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A STUDY OF THE FLOW OF NON-NEWTONIAN FLUID BETWEEN HEATED PARALLEL PLATES BY HAM

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Abstract. This paper presents the heat transfer of a third grade fluid between two heated parallel plates for two models: constant viscosity model and Reynold's model. In both cases the nonlinear energy and momentum equations have been solved by HAM. The graphs for the velocity and temperature profiles are plotted and discussed for various values of the emerging parameters in the problem. The main effect is governed by whether or not the fluid is non-Newtonian and the temperature effects are being referred to have a less dominant role.

Key words : Non-Newtonian fluid, comparison of perturbation methods, homotopy analysis method, homotopy perturbation method.
AMS SUBJECT : Primary 74G10 , 74H10 , 81Q15.

1. Introduction

In general, the flowing mixtures consist of solid particles in a fluid such as coal based slurries which exhibit non-Newtonian characteristics. These mixtures are important in a variety of industrial applications and heat transfer plays an important role in handling and processing of these mixtures. There are properties of fluid behavior which cannot be explained on the basis of the classical, linearly viscous models. Several constitutive equations have been suggested to characterize such non-Newtonian behaviors. Among those are; the fluids of the different types of grade n [1] and the incompressible and homogeneous fluids of grade 1 being the linearly viscous Newtonian fluids. For example, it has been shown that the substantial performance benefits can be obtained if coal-water mixture is pre-heated [1,2].

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In this paper, we consider the model described by Szeri and Rajagopal in 1985 [2] for heat transfer flow of a third grade fluid between two parallel plates maintained at different temperatures located at $y = 0$ and $y = h$ planes respectively, of an orthogonal Cartesian coordinate system. Szeri and Rajagopal [2] examined this model for the two cases via a similarity transformation and concluded that the temperature dependence was not important for third grade fluid for the considered parameters and the variable viscosity solutions were not too distinct from that of constant viscosity. With a slight modification, we consider this model with HAM [9, 10] and compare our results with those produced by PEM and HPM in [1] and with those produced by Szeri and Rajagopal in [2]. In each case, the nonlinear momentum and energy equations have been solved using HAM [9, 10]. The graphs for the velocity and temperature profiles are presented and discussed for the various values of parameters appearing in the problem.

There have been several studies involving heat and mass transfer in the non-Newtonian fluids, but most of them seem to lack a systematic and rational treatment of thermodynamics of the problem; while the stress constitutive equation is altered to account for non-Newtonian behavior. The constitutive equations for the specific Helmholtz free energy or the heat flux vector are left unchanged. Although this might be correct for a particular fluid yet it does not seem proper to assume the same a priori.

In this paper, we attempt a thermodynamically consistent study of the heat transfer problem under consideration. We consider two models in our approach: constant viscosity model and Reynold's model [1, 2]. The governing differential equations for the velocity and temperature are non-linear whose exact solutions are not available. Therefore, asymptotic methods prove a powerful tool to obtain approximate solutions of these equations.

Among various asymptotic methods, the homotopy perturbation method provided by He in [7] is a coupling of the traditional perturbation methods and the homotopy concept used in the topology. He in [7] and others [3, 5, 8] not only applied this method successfully to obtain the solution of currently important problems in science and technology, but also have shown its effectiveness and reliability. In this paper, we have used HAM [9, 10] proposed by Liao to solve the problem under discussion. The HAM has some advantages over the traditional perturbation techniques, such as freedom of choice of initial guess, linear operator, auxiliary parameter and the auxiliary function. We have also given a comparison of different perturbation techniques with illustrative examples in [6].

2. Analysis of the Problem

We start our analysis of the problem by refereing to the papers [1, 2] and consider the following two models without further assumptions and transformations.

2.1. **The Constant Viscosity Model.** In this case, we have two nonlinear equations as [1]:

$$\frac{d^2w}{dx^2}\left[1+6\Lambda\left(\frac{dw}{dx}\right)^2\right]=-2, \tag{1}$$

along with boundary condition, $w\left(0\right)=0$, and $w\left(1\right)=0$.

$$\frac{d^2v}{dx^2}+\lambda\left(\frac{dw}{dx}\right)^2\left[\Omega+2\Lambda\left(\frac{dw}{dx}\right)^2\right]=0, \tag{2}$$

along with the boundary conditions $v\left(0\right)=0$, and $v\left(1\right)=1$.

We expand w in powers of parameter η, such as

$$\aleph_1\left(x;\eta\right)=w_0+\eta w_1+\eta^2 w_2+\ldots=w\left(x\right). \tag{3}$$

To find the initial guess we put $\Lambda=0$ in equation (1), and comparing the co-efficients of $\eta^0,\eta^1,\ldots$, we get,

$$\eta^0: w_0^{''}=-2, \tag{4}$$

$$\eta^1: w_1^{''}=0, \tag{5}$$

and so on. The expression (3) depending upon the parameter $\eta\in[0,1]$ is used for comparison reason to get the linear equations. Here $\eta\in[0,1]$ indicates the embedding parameter. The equation (4) has the solution using the given conditions $w_0\left(0\right)=0$, and $w_0\left(1\right)=0$, and becomes the initial guess/initial approximation for our analysis as,

$$w_0\left(x\right)=x-x^2, \tag{6}$$

The homotopy analysis method (HAM) is based on a continuous mapping $w\left(x\right)\rightarrow\aleph_1\left(x;\eta\right)$ such that when the embedding parameter η varies from 0 to 1, $\aleph_1\left(x;\eta\right)$ varies from the initial guess $w_0\left(x\right)$ to the exact solution $w\left(x\right)$.
The nonlinear operator is defined as,

$$N\left[\aleph_1\left(x;\eta\right)\right]=\frac{\partial^2\aleph_1\left(x;\eta\right)}{\partial x^2}+6\Lambda\left[\frac{\partial^2\aleph_1\left(x;\eta\right)}{\partial x^2}\right]\left[\frac{\partial\aleph_1\left(x;\eta\right)}{\partial x}\right]^2+2=0. \tag{7}$$

Let $H\neq 0$ and $\tau\neq 0$, stands for so called auxiliary function and auxiliary parameter respectively, so that as we use the embedding parameter $\eta\in[0;1]$, we get a series of equations as,

$$\left(1-\eta\right)L\left[\aleph_1\left(x;\eta\right)-w_0\left(x\right)\right]=\tau\eta HN\left[\aleph_1\left(x;\eta\right)\right], \tag{8}$$

with the prescribed boundary conditions. It should be noted that we have a number of choices for the auxiliary function H, the auxiliary parameter τ, the initial guess $w_0(x)$ and the auxiliary linear operator L. It is such an independence to play the most important role and it establishes the cornerstone of the validity and flexibility of the homotopy analysis method. For our convenience, we shall take the value of the auxiliary function $H = 1$ and auxiliary parameter $\tau = -1$. The rules of expression for these quantities are discussed in details in [9]. When $\eta = 0$, then equation (8) transformed to,

$$L\left[\aleph_1(x;0) - w_0(x)\right] = 0, \tag{9}$$

subject to $\aleph_1(0;0) = 0$. Thus, the equation (9) is

$$\aleph_1(x;0) = w_0(x). \tag{10}$$

When $\eta = 1$; equation (8) becomes,

$$\tau HN\left[\aleph_1(x;1)\right] = 0, \tag{11}$$

with $\aleph_1(0;1) = 0$. Since $H \neq 0$, and $\tau \neq 0$ and by definition, equation (7) and equation (11) are corresponding to the original equation (1) and (2), provided that

$$\aleph_1(x;1) = w(x). \tag{12}$$

Therefore, according to equation (10) and (12), $\aleph_1(x;\eta)$ varies from the initial guess $w_0(x)$ to the exact solution $w(x)$ as the embedding parameter η increases from 0 to 1. The equation (8) is called the zero-order deformation equation having independence to choose the auxiliary function H, the auxiliary parameter τ, the initial approximation $w_0(x)$, and the auxiliary linear operator$L$. Now it is assume that all of them are properly chosen so that the solution $\aleph_1(x;\eta)$ of the zero-order deformation equation (8) exists for $0 \leq \eta \leq 1$. We present the solution of the problem in the form as

$$\aleph_1(x;\eta) = w_0(x) + \sum_{m=1}^{\infty} w_m(x)\,\eta^m. \tag{13}$$

Now, we suppose that the auxiliary function H, the auxiliary parameter τ, the initial linear operator L, and the initial approximation $w_0(x)$ are correctly chosen that series (13) converges at $\eta = 1$, then

$$\aleph_1(x;1) = w_0(x) + \sum_{m=1}^{\infty} w_m(x). \tag{14}$$

The above equation gives us a relationship between the initial guess $w_0(x)$ and the exact solution $w(x)$ by means of the terms $w_m(x)$ for $m = 1, 2, 3, \ldots,$ which are unknown up to now.

2.1.1. *High –Order deformation equation:-.* Define the vector

$$\overrightarrow{w_n} = \{w_0, w_1, w_2, \ldots, w_n\},$$

then from the given definition (14), the main equation and related initial condition of $w_m(x)$ can be deduced from the zero-order deformation equation (8). Differentiating the equation (8) m-times with respect to the embedding parameter η, putting $\eta = 1$, and at last dividing by $m!$, we get the so called mth order deformation equation as,

$$L\left[w_m(x) - \sigma_m w_{m-1}(x)\right] = \tau H \Re_m\left(\overrightarrow{w}_{m-1}\right), \qquad (15)$$

where

$$\Re_m\left(\overrightarrow{w}_{m-1}\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N\left[\aleph_1(x;\eta)\right]}{\partial \eta^{m-1}}\bigg|_{\eta=0}. \qquad (16)$$

From the above methodology and using equation (7), we get

$$\begin{aligned} \Re_1\left(\overrightarrow{w}_0\right) &= \left[\frac{\partial^2\left(w_0 + \eta w_1 + \ldots.\right)}{\partial x^2}\right]\bigg|_{\eta=0} \\ &+ \left[6\Lambda\left(\frac{\partial^2\left(w_0 + \eta w_1 + \ldots\right)}{\partial x^2}\right)\left(\frac{\partial\left(w_0 + \eta w_1 + \ldots\right)}{\partial x}\right)^2 + 2\right]\bigg|_{\eta=0} = 0, \end{aligned}$$

and from equation (6), $w_0(x) = x - x^2$. Since

$$\Re_1\left(\overrightarrow{w}_0\right) = -12\Lambda\left(1 + 4x^2 - 4x\right), \qquad (17)$$

then using equation (17) for equation (15) along with the conditions $w_1(0) = 0$, and $w_1(1) = 0$, we get

$$w(x) = x - x^2 + \Lambda\left(8\tau x^3 - 4\tau x^4 - 6\tau x^2 + 2\tau x\right) + \ldots \qquad (18)$$

Now, we come to the other nonlinear equations:

$$\frac{d^2 v}{dx^2} + \Omega\left(\frac{dw}{dx}\right)^2\left[\Omega + 2\Lambda\left(\frac{dw}{dx}\right)^2\right] = 0, \qquad (19)$$

with boundary conditions $v(0) = 0$, and $v(1) = 1$. We expands v in powers of parameter γ, such that

$$\aleph_2(x;\gamma) = v_0 + \gamma v_1 + \gamma^2 v_2 + \ldots = v(x). \qquad (20)$$

To find the initial guess, we put $\Omega = 0$, and $\Lambda = 0$ in equation (19) and get, $v''(x) = 0$, which has the solution, after using the given conditions,

$$v_0(x) = x. \qquad (21)$$

Let $\gamma \in [0,1]$ indicates the embedding parameter. The homotopy analysis method is based on continuous mapping $v(x) \to \aleph_2(x;\gamma)$ such that when the embedding parameter γ increases from 0 to 1, $\aleph_2(x;\gamma)$ varies from the initial guess $v_0(x)$ to the exact solution $v(x)$. The expression (20) is dependent upon the parameter $\gamma \in [0,1]$ is used for comparison reason to get the linear equations. The nonlinear operator from equation (19) is define as,

$$N[\aleph_2(x;\gamma)] = \frac{\partial^2 \aleph_2(x;\gamma)}{\partial x^2} + \Omega\left(\frac{\partial \aleph_1(x;\gamma)}{\partial x}\right)^2 \left[\Omega + 2\Lambda\left(\frac{\partial \aleph_1(x;\gamma)}{\partial x}\right)^2\right] = 0. \tag{22}$$

Let $H \neq 0$ and $\tau \neq 0$, indicate the so called auxiliary function and auxiliary parameter respectively. We use the embedding parameter $\gamma \in [0,1]$, so that we create a number of linear equations as,

$$(1-\gamma)L[\aleph_2(x;\gamma) - v_0(x)] = \tau\gamma H N[\aleph_2(x;\gamma)]. \tag{23}$$

When $\gamma = 0$, then equation (23) becomes,

$$L[\aleph_2(x;0) - v_0(x)] = 0. \tag{24}$$

When $\gamma = 1$; the equation (23) becomes;

$$\tau H N[\aleph_2(x;1)] = 0, \tag{25}$$

Since $H \neq 0$ and $\tau \neq 0$, we have $N[\aleph_2(x;1)]$. We express $N[\aleph_2(x;\gamma)]$ as,

$$\aleph_2(x;\gamma) = v_0(x) + \sum_{m=1}^{\infty} v_m(x)\gamma^m. \tag{26}$$

We also suppose that the auxiliary function H , the auxiliary parameter τ, the initial approximation $v_0(x)$, and the auxiliary linear operator L are correctly choose so that the above expression converges at $\gamma = 1$,

$$\aleph_2(x;1) = v_0(x) + \sum_{m=1}^{\infty} v_m(x), \tag{27}$$

2.1.2. *High Order deformation equation.* Define the vector

$$\overrightarrow{v_n} = \{v_0(x), v_1(x), v_2(x), \ldots v_n(x)\},$$

and we have

$$L[v_m(x) - \sigma_m v_{m-1}(x)] = \tau H \Re_m(\overrightarrow{v}_{m-1}),$$

then

$$v_m(x) = \sigma_m v_{m-1} + \tau H L^{-1} \Re_m(\overrightarrow{v}_{m-1}). \tag{28}$$

When $m = 1$, then equation (28) becomes

$$v_1(x) = v_0 + \tau H L^{-1} \Re_1(\overrightarrow{v}_0), \tag{29}$$

and by definition we have,

$$N[\aleph_2(x;\gamma)] = \frac{\partial^2 \aleph_2(x;\gamma)}{\partial x^2} + \Omega\left(\frac{\partial \aleph_1(x;\gamma)}{\partial x}\right)^2 \left[\Omega + 2\Lambda\left(\frac{\partial \aleph_1(x;\gamma)}{\partial x}\right)^2\right] = 0.$$

Also from equation (6), we get

$$\Re_1\left(\overrightarrow{v_0}\right) = \Omega\left(1 + 4x^2 - 4x\right) + 2\Lambda\left(1 - 8x + 24x^2 - 32x^3 + 16x^4\right). \qquad (30)$$

Putting the value of $\Re_1\left(\overrightarrow{v_0}\right)$ in equation (29) we have,

$$v_1(x) = \tau H L^{-1}\left[\Omega + 4\Omega x^2 - 4\Omega x + 2\Lambda - 16\Lambda x + 48\Lambda x^2 - 64\Lambda x^3 + 32\Lambda x^4\right],$$

and using the given conditions,

$$\begin{aligned} v_1(x) &= H\left[\Omega\left(\frac{\tau x^2}{2} + \frac{\tau x^4}{3} - \frac{2\tau x^3}{3} - \frac{\tau x}{6}\right)\right] \qquad (31) \\ &+ \left[\Lambda\left(\tau x^2 - \frac{8\tau x^3}{3} + 4\tau x^4 - \frac{16\tau x^5}{5} + \frac{16\tau x^6}{15} - \frac{\tau x}{5}\right)\right]. \end{aligned}$$

Then equation (27) becomes

$$\begin{aligned} v(x) &= x + H\left[\Omega\left(\frac{\tau x^2}{2} + \frac{\tau x^4}{3} - \frac{2\tau x^3}{3} - \frac{\tau x}{6}\right)\right] \qquad (32) \\ &+ H\left[\Lambda\left(\tau x^2 - \frac{8\tau x^3}{3} + 4\tau x^4 - \frac{16\tau x^5}{5} + \frac{16\tau x^6}{15} - \frac{\tau x}{5}\right)\right] + \ldots, \end{aligned}$$

2.2. **Reynold's Model.** In this case we have two nonlinear equations as

$$\frac{d^2v}{dx^2} + \Omega\left(\frac{dw}{dx}\right)^2\left[\exp(-Mv) + 2\Lambda\left(\frac{dw}{dx}\right)^4\right] = 0, \qquad (33)$$

with the boundary condition $v(0) = 0$, and $v(1) = 1$.

$$\frac{d^2w}{dx^2} + 6\Lambda\left(\frac{dw}{dx}\right)^2 e^{Mv} - Mv\left(\frac{dw}{dx}\right) + 2e^{-Mv} = 0, \qquad (34)$$

with boundary condition $w(0) = 0$, and $w(1) = 0$. Following the same ideal of HAM, we find the initial guess and put $M = 0$, $\Omega = 0$, and $\Lambda = 0$, in equation (33) so that we get,

$$v_0(x) = x, \qquad (35)$$

with the boundary condition $v_0(0) = 0$, and $v_0(1) = 1$.
We define

$$N\left[\aleph_2(x;\gamma)\right] = \frac{\partial^2\aleph_2}{\partial x^2} + \Omega\left(\frac{\partial\aleph_1}{\partial x}\right)^2\left[e^{-M\aleph_2} + 2\Lambda\left(\frac{\partial\aleph_1}{\partial x}\right)^4\right] = 0. \qquad (36)$$

We have the first component after the initial guess which is defined as,

$$v_1(x) = \tau H L^{-1}\Re_1\left(\overrightarrow{v}_0\right), \qquad (37)$$

where,

$$\Re_1\left(\overrightarrow{v_0}\right) = \Omega\left(1 + 4x^2 - 4x\right)e^{-Mx} + 2\Omega\Lambda\left(1 - 2x\right)^6.$$

Putting the value of $\Re_1(\overrightarrow{v_0})$ in equation (37), we get

$$v_1(x) = \tau H L^{-1}\left[\Omega\left(1+4x^2-4x\right)e^{-Mx} + 2\Omega\Lambda\left(1-2x\right)^6\right],$$

or

$$\begin{aligned} v_1(x) &= \tau H\Omega\left[\left(\frac{24}{M^4}-\frac{8}{M^3}+\frac{1}{M^2}\right)\left(e^{-Mx}-1\right)\right] \\ &+ \tau H\Omega\left[\left(\frac{1}{M}-\frac{8}{M^3}+\frac{16e^{-Mx}}{M^3}\right)x\right] \\ &- \tau H\Omega\left[\frac{2x^2}{M}+\frac{4x^3}{3M}\right]+\tau H\Omega\left[\frac{1}{M^2}-\frac{1}{3M}-\frac{\Lambda}{7}\right] \\ &+ \tau H\Omega\left[\Lambda(x^2-4x^3+10x^4-16x^5+16x^6-\frac{64}{7}x^7+\frac{16}{7})x^8\right] \\ &+ \tau H\Omega x\left[\frac{24}{M^4}-\frac{24e^{-M}}{M^4}-\frac{8e^{-M}}{M^3}-\frac{e^{-M}}{M^2}\right] \end{aligned} \tag{38}$$

$$\begin{aligned} v(x) &= x+\tau H\Omega\left[\left(\frac{24}{M^4}-\frac{8}{M^3}+\frac{1}{M^2}\right)\left(e^{-Mx}-1\right)\right] \\ &+ \tau H\Omega\left[\left(\frac{1}{M}-\frac{8}{M^3}+\frac{16e^{-Mx}}{M^3}\right)x\right] \\ &- \tau H\Omega\left[\frac{2x^2}{M}+\frac{4x^3}{3M}\right]+\tau H\Omega\left[\frac{1}{M^2}-\frac{1}{3M}-\frac{\Lambda}{7}\right] \\ &+ \tau H\Omega\left[\Lambda(x^2-4x^3+10x^4-16x^5+16x^6-\frac{64}{7}x^7+\frac{16}{7})x^8\right] \\ &+ \tau H\Omega x\left[\frac{24}{M^4}-\frac{24e^{-M}}{M^4}-\frac{8e^{-M}}{M^3}-\frac{e^{-M}}{M^2}\right]+\ldots, \end{aligned} \tag{39}$$

Now we come to the other nonlinear equation which is given by,

$$\frac{d^2w}{dx^2}+6\Lambda\left(\frac{dw}{dx}\right)^2 e^{Mv}-Mv\left(\frac{dw}{dx}\right)+2e^{-Mv}=0, \tag{40}$$

with boundary condition $w(0)=0$, and $w(1)=0$.
We make use of the initial guess as,

$$w_0(x)=x-x^2. \tag{41}$$

The nonlinear operator will be described as,

$$\begin{aligned} N\left[\aleph_1(x;\eta)\right] &= \frac{\partial^2\aleph_1(x;\eta)}{\partial x^2} + 6\Lambda\left(\frac{\partial\aleph_1(x;\eta)}{\partial x}\right)^2 e^{M\aleph_2(x;\eta)} \\ &- M\aleph_2(x;\eta)\left(\frac{\partial\aleph_1(x;\eta)}{\partial x}\right) + 2e^{-M\aleph_2(x;\eta)} = 0. \end{aligned} \quad (42)$$

The first approximation is given by

$$w_1(x) = \tau H L^{-1}\Re_1(\overrightarrow{w}_0), \quad (43)$$

where

$$\Re(\overrightarrow{w_0}) = w_0^{''} + 6\Lambda\left(w_0^{'}\right)^2 e^{Mv_0} - Mv_0\left(w_0^{'}\right) + 2e^{-Mv_0} = 0, \quad (44)$$

which yields

$$\Re(\overrightarrow{w_0}) = -2 + 6\Lambda\left(1 + 4x^2 - 4x\right)e^{Mx} - Mx + 2Mx^2 + 2e^{-Mx}. \quad (45)$$

Putting the value of $\Re(\overrightarrow{w_0})$ in equation (43), to get,

$$w_1(x) = \tau H L^{-1}\left[6\Lambda e^{Mx} - 2 + 24\Lambda(x^2 - x)e^{Mx} - Mx + 2Mx^2 + 2e^{-Mx}\right],$$

or

$$\begin{aligned} w_1(x) &= \tau H\left[\frac{M}{6}x^4 - \left(8\Lambda + \frac{M}{6}\right)x^3 + \left(-1 + \frac{24\Lambda e^{Mx}}{M^2} - \frac{12\Lambda}{M}\right)x^2\right] \\ &- \tau H\left[\frac{12\Lambda}{M} + \frac{2}{M^2}(12\Lambda e^{Mx} + 3\Lambda e^{M} - 3\Lambda + e^{-M} - 1)\right]x \\ &+ \tau H\left[\frac{48\Lambda}{M^3}\left(2e^{Mx} + e^{M} - 1\right) + (8\Lambda + 1) - \frac{144\Lambda}{M^4}(e^{M} - 1)\right]x \\ &+ \tau H\left[\frac{144\Lambda}{M^4}\left(e^{Mx} - 1\right) + \frac{48\Lambda}{M^3}\left(e^{Mx} - 1\right)\right] \\ &+ \tau H\left[\frac{2}{M^2}\left(3\Lambda e^{Mx} - 3\Lambda + e^{-Mx} - 1\right)\right]. \end{aligned} \quad (46)$$

Therefore, we finally have,

$$\begin{aligned}
w(x) &= x - x^2 \\
&+ \tau H \left[\frac{M}{6} x^4 - \left(8\Lambda + \frac{M}{6} \right) x^3 + \left(-1 + \frac{24\Lambda e^{Mx}}{M^2} - \frac{12\Lambda}{M} \right) x^2 \right] \\
&- \tau H \left[\frac{12\Lambda}{M} + \frac{2}{M^2} (12\Lambda e^{Mx} + 3\Lambda e^{M} - 3\Lambda + e^{-M} - 1) \right] x \\
&+ \tau H \left[\frac{48\Lambda}{M^3} \left(2e^{Mx} + e^{M} - 1 \right) + (8\Lambda + 1) - \frac{144\Lambda}{M^4} (e^{M} - 1) \right] x \\
&+ \tau H \left[\frac{144\Lambda}{M^4} \left(e^{Mx} - 1 \right) + \frac{48\Lambda}{M^3} \left(e^{Mx} - 1 \right) \right] \\
&+ \tau H \left[\frac{2}{M^2} \left(3\Lambda e^{Mx} - 3\Lambda + e^{-Mx} - 1 \right) \right] + \ldots \\
&,
\end{aligned} \tag{47}$$

3. Results and Discussion

In this section, we discuss the velocity and temperature distributions with the variation of parameters M and Λ for both the cases: Reynold's model and constant viscosity model keeping the Ω fixed. The values $H = 1, \tau = -1$ are fixed throughout for our convenience. For dimensionless velocity and temperature distributions, from the constant viscosity model and Reynold's model by HAM keeping $\Omega = 10$, it is clear from the figure 4 that for $M = 0, 1$ and 3, departure from symmetry is slight. To investigate the effects of M on the temperature distribution, we include viscous heating $\Omega = 10$. At this moderate rate of viscous heating, the temperature gives strong dependence on M only for the Newtonian fluid.

It is clear from the figures 4 and 3 that the temperature and velocity distributions remain sensibly invariant with respect to the viscosity index M in non-Newtonian fluids if the viscosity-temperature law for these fluids is given by Reynolds' formula.

In fig 2, we sketch velocity profile for constant viscosity model. We have come on the conclusion that the fluid which is third grade non-Newtonian , the temperature dependence of viscosity will not be significant up to which the velocity distribution is concerned. But there is a significant distinguish between the characteristics of non-Newtonian and Newtonian fluids.

We marled that from the point of view of velocity and temperature distributions in Poiseuille flow, the dependence of temperature is not important for third grade fluids for the range of considered parameters. Even if the fluid is non-Newtonian to some extent, the variable viscosity results are not much different from that of constant viscosity results.

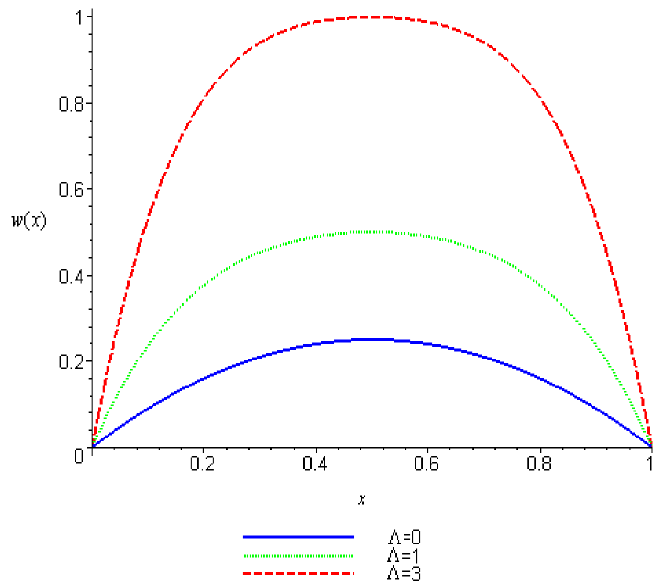

FIGURE 1. Dimensionless temperature distribution for Constant viscosity model by HAM keeping $\Omega = 10$, $H = 1$ and $\tau = -1$

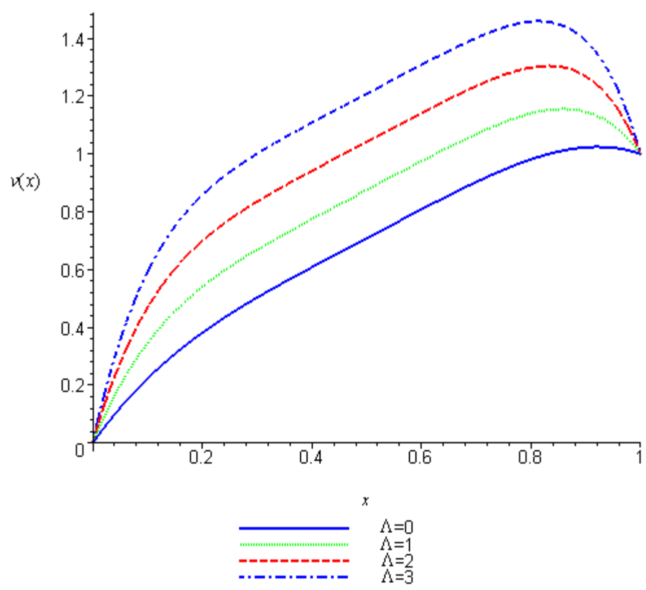

FIGURE 2. Dimensionless velocity distribution for Constant viscosity model by HAM keeping $\Omega = 10$, $H = 1$ and $\tau = -1$

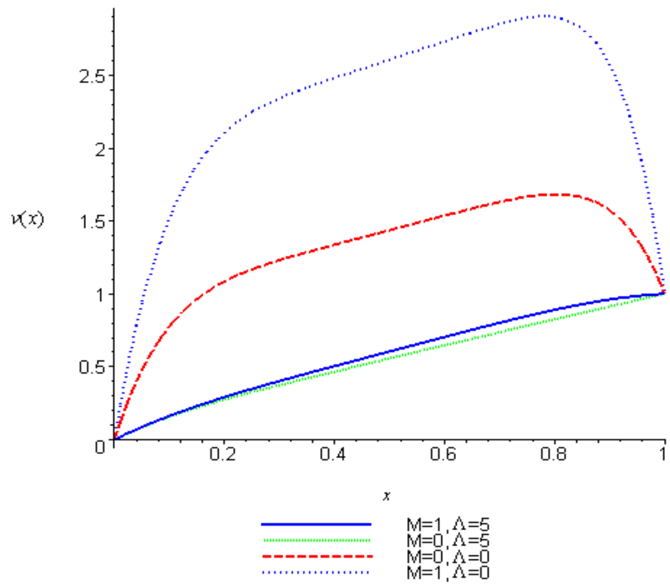

FIGURE 3. Dimensionless temperature distribution for Reynolds's model by HAM keeping $\Omega = 10$, $H = 1$ and $\tau = -1$

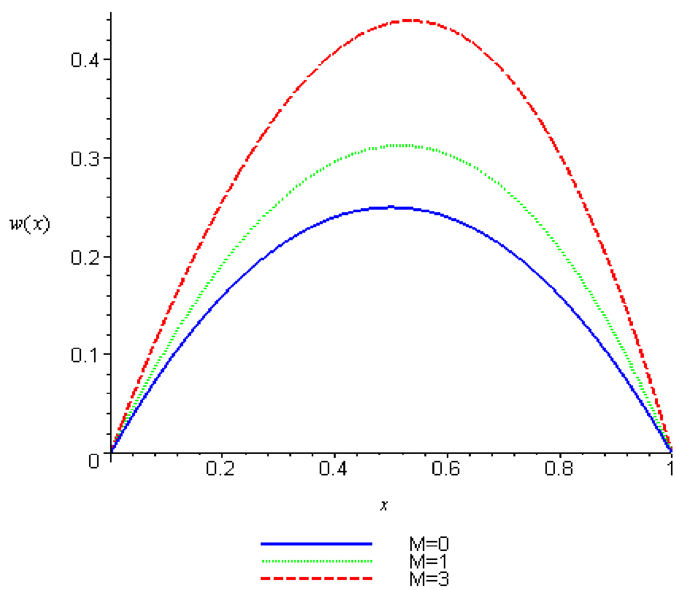

FIGURE 4. Dimensionless velocity distribution for Reynolds's model by HAM keeping $\Omega = 10$, $H = 1$ and $\tau = -1$

4. Conclusion

We observed that the flow of third grade fluid between heated parallel plates using HAM. It is clear from the results obtained by HAM that they are very close to the numerical results obtained in [2]. Hence, we conclude that this technique is an efficient and powerful technique to find the analytical solution for broad class of problems. Moreover, the advantage of HAM is the fast convergence of the solutions by means of the auxiliary parameter τ and the freedom to choose the initial guess in HAM, which provide us more accurate results than those obtained by HPM. We also observe that HPM is the special case of HAM under certain circumstances [9, 10].

References

[1] A. M. Siddiqui, H. A. Wahab , S. Bhatti, and M. Naeem, *Comparison of HPM and PEM for the flow of Non-Newtonian fluid between heater parallel plates*, Research Journal of Applied Sciences, Engineering and Technology 7 (20) (2014) 4226-4234.

[2] A. Z. Szeri and K. R. Rajagopal, *Flow of non Newtonian fluid between heated parallel plates*, Int. J. Nonlinear Mech. 20 (1985) 91-101.

[3] H. A. Wahab, M. Shakil, T. Khan, S. Bhatti and M. Naeem, *A comparative study of a system of Lotka-Voltera type of PDEs through perturbation methods*, Comput. ecol. softw. 3 (2013) 110-125.

[4] H. A. Wahab, T. Khan, M. Shakil, S. Bhatti and M. Naeem, *Analytical Treatment of System of KdV equations by homotopy perturbation method (HPM) and homotopy analysis method (HAM)*, Comput. ecol. softw. 4 (1) (2014) 63-81.

[5] H. A. Wahab, T. Khan, M. Shakil, S. Bhatti and M. Naeem, *Analytical approximate solutions of the systems of non Linear partial differential equations by homotopy perturbation method (HPM) and homotopy analysis method (HAM)*, J. Appl. Sci. Agri. 9 (4) (2014) 1855-1864.

[6] H. A. Wahab , S. Bhatti and M. Naeem, *A Comparison of perturbation techniques for nonlinear problems*, Journal of Prime Research in Mathematics 10(2015) 59-79.

[7] J. H. He, *Homotopy perturbation technique*, Comput. Methods in Appl. Mech. Eng. 178 (3-4)(1999) 257-262.

[8] M. Shakil, H. A. Wahab, M. Naeem, and S. Bhatti, *A quasi chemical approach for the modeling of predator-prey interactions*, Net. Bio. 4 (3) (2014) 130-150.

[9] S. J. Liao, Beyond Perturbation: *ntroduction to the homotopy analysis Method*, Chapman & Hall/ CRC Press [2003].

[10] S. J. Liao, *On the homotopy analysis method for nonlinear problems*, Appl. Math. Comput. 147 (2004) 499-513.