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(λ, μ)-FUZZY IDEALS IN TERNARY SEMIRINGS

T. ANITHA[1], D. KRISHNASWAMY[2]

Abstract. In this paper we introduce the notion of (λ, μ)-Fuzzy ternary subsemirings and (λ, μ)-Fuzzy ideals in ternary semirings which can be regarded as the generalization of fuzzy ternary subsemirings and fuzzy ideals in ternary semirings.

Key words : (λ, μ)-Fuzzy ternary subsemirings, (λ, μ)-Fuzzy ideals, (λ, μ)-characteristic function.
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1. Introduction

The notion of ternary algebraic system was introduced by Lehmer [2] in 1932. He investigated certain ternary algebraic systems called triplexes. In 1971, Lister [15] characterized additive semigroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. Dutta and Kar [8] introduced a notion of ternary semirings which is a generalization of ternary rings and semirings, and they studied some properties of ternary semirings [8, 9, 10, 11, 12, 13, 14, 7]. The theory of fuzzy sets was first studied by Zadeh [5] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory, etc. Kavikumar et. al.[3] and [4] studied fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals in ternary semirings. Ronnason Chinram et. al. [6] studied L-fuzzy ideals in ternary semirings. In this paper we introduce the notion of (λ, μ)-Fuzzy ternary subsemirings and (λ, μ)-Fuzzy ideals in ternary semirings which can be regarded as the generalization of fuzzy ternary subsemirings and fuzzy ideals in ternary semirings.

[1]Mathematics Wing, DDE, Annamalai University, Annamalainagar 608 002, India. Email: anitha81t@gmail.com
[2]Department of Mathematics, Annamalai University, Annamalainagar 608 002, India. Email: krishna_swamy2004@yahoo.co.in.

2. Preliminaries

In this section, we refer to some elementary aspects of the theory of semirings and ternary semirings and fuzzy algebraic systems that are necessary for this paper.

Definition 1. *A nonempty set S together with two binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) is called a semiring if $(S, +)$ is a commutative semigroup, $(S, \cdot)$ is a semigroup and multiplicative distributes over addition both from the left and the right, i.e., $a(b+c) = ab+ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in S$.*

Definition 2. *A nonempty set S together with a binary operation called, addition $+$ and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if $(S, +)$ is a commutative semigroup satisfying the following conditions:*
(i) $(abc)de = a(bcd)e = ab(cde)$,
(ii) $(a + b)cd = acd + bcd$,
(iii) $a(b + c)d = abd + acd$
and (iv) $ab(c + d) = abc + abd$ for all $a, b, c, d, e \in S$.

We can see that any semiring can be reduced to a ternary semiring. However, a ternary semiring does not necessarily reduce to a semiring by this example. We consider Z_0^-, the set of all non-positive integers under usual addition and multiplication, we see that Z_0^- is a commutative semigroup which is closed under the triple multiplication but is not closed under the binary multiplication. Moreover, Z_0^- is a ternary semiring but is not a semiring under usual addition and multiplication.
Throughout this paper S denotes a ternary semiring with zero.

Definition 3. *Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x = x + 0$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then 0 is called the zero element or simply the zero of the ternary semiring S. In this case we say that S is a ternary semiring with zero.*

Definition 4. *An additive subsemigroup T of S is called a ternary subsemiring of S if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.*

Definition 5. *An additive subsemigroup I of S is called a left [resp. right, lateral] ideal of S if $s_1s_2i \in I$ [resp. $is_1s_2 \in I, s_1is_2 \in I$] for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, right and lateral ideal of S, then I is called an ideal of S.*

It is obvious that every ideal of a ternary semiring with zero contains the zero element.

Definition 6. *Let S_1 and S_2 be ternary semirings. A mapping $f : S_1 \to S_2$ is said to be a homomorphism if $f(x + y) = f(x) + f(y)$ and $f(xyz) = f(x)f(y)f(z)$ for all $x, y, z \in S_1$.*

Let $f : S_1 \to S_2$ be an onto homomorphism of ternary semirings. Note that if I is an ideal of S_1, then $f(I)$ is an ideal of S_2. If S_1 and S_2 be ternary semirings with zero 0, then $f(0) = 0$.
Let X be a non-empty set. A map $A : X \to [0, 1]$ is called a fuzzy set in X.

Definition 7. *Let A, B and C be any three fuzzy subsets of a ternary semiring S. Then $A \cap B$, $A \cup B$, $A + B$, $A \cdot B \cdot C$ are fuzzy subsets of S defined by*

$$(A \cap B)(x) = min\{A(x),\ B(x)\}$$
$$(A \cup B)(x) = max\{A(x),\ B(x)\}$$
$$(A + B)(x) = \begin{cases} sup\{min\{A(y), B(z)\}\}\ if\ x = y + z \\ 0\ otherwise \end{cases}$$
$$(A \cdot B \cdot C)(x) = \begin{cases} sup\{min\{A(u), B(v), C(w)\}\}\ if\ x = uvw, \\ 0\ otherwise \end{cases}$$

Definition 8. *Let $a, b \in [0, 1]$ then $a \wedge b = min\{a, b\}$ and $a \vee b = max\{a, b\}$.*

Definition 9. *Let X be a nonempty set and let A be a fuzzy subset of X. Let $0 \leq t \leq 1$. Then the set $A_t = \{x \in X\ /\ A(x) \geq t\}$ is called a level set of X with respect to A.*

Definition 10. *Let A be a fuzzy set of a ternary semiring S. Then A is called a fuzzy ternary subsemiring of S if*
1. $A(x + y) \geq min\{A(x), A(y)\}$
2. $A(xyz) \geq min\{A(x), A(y), A(z)\}$ for all $x, y, z \in S$.

Definition 11. *A fuzzy set A of a ternary semiring S is called a fuzzy ideal of S if*
(i) $A(x + y) \geq min\{A(x), A(y)\}$
(ii) $A(xyz) \geq A(x)$
(iii) $A(xyz) \geq A(z)$ and
(iv) $A(xyz) \geq A(y)$ for all $x, y, z \in S$.
A fuzzy subset A with conditions (i) and (ii) is called a fuzzy right ideal of S. If A satisfies (i)and(iii), then it is called a fuzzy left ideal of S. Also if A satisfies (i)and(iv), then it is called a fuzzy lateral ideal of S. It is clear that A is a fuzzy ideal of a ternary semiring S if and only if $A(xyz) \geq max\{A(x), A(y), A(z)\}$ for all $x, y, z \in S$.

3. (λ, μ)- Fuzzy ideals

Based on the concept of (λ, μ)-fuzzy subrings and (λ, μ)-fuzzy ideals introduced by B.Yao [1], we introduce the following concepts which are the generalization of fuzzy sets. Throughout this paper λ and μ $(0 \leq \lambda < \mu \leq 1)$, are arbitrary, but fixed. In this section we introduce the notion of (λ, μ)-fuzzy ideals in ternary semirings.

Definition 12. *Let A be a fuzzy set of S. Then A is called a (λ, μ)-fuzzy ternary subsemiring of S if*
1. $A(x+y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$
2. $A(xyz) \vee \lambda \geq A(x) \wedge A(y) \wedge A(z) \wedge \mu$ for all $x, y, z \in S$.

Theorem 1. *Let A be a fuzzy set of a ternary semiring S. Then A is a (λ, μ)-fuzzy ternary subsemiring of S if and only if A_t is a ternary subsemiring of S for all $t \in (\lambda, \mu]$ whenever non-empty.*

Proof. Let A be a (λ, μ)-fuzzy ternary subsemiring of S and $t \in (\lambda, \mu]$. Let $x, y, z \in A_t$ then $A(x) \geq t,\ A(y) \geq t,\ A(z) \geq t$. Now $A(x+y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu \geq t$. Thus $A(x+y) \geq t$. Hence $x+y \in A_t$. Similarly $xyz \in A_t$. Hence A_t is a ternary subsemiring of S.
Conversely, let A_t be a ternary subsemiring of S for all $t \in (\lambda, \mu]$. If there exist $x, y, z \in S$ such that $A(x+y) \vee \lambda < t = A(x) \wedge A(y) \wedge \mu$ then $t \in (\lambda, \mu]$ and $x, y \in A_t$ with $x + y \notin A_t$ this contradicts to that A_t is a ternary subsemiring. Hence $A(x+y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$. Similarly we have $A(xyz) \vee \lambda \geq A(x) \wedge A(y) \wedge A(z) \wedge \mu$. Therefore A is a (λ, μ)-fuzzy ternary subsemiring of S. □

Remark 1. *Every fuzzy ternary subsemiring is a (λ, μ)-fuzzy ternary subsemiring by taking $\lambda = 0$ and $\mu = 1$. But the converse need not be true.*

Example 1. *Consider the set of integer modulo 6, non-positive integer $Z_6^- = \{0, -1, -2, -3, -4, -5\}$ with the addition modulo 6 and ternary multiplication modulo 6. Then $(Z_6^-, \oplus_6, \odot_6)$ is a ternary semiring. Let a fuzzy subset $A : Z_6^- \rightarrow [0, 1]$ be defined by $A(0) = 0.9,\ A(-1) = 0,\ A(-2) = 0.9,\ A(-3) = 0,\ A(-4) = 0.8$ and $A(-5) = 0$. Clearly, A is a $(\lambda = 0.3,\ \mu = 0.8)$-fuzzy ternary subsemiring. But A is not a fuzzy ternary subsemiring, since $A(-4 = -2 + (-2)) <\ A(-2) \wedge A(-2)$.*

Definition 13. *Let A be a fuzzy set of a ternary semiring S. A is called a (λ, μ)-fuzzy right (resp. left, lateral) ideal of S if*
1. $A(x+y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$
2. $A(xyz) \vee \lambda \geq A(x) \wedge \mu$ (resp. $A(xyz) \vee \lambda \geq A(z) \wedge \mu$, $A(xyz) \vee \lambda \geq A(y) \wedge \mu$) for all $x, y, z \in S$.

Definition 14. *Let A be a fuzzy set of a ternary semiring S. A is called a (λ, μ)-fuzzy ideal of S if*
1. $A(x+y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$
2. $A(xyz) \vee \lambda \geq [A(x) \vee A(y) \vee A(z)] \wedge \mu$ for all $x, y, z \in S$.

Theorem 2. *Let A be a fuzzy set of a ternary semiring S. Then A is a (λ, μ)-fuzzy right (resp. left, lateral) ideal of S if and only if A_t is a right (resp. left, lateral) ideal of S for all $t \in (\lambda, \mu]$ whenever nonempty.*

Proof. Let A be a (λ, μ)-fuzzy right ideal of S and $t \in (\lambda, \mu]$. By Theorem 1, A_t is a ternary subsemiring. Let $x \in A_t$ and $y, z \in S$. Then $A(xyz) \vee \lambda \geq A(x) \wedge \mu \geq t$. Thus $A(xyz) \geq t$, then $xyz \in A_t$. Hence A_t is a right ideal of S. Conversely, let A_t be a right ideal for all $t \in (\lambda, \mu]$. By Theorem 1, A is a (λ, μ)-fuzzy ternary subsemiring. If there exist $x, y, z \in S$ such that $A(xyz) \vee \lambda < t = A(x) \wedge \mu$, then $x \in A_t$ and $y, z \in S$ with $xyz \notin A_t$. This contradicts that A_t is a right ideal. Hence $A(xyz) \vee \lambda \geq A(x) \wedge \mu$. Therefore A is a (λ, μ)-fuzzy right ideal of S. □

Remark 2. *Every fuzzy right (resp. left, lateral) ideal is a (λ, μ)-fuzzy right (resp. left, lateral) ideal by taking $\lambda = 0$ and $\mu = 1$. But the converse need not be true.*

Example 2. *Let S be a ternary semiring consists of non-positive integers. Let*

$$A(x) = \begin{cases} 0.9 \ if \ x = -4 \\ 0.8 \ if \ x \in \langle -2 \rangle + \langle -5 \rangle \ \textit{and} \ x \neq -4 \\ 0.3 \ if \ x \notin \langle -2 \rangle + \langle -5 \rangle \ \textit{and} \ x \neq -3 \\ 0.2 \ if \ x = -3 \end{cases}$$

Clearly A is a $(0.3, 0.8)$-fuzzy right ideal. But A is not a fuzzy right ideal, since $A(-64 = -4 \cdot -4 \cdot -4) < A(-4)$.

Definition 15. *Let I be any subset of a ternary semiring S. The (λ, μ)-characteristic function of I denoted by χ_I^* is defined as*

$$\chi_I^* = \begin{cases} \mu \ if \ x \in I \\ \lambda \ otherwise \end{cases}$$

Theorem 3. *χ_I^* is a (λ, μ)-fuzzy ideal of S if and only if I is an ideal of S.*

Proof. Let χ_I^* be a (λ, μ)-fuzzy ideal of a ternary semiring S. Let $x, y \in I$. Then $\chi_I^*(x+y) \vee \lambda \geq \chi_I^*(x) \wedge \chi_I^*(y) \wedge \mu = \mu$ which implies $\chi_I^*(x+y) \geq \mu$. Thus $x + y \in I$. Similarly if $x \in I$; $y, z \in S$ we have $xyz,\ yzx,\ yxz \in I$. Therefore I is an ideal of S.

Conversely, if there exist $x, y \in S$ such that $\chi_I^*(x+y) \vee \lambda < \mu = \chi_I^*(x) \wedge \chi_I^*(y) \wedge \mu$ then $\chi_I^*(x+y) < \mu,\ \chi_I^*(x) = \mu,\ \chi_I^*(y) = \mu$. Thus $x, y \in I$ and $x + y \notin I$ which is a contradiction. Thus $\chi_I^*(x+y) \vee \lambda \geq \chi_I^*(x) \wedge \chi_I^*(y) \wedge \mu$. Similarly we prove that $\chi_I^*(xyz) \vee \lambda \geq [\chi_I^*(x) \vee \chi_I^*(y) \vee \chi_I^*(z)] \wedge \mu$. Therefore χ_I^* is a (λ, μ)-fuzzy ideal of S. □

Theorem 4. *Let $f : S_1 \to S_2$ be an epimorphism of ternary semirings. Let B be a fuzzy set of S_2. Then B is a (λ, μ)-fuzzy ideal of S_2 if and only if $f^{-1}(B)$ is a (λ, μ)-fuzzy ideal of S_1 where $[f^{-1}(B)](x) = B(f(x))$ for all $x \in S_1$.*

Proof. Let B be a (λ, μ)-fuzzy ideal of S_2. Let $x_1, x_2, x_3 \in S_1$.
Now $f^{-1}(B)(x_1 + x_2) \vee \lambda = B(f(x_1 + x_2)) \vee \lambda = B(f(x_1) + f(x_2)) \vee \lambda \geq B(f(x_1)) \wedge B(f(x_2)) \wedge \mu = f^{-1}(B)(x_1) \wedge f^{-1}(B)(x_2) \wedge \mu$.
Now $f^{-1}(B)(x_1x_2x_3) \vee \lambda = B(f(x_1x_2x_3)) \vee \lambda = B(f(x_1)f(x_2)f(x_3)) \vee \lambda \geq [B(f(x_1)) \vee B(f(x_2)) \vee B(f(x_3))] \wedge \mu = [f^{-1}(B)(x_1) \vee f^{-1}(B)(x_2) \vee f^{-1}(B)(x_3)] \wedge \mu$. Thus $f^{-1}(B)$ is a (λ, μ)-fuzzy ideal of S_1. Conversely let $f^{-1}(B)$ is a (λ, μ)-fuzzy ideal of S_1. Let $y_1, y_2, y_3 \in S_2$ such that $f(x_1) = y_1,\ f(x_2) = y_2$ and $f(x_3) = y_3$, where $x_1, x_2, x_3 \in S_1$. $B(y_1 + y_2) \vee \lambda = B(f(x_1) + f(x_2)) \vee \lambda = B(f(x_1 + x_2)) \vee \lambda = f^{-1}(B)(x_1 + x_2) \vee \lambda \geq f^{-1}(B)(x_1) \wedge f^{-1}(B)(x_2) \wedge \mu = B(f(x_1)) \wedge B(f(x_2)) \wedge \mu = B(y_1) \wedge B(y_2) \wedge \mu$. Similarly we have $B(y_1y_2y_3) \vee \lambda \geq [B(y_1) \vee B(y_2) \vee B(y_3)] \wedge \mu$. Thus B is a (λ, μ)-fuzzy ideal of S_2. □

Theorem 5. *Let $f : S_1 \to S_2$ be an epimorphism of ternary semirings and let A be a fuzzy set of S_1. If A is a (λ, μ)-fuzzy ideal of S_1 then $f(A)$ is a (λ, μ)-fuzzy ideal of S_2 where $f(A)(y) = sup\{A(x)/f(x) = y\}$.*

Proof. Let us assume that A is a (λ, μ)-fuzzy ideal of S_1. Let $y_1, y_2, y_3 \in S_2$ then there exist $x_1, x_2, x_3 \in S_1$ such that $f(x_1) = y_1,\ f(x_2) = y_2$ and $f(x_3) = y_3$. Now $f(A)(y_1 + y_2) \vee \lambda = \sup\limits_{x \in S_1}\{A(x)/f(x) = y_1 + y_2\} \vee \lambda$

$$\geq \sup_{x_1, x_2 \in S_1}\{A(x_1 + x_2)/f(x_1 + x_2) = y_1 + y_2\} \vee \lambda$$
$$= \sup_{x_1, x_2 \in S_1}\{A(x_1 + x_2) \vee \lambda/f(x_1) + f(x_2) = y_1 + y_2\}$$
$$\geq \sup\{A(x_1) \wedge A(x_2) \wedge \mu/f(x_1) = y_1,\ f(x_2) = y_2\}$$
$$= \sup\{A(x_1)/f(x_1) = y_1\} \wedge \sup\{A(x_2)/f(x_2) = y_2\} \wedge \mu$$
$$= f(A)(y_1) \wedge f(A)(y_2) \wedge \mu.$$

Now $f(A)(y_1y_2y_3) \vee \lambda \geq \sup\{A(x_1x_2x_3)/f(x_1x_2x_3) = y_1y_2y_3\} \vee \lambda$
$$= \sup\{A(x_1x_2x_3) \vee \lambda/f(x_1)f(x_2)f(x_3) = y_1y_2y_3\}$$
$$\geq \sup\{[A(x_1) \vee A(x_2) \vee A(x_3)] \wedge \mu/f(x_1) = y_1,\ f(x_2) = y_2,\ f(x_3) = y_3\}$$
$$= \{[\sup\{A(x_1)/f(x_1) = y_1\}] \vee [\sup\{A(x_2)/f(x_2) = y_2\}] \vee [\sup\{A(x_3)/f(x_3) = y_3\}]\} \wedge \mu$$
$$= [f(A)(y_1) \vee f(A)(y_2) \vee f(A)(y_3)] \wedge \mu.$$

Thus $f(A)$ is a (λ, μ)-fuzzy ideal of S_2. □

The following example shows that converse of the above theorem need not be true.

Example 3. *Let Z_0^- and Z_6^- be the ternary semirings of negative integers and negative integer modulo 6 respectively. The mapping f defined by $f : Z_0^- \to Z_6^-$, $f(x) = k$ (mod 6) where $k \equiv x$(mod 6), $-5 \leq k \leq 0$ is a homomorphism* *and* f *is onto. Let*

$$A(x) = \begin{cases} 0.8 \; if \; x = -18 \\ 0.4 \; if \; x \in \langle -9 \rangle \; \; and \; x \neq -18 \\ 0.7 \; if \; x \in \langle -3 \rangle \; \; and \; x \notin \langle -9 \rangle \\ 0.2 \; otherwise \end{cases}$$

Then

$$f(A)(x) = \begin{cases} 0.8 \; if \; x = 0 \\ 0.7 \; if \; x = -3 \\ 0.2 \; otherwise \end{cases}$$

Clearly, $f(A)$ *is a* (0.3, 0.8)*-fuzzy ideal of* Z_6^- *but* A *is not a* (0.3, 0.8)*-fuzzy ideal of* Z_0^-*, since* $A(-18 + (-18)) \vee 0.3 < A(-18) \wedge A(-18) \wedge 0.8$.

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