C : V(G) \to N \cup \{0\}$. The *value* $C(v)$ equals the number of pebbles placed at vertex v, and the quantity $\sum_{v \in V(G)} C(v)$ is called the *size* of C; the size of C is just the total number of pebbles assigned to vertices. A *pebbling move* [8] consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. Suppose C is a configuration of pebbles on a graph G. We say a pebble can be moved to a vertex v, the *target vertex*, if we can apply pebbling moves repeatedly (if necessary) so that in the resulting configuration the vertex v has at least one pebble.

Definition 1. [1] *The t-pebbling number of a vertex v in a graph G, $f_t(v, G)$, is the smallest positive integer n such that however n pebbles are placed on the vertices of the graph, t pebbles can be moved to v in finite number of*

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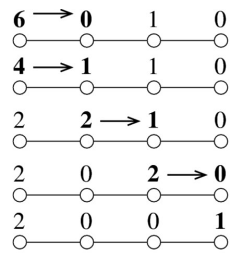

FIGURE 1. An illustration of moving one pebble to the end vertex of the path P_4 from a configuration of size 7

pebbling moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. The t-pebbling number of G, $f_t(G)$, is defined to be the maximum of the pebbling numbers of its vertices.

Thus the t-pebbling number of a graph G, $f_t(G)$, is the least n such that, for any configuration of n pebbles to the vertices of G, we can move t pebbles to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. Clearly, $f_1(G) = f(G)$, the pebbling number of G.

Fact 1. [9] *The pebbling number of a graph G satisfies*

$$f(G) \geq max\{2^{diam(G)}, |V(G)|\}.$$

If one pebble is placed on each vertex other than the vertex v, then no pebble can be moved to v. Also, if u is at a distance d from v, and $2^d - 1$ pebbles are placed on u, then no pebble can be moved to v. So it is clear that $f(G) \geq max\{|V(G)|, 2^D\}$, where D is the diameter of graph G. Furthermore, we know that $f(K_n) = n$ and $f(P_n) = 2^{n-1}$, where K_n is the complete graph with n vertices and P_n is the path with n vertices, so this bound is sharp.

With regard to t-pebbling number of graphs, we find the following theorems:

Theorem 1. [6] *Let K_n be the complete graph on n vertices where $n \geq 2$. Then $f_t(K_n) = 2t + n - 2$.*

Theorem 2. [1] *Let $K_1 = \{v\}$. Let $C_{n-1} = (u_1, u_2, \cdots, u_{n-1})$ be a cycle of length $n - 1$. Then the t-pebbling number of the wheel graph W_n is $f_t(W_n) = 4t + n - 4$ for $n \geq 5$.*

Theorem 3. [4] *For $G = K^*_{s_1,s_2,\cdots,s_r}$, $f_t(G) = \begin{cases} 2t + n - 2, & if\ 2t \leq n - s_1 \\ 4t + s_1 - 2, & if\ 2t \geq n - s_1 \end{cases}$.*

Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph on $nm + 1$ vertices, that is, a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}. The Jahangir graph

$J_{2,8}$ appears on Jahangir's tomb in his mausoleum. It lies in 5 kilometer northwest of Lahore, Pakistan, across the River Ravi [7].

Lourdusamy et al. proved the t-pebbling number of the Jahangir graphs $J_{2,m}$ and $J_{3,m}$ for $m \geq 3$ and $t \geq 1$ in [3, 5]. In the next section, we are going to prove the t-pebbling number of squares of even cycles and then we prove the t-pebbling number of squares of odd cycles in the third section. Before that, we give the defintion of p^{th} *power of a graph* G and the known results of the pebbling number of squares of cycles.

Definition 2. [11] *Let G be a connected graph. For u, $v \in V(G)$, we denote by $d_G(u,v)$ the distance between u and v in G. The p^{th} power of G, denoted by G^p, is the graph obtained from G by adding edge uv to G whenever $2 \leq d_G(u,v) \leq p$. That is, $E(G^p) = \{uv : 1 \leq d_G(u,v) \leq p\}$. Note that $G^1 = G$.*

In [9], Pachter et al. gave the pebbling numbers of squares of paths.

Theorem 4. [9] *The pebbling number of squares of paths is $f(P^2_{2k}) = 2^k$ and $f(P^2_{2k+1}) = 2^k + 1$.*

We have obtained the t-pebbling numbers of squares of paths in [2] for $t \geq 2$.

Theorem 5. [2] *The t-pebbling number of P^2_{2k+r} $(0 \leq r \leq 1)$ is $f_t(P^2_{2k}) = t(2^k)$ and $f(P^2_{2k+1}) = t(2^k) + 1$.*

Lourdusamy et al. gave the t-pebbling numbers of cycles:

Theorem 6. [6] *Let C_n denote a simple cycle with n vertices, where $n \geq 3$. Then $f_t(C_{2k}) = t2^k$ and $f_t(C_{2k+1}) = \frac{2^{k+2}-(-1)^{k+2}}{3} + (t-1)2^k$.*

Naturally, we want to know the t-pebbling numbers of squares of cycles. In [10, 11], the pebbling numbers of squares of cycles were obtained:

Theorem 7. [11] *The pebbling number of squares of even cycles is*

(i) For $2 \leq n \leq 6$, $f(C^2_{2n}) = 2n$.
(ii) For $k \geq 3$, $f(C^2_{4k+2}) = 2^{k+1}$.
(iii) For $k \geq 4$, $f(C^2_{4k}) = 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$.

Theorem 8. [10] *The pebbling number of squares of odd cycles is*

(i) For $2 \leq n \leq 6$, $f(C^2_{2n+1}) = 2n + 1$.
(ii) For $k \geq 3$, $f(C^2_{4k+3}) = 2^{k+1} + 1$.
(iii) For $k \geq 4$, $f(C^2_{4k+1}) = \lceil \frac{2^{k+2}}{3} \rceil + 1$.

Motivated by this, we compute the t-pebbling numbers of squares of cycles for $t \geq 2$ in this paper.

Notation 1. *Let $p(v)$ denote the number of pebbles on the vertex v and $p(A)$ denote the number of pebbles on the vertices of the subgraph A of G.*

2. The t-pebbling number of squares of even cycles

In this section, we prove the t-pebbling number of C_{2n}^2, where $n \geq 2$. Let $V(C_{2n}) = \{v, a_1, a_2, \cdots, a_{n-1}, x, b_{n-1}, b_{n-2}, \cdots, b_2, b_1\}$. By symmetry, Let v be the target vertex. Let $P_A : va_1a_2 \cdots a_{n-1}$ and $P_B : vb_1b_2 \cdots b_{n-1}$ be the paths. Also, we let $P_C : P_A \cup \{x\}$ and $P_D : P_B \cup \{x\}$. We always assume that $p(P_A^2) \geq p(P_B^2)$.

Since $C_4^2 \cong K_4$, $f_t(C_4^2) = f_t(K_4) = 2t + 2$ by Theorem 1.

Theorem 9. 1. $f_t(C_6^2) = \begin{cases} 6 & \text{if } t = 1 \\ 4t & \text{if } t \geq 2 \end{cases}$,

2. $f_t(C_8^2) = \begin{cases} 8 & \text{if } t = 1 \\ 4t + 2 & \text{if } t \geq 2 \end{cases}$.

Proof. Clearly, the results are true for $t = 1$ by Theorem 7.
Proof of (1): $f_t(C_6^2) = 4t$ for $t \geq 2$.

Claim 9.1. *$f_2(C_6^2) = 8$.*
Put 7 pebbles on the vertex x. Then we cannot move two pebbles to v. Thus, $f_2(C_6^2) \geq 8$.

Consider the distribution of 8 pebbles on the vertices of C_6^2. If $p(v) = 1$ or $p(u) \geq 2$ (where $uv \in E(C_6^2)$) then we can move two pebbles to v easily, since $p(C_6^2) - 2 \geq 6$ and $f(C_6^2) = 6$ (by Theorem 7). So, we assume $p(v) = 0$ and $p(u) \leq 1$ for all u (where $uv \in E(C_6^2)$). Clearly, $p(x) \geq 4$. If any two adjacent vertices of v have one pebble each on them then we can move two pebbles to v easily (by moving one pebble each to the adjacent (pebbled) vertices from x). Without loss of generality, let $p(a_1) = 1$ and $p(a_2) = p(b_1) = p(b_2) = 0$. Then $p(x) = 7$ and hence we can move two pebbles to v, since we can move three pebbles to a_1 from x. Assume $p(a_i) = 0$ and $p(b_j) = 0$ for all i, and j. Then also we can move two pebbles to v, since $p(x) = 8$ and $d_{C_6^2}(v, x) = 2$. Hence we have proved the claim.

We have to show that $f_t(C_6^2) = 4t$ for $t \geq 2$. Clearly, the result is true for $t = 2$ from Claim 9.1. Assume the result is true for $2 \leq t' < t$. Put $4t - 1$ pebbles on the vertex x and hence we cannot move t pebbles to v. Thus $f_t(C_6^2) \geq 4t$. Now, consider the distribution of $4t$ pebbles on the vertices of C_6^2. Either $p(P_C^2) \geq 2t$ or $p(P_B^2) \geq 2t$. Let $p(P_B^2) \geq 2t \geq 4$ and hence we can move one pebble to v from the vertices of P_B^2, since $P_B^2 \cong P_3^2$ and $f(P_B^2) = 3$ by Theorem 4. Then we have $4t - 3 \geq 4(t - 1)$ pebbles remaining on the vertices of C_6^2 and hence we can move the additional $t - 1$ pebbles to v by induction. Assume $p(P_B^2) \leq 2t - 1$ and so $p(P_C^2) \geq 2t + 1 \geq 5$. Clearly we can move one pebble to v at a cost of at most four pebbles, since $P_C^2 \cong P_4^2$ and $f(P_4^2) = 4$ by Theorem 4. Then we have $4t - 4 \geq 4(t - 1)$ pebbles remaining on the vertices of C_6^2 and hence we can move the additional $t-1$ pebbles to v by induction. Thus $f_t(C_6^2) \leq 4t$.

Proof of (2): $f_t(C_8^2) = 4t+2$ for $t \geq 2$.

Claim 9.2. $f_2(C_8^2) = 10$.

Put 7 pebbles on the vertex x and put one pebble each on the vertices a_3 and b_3. Then we cannot move two pebbles to v. Thus, $f_2(C_8^2) \geq 10$.

Consider the distribution of 10 pebbles on the vertices of C_8^2. If $p(v) = 1$ or $p(u) \geq 2$ (where $uv \in E(C_8^2)$) then we can move two pebbles to v easily, since $p(C_8^2) - 2 \geq 8$ and $f(C_8^2) = 8$ (by Theorem 7). So, we assume $p(v) = 0$ and $p(u) \leq 1$ for all u (where $uv \in E(C_8^2)$). Clearly, $p(x) + p(a_3) + p(b_3) \geq 6$. Either $p(P_C^2) \geq 5$ or $p(P_D^2) \geq 5$. Without loss of generality, let $p(P_C^2) \geq 5$. If $p(P_B^2) \geq 4$ then we can move one pebble to v, since $f(P_B^2) = 4$. Also we can move one more pebble to v from the vertices of P_C^2, since $f(P_5^2) = 5$. Let $p(P_B^2) = 3$ and so $p(P_C^2) = 7$. We can move one pebble to b_3, since either $p(a_3) \geq 2$ or $p(x) \geq 2$. Thus $p(P_B^2) + 1 = 4$ and $p(P_C^2) - 2 = 5$ and hence we can move two pebbles to v. Let $p(P_B^2) = 2$ and so $p(P_C^2) = 8$. Let $p(x) \geq 2$. If any two vertices of $P_B^2 - \{v\}$ have one pebble each on them, then clearly we can move one pebble to v. Then we have six pebbles remaining on the vertices of P_C^2 and hence we can move one more pebble to v, since $f(P_5^2) = 5$. Assume $p(x) \leq 1$ and so $p(a_3) \geq 5$. If $p(a_1) = 1$ or $p(a_2) = 1$ or both $p(b_3) = 1$ and $p(b_1) = 1$ or both $p(b_3) = 1$ and $p(b_2) = 1$ then clearly we can move one pebble to v. Then we have at least five pebbles remaining on the vertices of P_C^2 and hence we can move one more pebble to v, since $f(P_5^2) = 5$. If $p(a_3) = 8$ then we can move two pebbles to v, since $d_{c_8^2}(v, a_3) = 2$. Otherwise, we have $p(a_3) = 7$, $p(x) = 1$, $p(b_2) = 1$, and $p(b_1) = 1$. Hence we can move one pebble to v using two pebbles from a_3 and the pebbles on the vertices x, b_1, b_2 and then we move one more pebble to v from a_3, since $d_{c_8^2}(v, a_3) = 2$. Let $p(P_B^2) \leq 1$ and so P_C^2 contains at least nine pebbles. Thus we can move two pebbles to v, since $f_2(P_5^2) = 9$. Thus, $f_2(C_8^2) \leq 10$. Hence we have proved the claim.

We have to show that $f_t(C_8^2) = 4t+2$ for $t \geq 2$. Clearly, the result is true for $t = 2$ from Claim 9.2. Assume the result is true for $2 \leq t' < t$. Put $4t-1$ pebbles on the vertex x and one pebble each on the vertices a_3 and b_3 and hence we cannot move t pebbles to v. Thus $f_t(C_8^2) \geq 4t+2$. Now, consider the distribution of $4t+2$ pebbles on the vertices of C_8^2. Either $p(P_C^2) \geq 2t+1$ or $p(P_D^2) \geq 2t+1$. Without loss of generality, we let $p(P_D^2) \geq 2t+1 \geq 5$ and hence we can move one pebble to v at a cost of at most four pebbles, from the vertices of P_D^2, since $P_D^2 \cong P_5^2$ and $f(P_5^2) = 5$ by Theorem 4. Then we have $4t+2-4 \geq 4(t-1)+2$ pebbles remaining on the vertices of C_8^2 and hence we can move the additional $t-1$ pebbles to v by induction. Thus $f_t(C_8^2) \leq 4t+2$. □

Theorem 10. 1. $f_t(C_{10}^2) = \begin{cases} 10 & \text{if } t = 1 \\ 8t & \text{if } t \geq 2 \end{cases}$,

2. $f_t(C_{12}^2) = \begin{cases} 12 & \text{if } t = 1 \\ 8t+3 & \text{if } t \geq 2 \end{cases}$.

Proof. Clearly, the results are true for $t = 1$ by Theorem 7.
Proof of (1): $f_t(C_{10}^2) = 8t$ for $t \geq 2$.

Claim 10.1. $f_2(C_{10}^2) = 16$.
Let $p(x) = 15$ and so we cannot move two pebbles to v. Thus, $f_2(C_{10}^2) \geq 16$. Now, consider the distribution of 16 pebbles on the vertices of C_{10}^2. Note that we can move one pebble to v at a cost of at most 8 pebbles, since $p(C_{10}^2) = 16$ and $f(C_{10}^2) = 10$. Suppose if we have used only six or less pebbles to put one pebble to v, then we can move another one pebble to v, since $p(C_{10}^2) - 6 \geq 10$ and $f(C_{10}^2) = 10$. Assume that we have used seven or eight pebbles to put a pebble on the vertex v. Clearly, $p(P_A^2) \leq 1$ and so $p(P_B^2) \leq 1$. This implies that $p(x) \geq 14$, $p(a_1) = 0 = p(a_2)$ and $p(b_1) = 0 = p(b_2)$. Without loss of generality, let $p(a_3) = 1$. We can move 7 pebbles to a_3 from x and hence we move two pebbles to v, since $d_{C_{10}^2}(v, a_3) = 2$. Assume $p(P_A^2) = 0$ and so $p(P_B^2) = 0$. Then $p(x) \geq 16$ and hence we can move two pebbles to v, since $d_{C_{10}^2}(v, x) = 3$. Thus, $f_2(C_{10}^2) \leq 16$. Hence we have proved the claim.

We have to show that $f_t(C_{10}^2) = 8t$ for $t \geq 2$. Clearly, the result is true for $t = 2$ from Claim 10.1. Assume the result is true for $2 \leq t' < t$. Put $8t-1$ pebbles on the vertex x and hence we cannot move t pebbles to v. Thus $f_t(C_{10}^2) \geq 8t$. Now, consider the distribution of $8t$ pebbles on the vertices of C_{10}^2. Either $p(P_C^2) \geq 4t$ or $p(P_D^2) \geq 4t$. Without loss of generality, we let $p(P_C^2) \geq 4t \geq 8$ and hence we can move one pebble to v at a cost of at most eight pebbles, from the vertices of P_C^2, since $P_C^2 \cong P_6^2$ and $f(P_6^2) = 8$ by Theorem 4. Then we have $8t - 8 \geq 8(t-1)$ pebbles remaining on the vertices of C_{10}^2 and hence we can move the additional $t-1$ pebbles to v by induction. Thus $f_t(C_{10}^2) \leq 8t$.
Proof of (2): $f_t(C_{12}^2) = 8t + 3$ for $t \geq 2$.

Claim 10.2. $f_2(C_{12}^2) = 19$.
Let $p(a_5) = 13$ and $p(b_5) = 5$. Then we cannot move two pebbles to v. Thus, $f_2(C_{12}^2) \geq 19$. Now, consider the distribution of 19 pebbles on the vertices of C_{12}^2. Note that we can move one pebble to v at a cost of at most 8 pebbles, since $p(C_{12}^2) = 19$ and $f(C_{12}^2) = 12$. Suppose if we have used only seven or less pebbles to put one pebble to v, then we can move another one pebble to v, since $p(C_{12}^2) - 7 \geq 12$ and $f(C_{12}^2) = 12$. Assume that we have used exactly eight pebbles to put a pebble on the vertex v. Clearly, $p(P_A^2 - \{a_5\}) \leq 1$. Let $p(P_A^2 - \{a_5\}) = 1$ and so $p(a_5) + p(b_5) + p(x) \geq 17$. Let $p(a_1) = 1$ and so

$p(a_5) \leq 1$. This implies that $p(x) \geq 15$. If $p(x) \geq 16$ then we can move two pebbles to v, since $d_{C_{12}^2}(v,x) = 3$. Let $p(x) = 15$ and so either $p(a_5) = 1$ or $p(b_5) = 1$. Without loss of generality, let $p(a_5) = 1$. We move 3 pebbles to a_5 from x and hence we can move one pebble to v, since $p(a_1) = 1$. Then we can move another one pebble to v from x, since $p(x) - 6 \geq 8$ and $d_{C_{12}^2}(v,x) = 3$. Next, we assume $p(P_A^2 - \{a_5\}) = 0$ and so $p(a_5)+p(b_5)+p(x) = 19$. If $p(a_5) \leq 2$, then we can move two pebbles to v, since $p(P_D^2) \geq 17$ and $f_2(P_7^2) = 17$. So, we assume $p(a_5) \geq 3$ and $p(b_5) \geq 3$. Let $p(a_5) \geq 8$. Then we can move one pebble to v from a_5. If $p(b_5) \geq 8$ then we can move the another one pebble to v. So, we assume $p(b_5) \leq 7$. Let $p(b_5) = 6$ or 7. If $p(x) \geq 2$ then we move one pebble from x and three pebbles from b_5 to b_4 and hence we can move the another one pebble to v. Let $p(x) = 1$ and so $p(a_5) - 8 \geq 3$ and hence we can move another one pebble to v. Let $p(x) = 0$ and so $p(a_5) - 8 \geq 4$ and hence we can move another one pebble to v. Let $p(b_5) = 4$ or 5. If $p(x) \geq 4$ then we move two pebbles from x and two pebbles from b_5 to b_4 and hence we can move the another one pebble to v. Let $p(x) = 3$ and so $p(a_5) - 8 \geq 3$ and hence we can move another one pebble to v. Let $p(x) = 2$ and so $p(a_5) - 8 \geq 4$ and hence we can move another one pebble to v. Let $p(x) = 1$ and so $p(a_5) - 8 \geq 5$. Clearly we can move the another one pebble to v, since $p(b_5) = 4$ or 5. Let $p(x) = 0$ and so $p(a_5) - 8 \geq 6$. we move two pebbles to a_5 from b_5 and hence we can move one more pebble to v. Assume $p(a_5) \leq 7$ (similarly, we assume $p(b_5) \leq 7$). Then $p(x) \geq 5$. If $p(a_5) = 6$ or 7 then we move three pebbles from a_5 and one pebble from x to a_4. Thus we can move the first pebble to v. Then we can move another one pebble to v, since $p(P_D^2) \geq 9$ and $f(P_7^2) = 9$. Let $p(a_5) = 4$ or 5. Then we move two pebbles from a_5 and two pebbles from x to a_4. Thus we can move the first pebble to v. Then we can move another one pebble to v, since $p(P_D^2) \geq 9$ and $f(P_7^2) = 9$. Let $p(a_5) = 3$ and so $p(b_5) = 3$ and $p(x) = 13$. First, we move three pebbles from x and one pebble from a_5 to a_4 and then we move three pebbles from x and one pebble from b_5 to b_4. Thus we can move one pebble each to v from a_4 and b_4, since $d_{C_{12}^2}(v,a_4) = d_{C_{12}^2}(v,b_4) = 2$. Thus, $f_2(C_{12}^2) \leq 19$. Hence we have proved the claim.

We have to show that $f_t(C_{12}^2) = 8t + 3$ for $t \geq 2$. Clearly, the result is true for $t = 2$ from Claim 10.2. Assume the result is true for $2 \leq t' < t$. Put $8t - 3$ pebbles on the vertex a_5 and 5 pebbles on the vertex b_5 and hence we cannot move t pebbles to v. Thus $f_t(C_{12}^2) \geq 8t + 3$. Now, consider the distribution of $8t + 3$ pebbles on the vertices of C_{12}^2. Either $p(P_C^2) \geq 4t + 2$ or $p(P_D^2) \geq 4t + 2$. Without loss of generality, we let $p(P_C^2) \geq 4t + 2 \geq 10$ and hence we can move one pebble to v at a cost of at most eight pebbles, from the vertices of P_C^2, since $P_C^2 \cong P_7^2$ and $f(P_7^2) = 9$ by Theorem 4. Then we have $8t + 3 - 8 \geq 8(t-1) + 3$ pebbles remaining on the vertices of C_{12}^2

and hence we can move the additional $t-1$ pebbles to v by induction. Thus $f_t(C_{12}^2) \leq 8t+3$. □

Theorem 11. 1. $f_t(C_{4k+2}^2) = t2^{k+1}$ $(k \geq 3)$,

2. $f_t(C_{4k}^2) = (t-1)2^k + 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$ $(k \geq 4)$.

Proof. Consider the graph C_{2n}^2, where $n \geq 7$.

Proof of (1): Consider the graph C_{4k+2}^2 $(k \geq 3)$ and we place $t2^{k+1}-1$ pebbles on the vertex x. Then we cannot move t pebbles to v. Thus, $f_t(C_{4k+2}^2) \geq t2^{k+1}$. Now, consider the distribution of $t2^{k+1}$ pebbles on the vertices of C_{4k+2}^2. Clearly, the result is true for $t=1$ by Theorem 7. We assume the result is true for $1 \leq t' < t$. Without loss of generality, we assume that $p(P_C^2) \geq t2^k \geq 2^{k+1}$. Clearly, we can move one pebble to v, since $P_C^2 \cong P_{2(k+1)}^2$ and $f(P_{2(k+1)}^2) = 2^{k+1}$. Then we have $t2^{k+1} - 2^{k+1} \geq (t-1)2^{k+1}$ pebbles remaining on the vertices of C_{4k+2}^2 and hence we can move the additional $t-1$ pebbles to v by induction. Thus $f_t(C_{4k+2}^2) \leq t2^{k+1}$.

Proof of (2): Consider the graph C_{4k}^2 $(k \geq 4)$ and we place $(t-1)2^k + \lfloor \frac{2^{k+1}}{3} \rfloor$ pebbles on the vertex a_{n-1} and place $\lfloor \frac{2^{k+1}}{3} \rfloor$ pebbles on the vertex b_{n-1}. Then we cannot move t pebbles to v. Thus, $f_t(C_{4k}^2) \geq (t-1)2^k + 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$. Now, consider the distribution of $(t-1)2^k + 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$ pebbles on the vertices of C_{4k}^2. Clearly, the result is true for $t=1$ by Theorem 7. We assume the result is true for $1 \leq t' < t$. Without loss of generality, we assume that $p(P_C^2) \geq t2^{k-1} + 1 \geq 2^k + 1$. Clearly, we can move one pebble to v at a cost of at most 2^k pebbles, since $P_C^2 \cong P_{2k+1}^2$ and $f(P_{2k+1}^2) = 2^k + 1$. Then we have $(t-2)2^k + 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$ pebbles remaining on the vertices of C_{4k}^2 and hence we can move the additional $t-1$ pebbles to v by induction. Thus $f_t(C_{4k}^2) \leq (t-1)2^k + 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$. □

3. The t-pebbling number of squares of odd cycles

In this section, we prove the t-pebbling number of C_{2n+1}^2, where $n \geq 2$. Let $V(C_{2n+1}) = \{v, a_1, a_2, \cdots, a_{n-1}, x, y, b_{n-1}, b_{n-2}, \cdots, b_2, b_1\}$. By symmetry, Let v be the target vertex. Let $P_A : va_1a_2 \cdots a_{n-1}$ and $P_B : vb_1b_2 \cdots b_{n-1}$ be the paths. Also, we let $P_C : P_A \cup \{x\}$ and $P_D : P_B \cup \{y\}$. We always assume that $p(P_A^2) \geq p(P_B^2)$.

Since $C_5^2 \cong K_5$, $f_t(C_5^2) = f_t(K_5) = 2t+3$ by Theorem 1.

Theorem 12. 1. $f_t(C_7^2) = \begin{cases} 7 & \textit{if } t = 1 \\ 4t+1 & \textit{if } t \geq 2 \end{cases}$,

2. $f_t(C_9^2) = \begin{cases} 9 & \textit{if } t = 1 \\ 4t+3 & \textit{if } t \geq 2 \end{cases}$.

Proof. Clearly, the results are true for $t = 1$ by Theorem 8.
Proof of (1): $f_t(C_7^2) = 4t + 1$ for $t \geq 2$.

Claim 12.1. $f_2(C_7^2) = 9$.
Put 5 pebbles at x and 3 pebbles at y. Then we cannot move two pebbles to v and hence $f_2(C_7^2) \geq 9$. Now, consider the distribution of 9 pebbles on the vertices of C_7^2. If $p(v) = 1$ or $p(u) \geq 2$ (where $uv \in E(C_7^2)$) then we can move two pebbles to v easily, since $p(C_7^2) - 2 \geq 7$ and $f(C_7^2) = 7$ (by Theorem 7). So, we assume $p(v) = 0$ and $p(u) \leq 1$ for all u (where $uv \in E(C_7^2)$). Clearly, $p(x) \geq 3$ or $p(y) \geq 3$. Let $p(x) \geq 3$. If $p(a_1) = 1$ and $p(a_2) = 1$ then we can move two pebbles to v easily. Let $p(a_1) = 1$ and $p(a_2) = 0$ and so $p(x) \geq 4$. If $p(y) \geq 2$ then we move one pebble to a_2 and hence we can move two pebbles to v. Assume $p(y) \leq 1$ and so $p(x) \geq 6$. We move three pebbles to a_1 and hence we can move two pebbles to v. Similarly, we are done if $p(a_1) = 0$ and $p(a_2) = 1$. Let $p(a_1) = p(a_2) = 0$. Clearly, $p(x) + p(y) = 9$ and hence we can move two pebble to v easily. Thus $f_2(C_7^2) \leq 9$. Hence we have proved the claim.

We have to show that $f_t(C_7^2) = 4t + 1$ for $t \geq 2$. Clearly, the result is true for $t = 2$ from Claim 12.1. Assume the result is true for $2 \leq t' < t$. Put $4t - 1$ pebbles on the vertex x and one pebble at y. Then we cannot move t pebbles to v. Thus $f_t(C_7^2) \geq 4t + 1$. Now, consider the distribution of $4t + 1$ pebbles on the vertices of C_7^2. Either $p(P_C^2) \geq 2t + 1$ or $p(P_D^2) \geq 2t + 1$. Without loss of generality, we let $p(P_C^2) \geq 2t + 1 \geq 5$ and hence we can move one pebble to v from the vertices of P_C^2, since $P_C^2 \cong P_4^2$ and $f(P_4^2) = 4$ by Theorem 4. Then we have $4t - 3 \geq 4(t - 1) + 1$ pebbles remaining on the vertices of C_7^2 and hence we can move the additional $t - 1$ pebbles to v by induction. Thus $f_t(C_7^2) \leq 4t + 1$.
Proof of (2): $f_t(C_9^2) = 4t + 3$ for $t \geq 2$.

Claim 12.2. $f_2(C_9^2) = 11$.
Put 7 pebbles on the vertex x and put one pebble each on the vertices y, a_3 and b_3. Then we cannot move two pebbles to v. Thus, $f_2(C_9^2) \geq 11$.

Consider the distribution of 11 pebbles on the vertices of C_9^2. If $p(v) = 1$ or $p(u) \geq 2$ (where $uv \in E(C_9^2)$) then we can move two pebbles to v easily, since $p(C_9^2) - 2 \geq 9$ and $f(C_9^2) = 9$ (by Theorem 7). So, we assume $p(v) = 0$ and $p(u) \leq 1$ for all u (where $uv \in E(C_8^2)$). Clearly, $p(x)+p(y)+p(a_3)+p(b_3) \geq 7$. Either $p(P_C^2) \geq 6$ or $p(P_D^2) \geq 6$. Without loss of generality, let $p(P_C^2) \geq 6$. If $p(P_B^2) \geq 4$ then we can move one pebble to v, since $f(P_B^2) = 4$. Also we can move one more pebble to v from the vertices of P_C^2, since $f(P_5^2) = 5$. Let $p(P_B^2) = 3$. If $p(y) \geq 2$ or $p(x) \geq 2$, then we can move one pebble to b_3. Thus $p(P_B^2) + 1 = 4$ and $p(P_C^2) - 2 = 5$ and hence we can move two pebbles to v. Assume $p(y) \leq 1$ and $p(x) \leq 1$ and so $p(a_3) \geq 4$. If both $p(a_1) = 1$ and $p(a_2) = 1$ then we can move two pebbles to v easily. Let $p(a_1) = 1$ and

$p(a_2) = 0$. We move the first pebble to v through a_1 from a_3. If $p(x) = 1$ or $p(y) = 1$ then we move one pebble to b_3 and hence we can move one more pebble to v, since $p(P_B^2) + 1 = 4$. Assume $p(x) = p(y) = 0$ and so $p(a_3) \geq 7$. Clearly, we can move two pebbles to v. Similarly, we are done if $p(a_2) = 1$ and $p(a_1) = 0$. Let $p(a_1) = p(a_2) = 0$ and so $p(a_3) \geq 6$. If $p(x) = 1$ or $p(y) = 1$ then we move one pebble to b_3 and hence we can move one pebble each from p_B^2 and a_3, since $p(a_3) - 2 \geq 4$ and $d_{C_9^2}(v, a_3) = 2$. Assume $p(x) = p(y) = 0$ and so $p(a_3) = 8$. Thus we can move two pebbles to v, since $d_{C_9^2}(v, a_3) = 2$. Let $p(P_B^2) = 2$. Clearly, we can move two pebbles to v if $p(y) \geq 2$ or $p(x) \geq 2$. Assume $p(x) \leq 1$ and $p(x) \leq 1$. If any two vertices of $P_B^2 - \{v\}$ have one pebble each on them, then clearly we can move one pebble to v. Then we have five pebbles remaining on the vertices of P_C^2 and hence we can move one more pebble to v, since $f(P_5^2) = 5$. Let $p(b_3) = 2$. If $p(y) \geq 2$ then we can move two pebbles to v easily. Let $p(y) = 1$. If $p(x) \geq 2$ then we move one pebble to y from x and then we move one pebble each to b_2 from b_3 and y. Thus we can move two pebbles to v. Assume $p(x) \leq 1$ and so $p(a_3) \geq 5$. Now, we move one pebble to y from a_3 and hence we can move two pebbles to v. Let $p(y) = 0$. Clearly, we can move one pebble to b_2 from a_3 or x and hence we can move two pebbles to v. Let $p(P_B^2) = 1$ and so $p(y) \leq 4$. If $p(y) \leq 3$ then we can move two pebbles to v easily. Let $p(y) = 2$. we move one pebble to a_3 and hence we can move two pebbles to v. If $p(y) \leq 1$ then clearly we can move one pebble to v from P_C^2. Let $p(p_B^2) = 0$ and so $p(y) \leq 5$. Then we can move two pebbles to v. Thus, $f_2(C_9^2) \leq 11$. Hence we have proved the claim.

We have to show that $f_t(C_9^2) = 4t + 3$ for $t \geq 2$. Clearly, the result is true for $t = 2$ from Claim 12.2. Assume the result is true for $2 \leq t' < t$. Put $4t - 1$ pebbles on the vertex x and one pebble each on the vertices y, a_3 and b_3 and hence we cannot move t pebbles to v. Thus $f_t(C_9^2) \geq 4t + 3$. Now, consider the distribution of $4t + 3$ pebbles on the vertices of C_9^2. Either $p(P_C^2) \geq 2t + 2$ or $p(P_D^2) \geq 2t + 2$. Without loss of generality, we let $p(P_C^2) \geq 2t + 2 \geq 6$ and hence we can move one pebble to v at a cost of at most four pebbles, from the vertices of P_C^2, since $P_C^2 \cong P_5^2$ and $f(P_5^2) = 5$ by Theorem 4. Then we have $4t + 3 - 4 \geq 4(t - 1) + 3$ pebbles remaining on the vertices of C_9^2 and hence we can move the additional $t - 1$ pebbles to v by induction. Thus $f_t(C_9^2) \leq 4t + 3$. □

Theorem 13. 1. $f_t(C_{11}^2) = \begin{cases} 11 & \text{if } t = 1 \\ 8t + 1 & \text{if } t \geq 2 \end{cases}$,

2. $f_t(C_{13}^2) = \begin{cases} 13 & \text{if } t = 1 \\ 8t + 4 & \text{if } t \geq 2 \end{cases}$.

Proof. Clearly, the results are true for $t = 1$ by Theorem 8.
Proof of (1): $f_t(C_{11}^2) = 8t + 1$ for $t \geq 2$.

Claim 13.1. *$f_2(C_{11}^2) = 17$.*
Put 15 pebbles at x and 1 pebble at y. Then we cannot move two pebbles to v and hence $f_2(C_{11}^2) \geq 17$. Now, consider the distribution of 17 pebbles on the vertices of C_9^2. Note that, we can move one pebble to v at a cost of at most eight pebbles, since $p(C_{11}^2) = 17$ and $f(C_{11}^2) = 11$. If we have used only six or less pebbles to put the first pebble at v then we can move the second pebble to v easily, since $p(C_{11}^2) - 6 \geq 11$. Suppose we have used seven or eight pebbles to put the first pebble at v. Clearly, $p(P_A^2) \leq 2$. Let $p(a_1) = 1$ and $p(a_4) = 1$ and so $p(x) \leq 1$, $p(b_1) = p(b_2) = 0$, either $p(b_4) = 1$ or $p(b_3) = 1$. If $p(x) = 1$ then we move one pebble each to a_4 and x from y. Then we can move one pebble to v through a_3 and a_1 from a_4 and x. Thus $p(y) - 4 \geq 8$ and hence we can move another one pebble to v, since $d_{C_{11}^2}(v, y) = 3$. Assume $p(x) = 0$ and so $p(y) \geq 14$. We move three pebbles to a_4 and hence we can move one pebble each to v from a_4 and y, since $p(y) - 6 \geq 8$. Let $p(a_3) = 1$ and $p(a_4) = 1$. Clearly, $p(x) \leq 1$, $p(b_1) = p(b_2) = 0$, either $p(b_4) = 1$ or $p(b_3) = 1$. If $p(x) = 1$ then we move one pebble each to a_4 and x from y. Then we can move one pebble to v through a_3, and a_2 from a_4 and x. Thus $p(y) - 4 \geq 8$ and hence we can move another one pebble to v, since $d_{C_{11}^2}(v, y) = 3$. Assume $p(x) = 0$ and so $p(y) \geq 14$. We move three pebbles to a_4 and hence we can move one pebble each to v from a_4 and y, since $p(y) - 6 \geq 8$. Let $p(P_A^2) = 1$. If $p(P_B^2) = 0$ then we can move two pebbles to v, since $p((P_C \cup \{y\})^2) = 17$ and $f_2(P_7^2) = 17$. Assume $p(P_B^2) = 1$. Since $p(x) + p(y) = 15$, we can move two pebbles to v (by moving pebbles from x and y to a_4 or b_4) if $p(a_4) = 1$ or $p(b_4) = 1$. Let $p(a_3) = 1$. Clearly, we can move four pebbles to a_2 if $p(x) \geq 1$. Otherwise, we can move two pebbles to v, since $p(P_D^2) = 16$ and $f_2(P_6^2) = 16$. Let $p(a_1) = 1$ and so $1 \leq p(x) \leq 2$. Assume $p(x) = 2$ This implies that $p(y) \geq 13$. We move one pebble each to a_3 from y and x and hence we can move one pebble to v through a_1. Then we can move the another one pebble to v, since $p(y) - 4 \geq 8$ and $d_{C_{11}^2}(v, y) = 3$. Let $p(x) = 1$ and so $p(b_3) = 1$. We move three pebbles to b_3 from y and hence we can move one pebble each from b_3 and y, since $p(y) - 6 \geq 8$. Thus $f_2(C_{11}^2) \leq 17$. Hence we have proved the claim.

We have to show that $f_t(C_{11}^2) = 8t + 1$ for $t \geq 2$. Clearly, the result is true for $t = 2$ from Claim 13.1. Assume the result is true for $2 \leq t' < t$. Put $8t - 1$ pebbles on the vertex x and one pebble at y. Then we cannot move t pebbles to v. Thus $f_t(C_{11}^2) \geq 8t + 1$. Now, consider the distribution of $8t + 1$ pebbles on the vertices of C_{11}^2. Either $p(P_C^2) \geq 4t + 1$ or $p(P_D^2) \geq 4t + 1$. Without loss of generality, we let $p(P_C^2) \geq 4t + 1 \geq 9$ and hence we can move one pebble to v at a cost of at most eight pebbles from the vertices of P_C^2, since $P_C^2 \cong P_6^2$ and $f(P_6^2) = 8$ by Theorem 4. Then we have at least $8t - 7 = 8(t - 1) + 1$ pebbles

remaining on the vertices of C_{11}^2 and hence we can move the additional $t-1$ pebbles to v by induction. Thus $f_t(C_{11}^2) \leq 8t+1$.
Proof of (2): $f_t(C_{13}^2) = 8t+4$ for $t \geq 2$.

Claim 13.2. *$f_2(C_{13}^2) = 20$.*
Let $p(a_5) = 13$, $p(b_5) = 5$ and $p(x) = 1$. Then we cannot move two pebbles to v. Thus, $f_2(C_{13}^2) \geq 20$. Now, consider the distribution of 20 pebbles on the vertices of C_{13}^2. Note that we can move one pebble to v at a cost of at most 8 pebbles, since $p(C_{13}^2) = 20$ and $f(C_{13}^2) = 13$. Suppose if we have used only seven or less pebbles to put one pebble to v, then we can move another one pebble to v, since $p(C_{13}^2) - 7 \geq 13$ and $f(C_{13}^2) = 13$. Assume that we have used exactly eight pebbles to put a pebble on the vertex v. Clearly, $p(P_A^2 - \{a_5\}) \leq 2$. Assume $p(P_A^2) - \{a_5\} = 2$. Let $p(a_1) = 1$ and $p(a_2) = 1$ and so either $p(a_5) \leq 1$ or $p(x) \leq 1$. Assume $p(a_5) = 1$ and so $p(x) = 0$, and $p(y) \geq 13$. We move three pebbles to a_5 and hence we can move one pebble to v through a_1 or a_2. If $p(P_B^2) \geq 2$ then we can move another one pebble to v, since $p(P_D^2) \geq 9$ and $f(P_7^2) = 9$. If $p(P_B^2) \leq 1$ then $p(y) \geq 8$ and hence we can move the another one pebble to v, since $d_{C_{13}^2}(v, y) = 3$. Let $p(a_1) = 1$ and $p(a_4) = 1$ and so $p(a_5) \leq 1$ and $p(x) \leq 5$. Assume $p(x) = 4$ or 5. First, we move one pebble to a_4 from x and also we move one pebble each to a_5 from x and y and then we move one pebble each to a_3 from a_4 and a_5 and hence we can move one pebble to v through a_1. Clearly, we can move another one pebble to v, since $p(P_D^2) - 2 \geq 10$ and $f(P_7^2) = 9$. Assume $p(x) = 2$ or 3. First, we move one pebble to a_4 from x and also we move two pebbles to a_5 from y and then we move one pebble each to a_3 from a_4 and a_5 and hence we can move one pebble to v through a_1. Clearly, we can move another one pebble to v, since $p(P_D^2) - 4 \geq 10$ and $f(P_7^2) = 9$. Assume $p(x) \leq 1$. If $p(x) = 0$ or $p(a_5) = 0$ then clearly we can move two pebbles to v, since $p(P_D^2) \geq 17$ and $f_2(P_7^2) = 17$. Assume $p(x) = 1$ and $p(a_5) = 1$. We move one pebble each to a_5 and x from y and then we move one pebble each to a_3 from a_4 and a_5 and hence we can move one pebble to v through a_1 from a_3. Then we can move one more pebble to v, since $p(y) - 4 \geq 8$. Let $p(a_3) = 1$ and $p(a_4) = 1$ and so $p(a_5) \leq 1$ and $p(x) \leq 5$. In a similar way, we can move two pebbles to v for this case. Clearly, we can move two pebbles to v if $p(a_3) = 2$ or $p(a_4) = 2$. Assume $p(P_A^2 - \{a_5\}) = 1$.

Let $p(P_A^2 - \{a_5\}) = 1$ and so $p(a_5) + p(b_5) + p(x) + p(y) \geq 18$. Let $p(a_1) = 1$ and so $p(a_5) \leq 1$. This implies that $p(x) + p(y) \geq 16$. Either $p(x) \geq 8$ or $p(y) \geq 8$. Assume $p(x) \geq 8$. Let $p(a_5) = 1$. First we move 3 pebbles to a_5 from x and hence we can move one pebble to v, since $p(a_1) = 1$. Let $p(x) = 8, 9,$ or 10. After using six pebbles from x, we move $\lfloor \frac{p(x)}{2} \rfloor$ pebbles to y. Clearly, $p(y) + \lfloor \frac{p(x)-6}{2} \rfloor \geq 8$ and hence we can move one more pebble to v. Let $p(x) \geq 12$. After using six pebbles from x, we move $\lfloor \frac{p(y)}{2} \rfloor$ pebbles to x.

Clearly, $p(x) + \lfloor \frac{p(y)}{2} \rfloor \geq 8$ and hence we can move one more pebble to v. Let $p(x) = 11$. If $p(P_B^2 - \{b_5\}) = 1$ then we can move one more pebble to v, since $p(x) = 5$ and $p(y) \geq 5$. Assume $p(P_B^2 - \{b_5\}) = 0$. Then $p(y) + \lfloor \frac{p(x)-6}{2} \rfloor \geq 8$ and hence we can move one more pebble to v. Assume $p(x) \leq 7$. Clearly, we can move two pebbles to v if $p(x) \geq 2$. Let $p(x) \leq 1$. We move three pebbles to a_5 and then we move one pebble to v through a_1. Then we can move the another one pebble to v, since $p(y) - 6 \geq 8$. Let $p(a_5) = 0$ and so $p(x) + p(y) \geq 18$. Let $p(x) \geq 9$. Clearly we can move one pebble to v easily if $p(x) \leq 12$ (by moving pebbles from x to y after using eight pebbles from x) or $p(x) \geq 14$ (by moving pebbles from y to x after using eight pebbles from x). Assume $p(x) = 13$. Clearly, we are done if $p(P_B^2) = 1$. Assume $p(P_B^2) = 0$. We move two pebbles to y and hence we can move one pebble each to v from y and x, since $p(y) + 2 = 8$ and $p(x) - 4 = 9$. In a similar way, we can move one pebble to v if $p(a_i) = 1$ for some $i = 2, 3, 4$.

Let $p(P_A^2 - \{a_5\}) = 0$ and so $p(a_5) + p(b_5) + p(x) + p(y) = 20$. Without loss of generality, let $p(a_5) + p(x) \geq 10$ and so $p(b_5) + p(y) \leq 10$. If $p(a_5) + p(x) \leq 13$, then we move $\lfloor \frac{p(a_5)+p(x)-9}{2} \rfloor$ pebbles to b_5 and y, and hence we can move two pebbles to v, since $p(a_5) + p(x) \geq 9$, $p(b_5) + p(y) + \lfloor \frac{p(a_5)+p(x)-9}{2} \rfloor \geq 9$ and $f(P_7^2) = 9$. If $p(a_5) + p(x) \geq 15$, then we move $\lfloor \frac{p(b_5)+p(y)}{2} \rfloor$, pebbles to a_5 and x, and hence we can move two pebbles to v, since $p(a_5) + p(x) + \lfloor \frac{p(b_5)+p(y)}{2} \rfloor \geq 17$ and $f_2(P_7^2) = 17$. Let $p(a_5) + p(x) = 14$ and so $p(b_5) + p(y) = 6$. If both $p(b_5)$ and $p(y)$ are even then we can move three pebble to x and hence we can move two pebbles to v, since $f_2(P_7^2) = 17$. Assume both $p(b_5)$ and $p(y)$ are odd. Let $p(b_5) = 3$ and $p(y) = 3$. If $p(x) \geq 1$ then we can move one pebble to v through b_2 using at most five pebbles from the vertices a_5 and x. Hence we can move one more pebble to v, since $p(a_5) + p(x) - 5 \geq 9$ and $f(P_7^2) = 9$. Assume $p(x) = 0$. Then we can move one pebble to v through b_2 using six pebbles from the vertex a_5 and then we can move one pebble to v, since $p(a_5) = 8$ and $d_{C_{13}^2}(v, a_5) = 3$. Similarly, we can move two pebbles to v if $p(b_5) = 1$ and $p(y) = 5$ or $p(b_5) = 5$ and $p(y) = 1$. Thus, $f_2(C_{13}^2) \leq 20$. Hence we have proved the claim.

We have to show that $f_t(C_{13}^2) = 8t + 4$ for $t \geq 2$. Clearly, the result is true for $t = 2$ from Claim 13.2. Assume the result is true for $2 \leq t' < t$. Put $8t - 3$ pebbles on the vertex a_5, 5 pebbles on the vertex b_5 and one pebble at x. Then we cannot move t pebbles to v. Thus $f_t(C_{13}^2) \geq 8t + 4$. Now, consider the distribution of $8t + 4$ pebbles on the vertices of C_{13}^2. Either $p(P_C^2) \geq 4t + 2$ or $p(P_D^2) \geq 4t + 2$. Without loss of generality, we let $p(P_C^2) \geq 4t + 2 \geq 10$ and hence we can move one pebble to v at a cost of at most eight pebbles, from the vertices of P_C^2, since $P_C^2 \cong P_7^2$ and $f(P_7^2) = 9$ by Theorem 4. Then we have $8t + 4 - 8 \geq 8(t - 1) + 4$ pebbles remaining on the vertices of C_{13}^2

and hence we can move the additional $t-1$ pebbles to v by induction. Thus $f_t(C_{13}^2) \leq 8t+4$. $\square$

Theorem 14. *(1)* $f_t(C_{4k+3}^2) = t2^{k+1}+1$ $(k \geq 3)$, *(2)* $f_t(C_{4k+1}^2) = (t-1)2^k + \lceil \frac{2^{k+2}}{3} \rceil + 1 = (t-1)2^k + \begin{cases} 2\lceil \frac{2^{k+1}}{3} \rceil + 1 & \text{if } k \text{ is even} \\ 2\lfloor \frac{2^{k+1}}{3} \rfloor + 2 & \text{if } k \text{ is odd} \end{cases}$ $(k \geq 4)$.

Proof. Consider the graph C_{2n+1}^2, where $n \geq 7$.
Proof of (1): Consider the graph C_{4k+3}^2 $(k \geq 3)$. Put $t2^{k+1}-1$ pebbles on the vertex x and one pebble on the vertex x. Then we cannot move t pebbles to v and thus $f_t(C_{4k+3}^2) \geq t2^{k+1}+1$. Next, we have to show that $f_t(C_{4k+3}^2) \leq t2^{k+1}+1$. Clearly, the result is true for $t=1$ by Theorem 8. Assume the result is true for $1 \leq t' \leq t$. Consider the distribution of $t2^{k+1}+1$ pebbles on the vertices of C_{4k+3}^2. Without loss of generality, we assume that $p(P_C^2) \geq t2^k+1 \geq 2^{k+1}+1$. Clearly, we can move one pebble to v, since $P_C^2 \cong P_{2(k+1)}^2$ and $f(P_{2(k+1)}^2) = 2^{k+1}$. Then we have $t2^{k+1}-2^{k+1}+1 \geq (t-1)2^{k+1}+1$ pebbles remaining on the vertices of C_{4k+3}^2 and hence we can move the additional $t-1$ pebbles to v by induction. Thus $f_t(C_{4k+3}^2) \leq t2^{k+1}+1$.
Proof of (2): Consider the graph C_{4k+1}^2 $(k \geq 4)$. If k is even, then we place $(t-1)2^k + \lceil \frac{2^{k+1}}{3} \rceil$ pebbles on the vertex a_{n-1} and place $\lceil \frac{2^{k+1}}{3} \rceil$ pebbles on the vertex b_{n-1}. If k is odd, then we place $(t-1)2^k + \lfloor \frac{2^{k+1}}{3} \rfloor$ pebbles on the vertex a_{n-1}, place $\lfloor \frac{2^{k+1}}{3} \rfloor$ pebbles on the vertex b_{n-1} and place one pebble at x. Then, we cannot move t pebbles to v. Thus, $f_t(C_{4k+1}^2) \geq (t-1)2^k + \lceil \frac{2^{k+2}}{3} \rceil + 1$. Now, consider the distribution of $(t-1)2^k + \lceil \frac{2^{k+2}}{3} \rceil + 1$ pebbles on the vertices of C_{4k+1}^2. Clearly, the result is true for $t=1$ by Theorem 8. We assume the result is true for $1 \leq t' < t$. Without loss of generality, we assume that $p(P_C^2) \geq t2^{k-1}+1 \geq 2^k+1$. Clearly, we can move one pebble to v at a cost of at most 2^k pebbles, since $P_C^2 \cong P_{2k+1}^2$ and $f(P_{2k+1}^2) = 2^k+1$. Then we have at least $(t-2)2^k + \lceil \frac{2^{k+2}}{3} \rceil + 1$ pebbles remaining on the vertices of C_{4k+1}^2 and hence we can move the additional $t-1$ pebbles to v by induction. Thus $f_t(C_{4k+1}^2) \leq (t-1)2^k + \lceil \frac{2^{k+2}}{3} \rceil + 1$. $\square$

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$G = C_n^2$	$f(G)$	$f_t(G)$ $(t \geq 2)$
For $n = 4$	4	$2t + 2$
For $n = 6$	6	$4t$
For $n = 8$	8	$4t + 2$
For $n = 10$	10	$8t$
For $n = 12$	12	$8t + 3$
For $n = 4k + 2$ $(k \geq 3)$	2^{k+1}	$t(2^{k+1})$
For $n = 4k$ $(k \geq 4)$	$2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$	$(t-1)2^k + 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$

Table 1. The t-pebbling numbers of squares of even cycles

$G = C_n^2$	$f(G)$	$f_t(G)$ $(t \geq 2)$
For $n = 5$	5	$2t + 3$
For $n = 7$	7	$4t + 1$
For $n = 9$	9	$4t + 3$
For $n = 11$	11	$8t + 1$
For $n = 13$	13	$8t + 4$
For $n = 4k + 3$ $(k \geq 3)$	$2^{k+1} + 1$	$t2^{k+1} + 1$
For $n = 4k + 1$ $(k \geq 4)$	$\lceil \frac{2^{k+2}}{3} \rceil + 1$	$(t-1)2^k + \lceil \frac{2^{k+2}}{3} \rceil + 1$

Table 2. The t-pebbling numbers of squares of odd cycles

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