Journal of Prime Research in Mathematics Vol. **11**(2015), 123-136

EXISTENCE AND NON EXISTENCE OF MEAN CORDIAL LABELING OF CERTAIN GRAPHS

R. PONRAJ[1], S. SATHISH NARAYANAN[2]

ABSTRACT. Let f be a function from the vertex set $V(G)$ to $\{0, 1, 2\}$. For each edge uv assign the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a mean cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, 2\}$, where $v_f(x)$ and $e_f(x)$ respectively denote the number of vertices and edges labeled with x $(x = 0, 1, 2)$. A graph with a mean cordial labeling is called a mean cordial graph. In this paper we investigate mean cordial labeling behavior of prism, $K_2 + \overline{K_m}$, $\overline{K_n} + 2K_2$, book B_m and some snake graphs.

Key words : prism, corona, flower graph, complete graph, path.
AMS SUBJECT : 05C78.

1. INTRODUCTION

Throughout this paper, all graphs are finite, undirected and have no loops and multiple edges. Unless otherwise specified the graph G has vertex set $V = V(G)$ and $E = E(G)$ and we write p for $|V|$ and q for $|E|$. Graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labeling plays an important role of various fields of science and few of them are astronomy, coding theory, x-ray crystallography, radar, circuit design, communication network addressing, database management, secret sharing schemes, and models for constraint programming over finite domains [3]. The notion of mean cordial graphs was introduced and studied by Ponraj, Sundaram and sivakumar [4]. In [4] and [5] it is proved that path P_n, comb $P_n \odot K_1$, double comb $P_n \odot 2K_1$, $P_m \cup P_n$ are mean cordial. Additionally, Albert William, Indra Rajasingh and Roy [1] studied

[1]Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627 412, Tamil Nadu, India. Email: ponrajmaths@gmail.com
[2]Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627 412, Tamil Nadu, India. Email: sathishrvss@gmail.com.

mean cordial labeling behaviour of several graphs like caterpillar, banana tree, $S(B_{n,n})$, subdivision of ladder. In this paper, we shall discuss mean cordial labeling behavior of some more graphs like prism, $K_2 + \overline{K_m}$, $\overline{K_n} + 2K_2$, book B_m, alternate triangular snake $A(Q_n)$, book with n pentagonal pages, $P_n \odot K_2$, $C_n \odot K_2$. Let x be any real number. Then the symbol $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. A general reference for graph theoretic ideas can be seen from [2].

2. Preliminary Results

In this section we give some basic definitions and results which are needed for the next section.

Definition 1. Let f be a function from $V(G)$ to $\{0, 1, 2\}$. For each edge uv of G assign the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a mean cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, 2\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with x $(x = 0, 1, 2)$ respectively. A graph which admits a mean cordial labeling is called a mean cordial graph.

Definition 2. The *Join* of two graphs $G_1 + G_2$ is obtained from G_1 and G_2 and whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

Definition 3. The *Cartesian product* graph $G_1 \square G_2$ is defined as follows: Consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \square G_2$ whenever $[u_1 = v_1$ and $u_2 v_2 \in E(G_2)]$ or $[u_2 = v_2$ and $u_1 v_1 \in E(G_1)]$.

Definition 4. The *Corona* of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.

3. Main Results

Now we investigate the mean cordial labeling behavior of $K_2 + \overline{K_m}$.

Theorem 1. $K_2 + \overline{K_m}$ is not mean cordial.

Proof. Let $V(K_2 + \overline{K_m}) = \{u, v\} \cup \{u_i : 1 \leq i \leq m\}$ and $E(K_2 + \overline{K_m}) = \{uu_i, vu_i : 1 \leq i \leq m\} \cup \{uv\}$. Note that the order and size of $K_2 + \overline{K_m}$ are $m + 2$ and $2m + 1$ respectively. Suppose $K_2 + \overline{K_m}$ is mean cordial. We divide this proof into three cases.
Case 1. $m \equiv 0 \pmod 3$.

Let $m = 3t$. Then $p = 3t + 2$. If $v_f(0) = t$ then $e_f(0) \le 1 + (t-1) + (t-1) = 2t-1$, a contradiction. If $v_f(0) = t+1$ then $e_f(0) \le 1+(t-2)+(t-2) = 2t-3$, again a contradiction.
Case 2. $m \equiv 1 \pmod 3$.
Let $m = 3t + 1$. Then $p = 3t + 3$. Also $v_f(0) = t + 1$. Here, $e_f(0) \le 1 + (t-1) + (t-1) = 2t - 1$, a contradiction.
Case 3. $m \equiv 2 \pmod 3$.
Let $m = 3t + 2$. Here $p = 3t + 4$ and $q = 6t + 5$.
Subcase 1. $v_f(0) = t + 2$.

If u and v are labeled with 0 then there is no possibility for an edge with label 2. If either $f(u)$ or $f(v)$ is 0. In this case $e_f(0) = t + 1$, a contradiction to q. If both $f(u) \neq 0$ and $f(v) \neq 0$ then there is no possibility for an edge label 0. That is $e_f(0) = 0$, a contradiction to the size of $K_2 + \overline{K_m}$.
Subcase 2. $v_f(0) = t + 1$.
Similar to Subcase 1.
Hence $K_2 + \overline{K_m}$ is not mean cordial. □

The next investigation is about $\overline{K_n} + 2K_2$.

Theorem 2. $\overline{K_n} + 2K_2$ is not mean cordial.

Proof. The order and size of $\overline{K_n} + 2K_2$ are $n + 4$ and $4n + 2$, respectively. Suppose $\overline{K_n} + 2K_2$ is mean cordial. Let f denote the desired labeling. We divide this proof into three cases.
Case 1. $n \equiv 0$ $(mod\ 3)$.
Let $n = 3t$. In this case $p = 3t + 4$ where p denote the number of vertices in $\overline{K_n}+2K_2$. Suppose $v_f(0) = t+2$ then $e_f(0) \le 2(t-1)+2(t-1)+2 = 4t-6$, a contradiction. Also if $v_f(0) = t+1$ then $e_f(0) \le 2(t-3)+2(t-3)+2 = 4t-10$, again a contradiction.
Case 2. $m \equiv 1$ $(mod\ 3)$.
Similar to case 1.
Case 3. $m \equiv 2$ $(mod\ 3)$.
Similar to case 1.
Therefore $\overline{K_n} + 2K_2$ is not mean cordial. □

A *Mongolian tent* $MT_{m,n}$ is a graph obtained from $P_m \Box P_n$, n odd, by adding one extra vertex above the grid and joining every other vertex of the top row of $P_m \Box P_n$ to the new vertex. Since there are n colomns, the top row consists of n vertices.

Theorem 3. The mongolian tent $MT_{m,n}$ (n odd) is mean cordial if and only if $m \equiv 0 \pmod 3$ or $n \equiv 0 \pmod 3$.

Proof. The order and size of $MT_{m,n}$ is $mn + 1$ and $2mn - m$ respectively.
Case 1. $m \equiv 0 \pmod 3$, $n \equiv 0 \pmod 3$.

Let $m = 3t_1$ and $n = 3t_2$. Assign the label to the vertices of first t_1 rows by 0 and next t_1 rows by 1 and last t_1 rows by 2. Finally assign the label 0 to the top vertex u. In this case, f is a mean cordial labeling follows from table 1.

i	0	1	2
$v_f(i)$	$3t_1t_2 + 1$	$3t_1t_2$	$3t_1t_2$
$e_f(i)$	$6t_1t_2 - t_1$	$6t_1t_2 - t_1$	$6t_1t_2 - t_1$

TABLE 1

Case 2. $m \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$.
Let $m = 3t_1$ and $n = 3t_2 + 1$. Assign the labels to the vertices as in Case 1. Here, f is a mean cordial labeling follows from table 2.

i	0	1	2
$v_f(i)$	$3t_1t_2 + t_1 + 1$	$3t_1t_2 + t_1$	$3t_1t_2 + t_1$
$e_f(i)$	$6t_1t_2 + t_1$	$6t_1t_2 + t_1$	$6t_1t_2 + t_1$

TABLE 2

Case 3. $m \equiv 0 \pmod 3$, $n \equiv 2 \pmod 3$.
Let $m = 3t_1$ and $n = 3t_2 + 2$. Assign the labels to the vertices as in Case 1. Then f is a mean cordial labeling follows from table 3.

i	0	1	2
$v_f(i)$	$3t_1t_2 + 2t_1 + 1$	$3t_1t_2 + 2t_1$	$3t_1t_2 + 2t_1$
$e_f(i)$	$6t_1t_2 + 3t_1$	$6t_1t_2 + 3t_1$	$6t_1t_2 + 3t_1$

TABLE 3

Case 4. $n \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$.
Let $n = 3t_1$ and $m = 3t_2 + 1$. Consider the first row. Assign the labels to vertices of this row by 0 until we use $3t_1t_2 + t_1$ zeros. If we could not put all the $3t_1t_2 + t_1$ zeros then we move to the second row and assign the label 0 to the vertices of the second row from right to left until we put all the $3t_1t_2 + t_1$ zeros. If not we move to the third row and so on. Next consider the vertex which is not labeled but whose predecessor is labeled by 0. Start from this vertex, assign the label 1 to the vertices of that row. If we use all the $3t_1t_2+t_1$ $1's$ then stop. Otherwise we move to next row and assign the label 1 to the vertices of the row from right to left. Proceeding like this until we use $3t_1t_2+t_1$ $1's$. Assign the label 2 to all the remaining vertices other than the top vertex. Finally assign 0 to the top vertex.
The values of $v_f(i)$ and $e_f(i)$ $(0, 1, 2)$ are given in table 4.
Case 5. $n \equiv 0 \pmod 3$, $m \equiv 2 \pmod 3$.

i	0	1	2
$v_f(i)$	$3t_1t_2+t_1+1$	$3t_1t_2+t_1$	$3t_1t_2+t_1$
$e_f(i)$	$6t_1t_2+2t_1-t_2-1$	$6t_1t_2+2t_1-t_2$	$6t_1t_2+2t_1-t_2$

TABLE 4

Let $n=3t_1$ and $m=3t_2+2$. Assign the labels to the vertices as in the argument of case 4. The following table 5 shows that f is a mean cordial labeling.

i	0	1	2
$v_f(i)$	$3t_1t_2+2t_1+1$	$3t_1t_2+2t_1$	$3t_1t_2+2t_1$
$e_f(i)$	$6t_1t_2+4t_1-t_2-1$	$6t_1t_2+4t_1-t_2-1$	$6t_1t_2+4t_1-t_2$

TABLE 5

A Mean cordial labeling of $MT_{5,9}$ is given in figure 1.

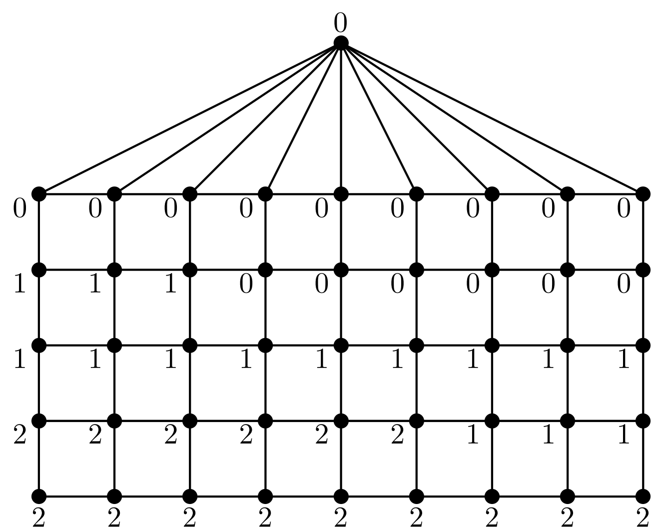

Figure 1

Case 6. $m \equiv 1 \pmod 3$, $n \equiv 1 \pmod 3$.
Let $m=3t_1+1$ and $m=3t_2+1$. In this case, $v_f(0)=3t_1t_2+t_1+t_2+1$ or $3t_1t_2+t_1+t_2$. Then $e_f(0) \le 6t_1t_2+t_1+2t_2-3$, a contradiction to the size of $MT_{m,n}$.
Case 7. $m \equiv 1 \pmod 3$, $n \equiv 2 \pmod 3$.
Let $m=3t_1+1$ and $m=3t_2+2$. Here, $v_f(0)=v_f(1)=v_f(2)=3t_1t_2+2t_1+t_2+1$. But $e_f(0) \le 6t_1t_2+3t_1+2t_2-1$, a contradiction to the size of $MT_{m,n}$.
Case 8. $m \equiv 2 \pmod 3$, $n \equiv 1 \pmod 3$.

Let $m = 3t_1 + 2$ and $m = 3t_2 + 1$. Then $v_f(0) = v_f(1) = v_f(2) = 3t_1t_2 + t_1 + 2t_2 + 1$. But $e_f(0) \leq 6t_1t_2 + t_1 + 4t_2 - 1$, a contradiction to the size of $MT_{m,n}$.
Case 9. $m \equiv 2 \pmod 3$, $n \equiv 2 \pmod 3$.
Let $m = 3t_1 + 2$ and $m = 3t_2 + 2$. Here, $v_f(0) = 3t_1t_2 + 2t_1 + 2t_2 + 2$ or $3t_1t_2 + 2t_1 + 2t_2 + 1$. But $e_f(0) \leq 6t_1t_2 + 3t_1 + 4t_2 + 1$ or $e_f(0) \leq 6t_1t_2 + 3t_1 + 4t_2 - 1$, a contradiction to the size of $MT_{m,n}$. □

Here we examine the mean cordial labeling behavior of *prism*.

Theorem 4. The prism $P_n \square P_2$ is not mean cordial.

Proof. We observe that prisms $P_n \square P_2$ consists of $2n$ vertices and $3n$ edges.
Case 1. $n \equiv 0$ (mod 3).
Let $n = 3t$. In this case $v_f(0) = 2t$. Clearly, $e_f(0) \leq 2t$, a contradiction.
Case 2. $n \equiv 1$ (mod 3).
Let $n = 3t + 1$. If $v_f(0) = 2t + 1$ or $2t$ then $e_f(0) \leq 2t + 1$ or $e_f(0) \leq 2t$ respectively, a contradiction.
Case 3. $n \equiv 2$ (mod 3).
Let $n = 3t + 2$. If $v_f(0) = 2t + 1$ or $2t + 2$ then $e_f(0) \leq 2t + 2$ or $e_f(0) \leq 2t + 1$ respectively, a contradiction. □

The *book* B_m is the graph $S_m \square P_2$, where S_m is a star with $m + 1$ vertices.

Theorem 5. The book B_m is mean cordial if and only if $m = 1$.

Proof. First we note that the number of vertices and edges of B_m are $2m + 2$ and $3m + 1$ respectively. One can easily check that the graph B_1 with a vertex labeling in the order 0, 0, 2, 1 (any direction) is mean cordial. Conversely, suppose $m \geq 2$ and B_m is mean cordial with a vertex labeling f.
Case 1. $m \equiv 0$ (mod 3).
Let $m = 3t$. Then $p = 6t + 2$ and $q = 9t + 1$. Here $v_f(0) = 2t + 1$ or $2t$. This implies $e_f(0) \leq 3t - 1$ or $3t - 2$, a contradiction.
Case 2. $m \equiv 1$ (mod 3).
Let $m = 3t + 1$. Here $p = 6t + 4$, $q = 9t + 4$. Also $v_f(0) = 2t + 1$ or $2t + 2$. If $v_f(0) = 2t + 1$, then $e_f(0) \leq 3t - 1$, a contradiction. Therefore $v_f(0)$ should necessarily be $2t + 2$. Then $e_f(0) \leq 3t + 1$. If $e_f(0) = 3t + 1$ then the labels of the central vertices of the two copies of S_m should be labeled with 0. Also, note that $v_f(0) = 3t$. This shows that $e_f(2) \leq 2t + 1$, a contradiction.
Case 3. $m \equiv 2$ (mod 3).
Let $m = 3t + 2$. In this case, $p = 6t + 6$, $q = 9t + 7$. Clearly $v_f(0) = v_f(1) = v_f(2) = 2t + 2$. But $e_f(0) \leq 3t + 1$, a contradiction. □

The *helm* H_n is the graph obtained from a wheel by attaching a pendant edge at each vertex of the n-cycle. A *flower* Fl_n is the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm.

Theorem 6. Flower graphs Fl_n are not mean cordial.

Proof. The number of vertices and edges of Fl_n are $2n+2$ and $3n+1$ respectively. Suppose f is a mean cordial labeling of Fl_n.
Case 1. $n \equiv 0$ $(mod\ 3)$.
Take $n = 3t$. Then $p = 6t+1$ and $q = 12t$. This implies $v_f(0) = 2t+1$ or $2t$. Hence $e_f(0) \leq 3t$ or $3t-2$, a contradiction.
Case 2. $n \equiv 1$ $(mod\ 3)$.
Put $n = 3t+1$. This implies $p = 6t+3$ and $q = 12t+4$. Therefore $v_f(0) = 2t+1$. This forces $e_f(0) \leq 3t$, a contradiction.
Case 3. $n \equiv 2$ $(mod\ 3)$.
Let $n = 3t+2$. Here $p = 6t+5$ and $q = 12t+8$. Then $v_f(0) = 2t+2$ or $2t+1$. Hence $e_f(0) \leq 3t+1$ or $3t$, a contradiction. □

The *sunflower graph* SF_n is obtained by taking a wheel with central vertex v_0 and the cycle $C_n : v_1, v_2, \ldots v_n, v_1$ and new vertices $w_1, w_2, \ldots, w_n$ where w_i is joined by vertices $v_i, v_{i+1\ (\text{mod}\ n)}$.

Theorem 7. The sunflower graph SF_n is not mean cordial.

Proof. First, we observe that the order and size of SF_n are $2n+1$ and $4n$, respectively. Assume that f is a mean cordial labeling of SF_n.
Case 1. $n \equiv 0$ $(mod\ 3)$.
Let $n = 3t$. Then $p = 6t+1$ and $q = 12t$. It is clear that $v_f(0) = 2t+1$ or $2t$. If $v_f(0) = 2t+1$ then $e_f(0) \leq 4t-1$ and if $v_f(0) = 2t$ then $e_f(0) \leq 4t-3$, a contradiction.
Case 2. $n \equiv 1$ $(mod\ 3)$.
Take $n = 3t+1$. In this case $p = 6t+3$ and $q = 12t+4$. Therefore $v_f(0) = 2t+1$. Then $e_f(0) \leq 4t-1$, a contradiction.
Case 3. $n \equiv 2$ $(mod\ 3)$.
Put $n = 3t+2$. Here $p = 6t+5$ and $q = 12t+8$. Also $v_f(0) = 2t+2$ or $2t+1$. If $v_f(0) = 2t+2$ then $e_f(0) \leq 4t+1$ and if $v_f(0) = 2t+1$ then $e_f(0) \leq 4t-1$, a contradiction. □

Let G_n be a graph with vertex set $V(G_n) = \{u, v, u_i, v_i, w_i : 1 \leq i \leq n\}$ and $E(G_n) = \{uv, uu_i, u_iw_i, w_iv_i, v_iv : 1 \leq i \leq n\}$ then the graph G_n is called a *book with n pentagonal pages.*

Theorem 8. G_n is mean cordial if and only if $n \equiv 1$ $(mod\ 3)$.

Proof. The order and size of the graph G_n are $3n+2$ and $4n+1$, respectively.
Case 1. $n \equiv 0$ $(mod\ 3)$.
Let $n = 3t$. Let f be a mean cordial labeling of G_n. Then $p = 9t+2$ and $q = 12t+1$. In this case $v_f(0) = 3t+1$ or $3t$. If $v_f(0) = 3t+1$ then $e_f(0) \leq 4t-1$ and if $v_f(0) = 3t$ then $e_f(0) \leq 4t-2$, a contradiction.
Case 2. $n \equiv 1$ $(mod\ 3)$.

Take $n = 3t + 1$. Here $p = 9t + 5$ and $q = 12t + 5$. Define a function $f : V(G_n) \to \{0, 1, 2\}$ as follows: $f(u) = 0$, $f(v) = 0$, $f(u_i) = f(v_i) = f(w_i) = 0$, $1 \le i \le t$,

$$\begin{array}{lcll} f(u_{t+i}) & = & 1 & 1 \le i \le 2t+1 \\ f(v_{t+i}) & = & 1 & 1 \le i \le t \\ f(v_{2t+i}) & = & 2 & 1 \le i \le t+1 \\ f(w_{t+i}) & = & 2 & 1 \le i \le 2t+1. \end{array}$$

The following table 6 shows that the above vertex labeling f is a mean cordial labeling.

i	0	1	2
$v_f(i)$	$3t+2$	$3t+1$	$3t+2$
$e_f(i)$	$4t+1$	$4t+2$	$4t+2$

TABLE 6

Case 3. $n \equiv 2$ $(mod\ 3)$.
Put $n = 3t + 2$. Here $p = 9t + 8$ and $q = 12t + 9$. Then $v_f(0) = 3t + 3$ or $3t + 2$. It follows that $e_f(0) \le 4t + 2$ or $4t + 1$, a contradiction. □

Theorem 9. *$P_n \odot K_2$ is mean cordial if and only if $n \equiv 0 \pmod 3$.*

Proof. Let $P_n : u_1, u_2, \ldots, u_n$ be a path. Let $V(P_n \odot K_2) = V(P_n) \cup \{u_i, v_i : 1 \le i \le n\}$ and $E(P_n \odot K_2) = E(P_n) \cup \{u_i v_i, v_i w_i, w_i u_i : 1 \le i \le n\}$. The order and size of $P_n \odot K_2$ are $3n$ and $4n - 1$ respectively.
Case 1. $n \equiv 0 \pmod 3$.
Let $n = 3t$. Define a map $f : V(P_n \odot K_2) \to \{0, 1, 2\}$ by

$$\begin{array}{lclclcll} f(u_i) & = & f(v_i) & = & f(w_i) & = & 0 & 1 \le i \le t \\ f(u_{t+i}) & = & f(v_{t+i}) & = & f(w_{t+i}) & = & 1 & 1 \le i \le t \\ f(u_{2t+i}) & = & f(v_{2t+i}) & = & f(w_{2t+i}) & = & 2 & 1 \le i \le t. \end{array}$$

The values of $v_f(i)$ and $e_f(i)$ $(i = 0, 1, 2)$ are given in table 7. Clearly the

i	0	1	2
$v_f(i)$	$3t$	$3t$	$3t$
$e_f(i)$	$4t-1$	$4t$	$4t$

TABLE 7

above vertex labeling is a mean cordial labeling.
Case 2. $n \equiv 1 \pmod 3$.
Put $n = 3t + 1$. Suppose there exisits a mean cordial labeling f then $v_f(0) = v_f(1) = v_f(2) = 3t + 1$. But $e_f(0) \le 4t$, a contradiction.
Case 3. $n \equiv 2 \pmod 3$.

Let $n = 3t + 2$. If f is a mean cordial labeling then $v_f(0) = v_f(1) = v_f(2) = 3t + 2$. In this case $e_f(0) \leq 4t + 1$, a contradiction.

□

Theorem 10. $C_n \odot K_2$ is not mean cordial.

Proof. Let $C_n : u_1, u_2, \ldots, u_n, u_1$ be a cycle. Let $V(C_n \odot K_2) = V(C_n) \cup \{u_i, v_i : 1 \leq i \leq n\}$ and $E(C_n \odot K_2) = E(C_n) \cup \{u_i v_i, v_i w_i, w_i u_i : 1 \leq i \leq n\}$. Here $p = 3n$ and $q = 4n$. Suppose f is a mean cordial labeling of $C_n \odot K_2$.
Case 1. $n \equiv 0 \pmod 3$.
Put $n = 3t$. In this case $v_f(0) = v_f(1) = v_f(2) = 3t$. Then $e_f(0) \leq 4t - 1$, a contradiction.
Case 2. $n \equiv 1 \pmod 3$.
Take $n = 3t + 1$. Here $v_f(0) = v_f(1) = v_f(2) = 3t + 1$. But $e_f(0) \leq 4t$, a contradiction.
Case 3. $n \equiv 0 \pmod 3$.
Let $n = 3t + 2$. Then $v_f(0) = v_f(1) = v_f(2) = 3t + 2$ and $e_f(0) \leq 4t + 1$, a contradiction. □

Let P_n be the path $u_1, u_2, \ldots, u_n$. The *quadrilateral snake* Q_n is the graph with vertex set $V(Q_n) = \{v_i, w_i : 1 \leq i \leq n - 1\} \cup V(P_n)$ and edge set $E(Q_n) = \{u_i v_i, v_i w_i, w_i u_{i+1} : 1 \leq i \leq n - 1\} \cup E(P_n)$.

Theorem 11. *Q_n is mean cordial.*

Proof. The order and size of Q_n are $3n - 2$ and $4n - 4$ respectively.
Case 1. $n \equiv 0$ (*mod* 3).
Put $n = 3t$. Define a map $f : V(Q_n) \to \{0, 1, 2\}$ by

$$
\begin{array}{lllllll}
f(w_i) & = & f(v_i) & = & f(u_i) & = & 0 \quad 1 \leq i \leq t \\
 & & f(u_{t+i}) & = & f(v_{t+i}) & = & 1 \quad 1 \leq i \leq t \\
 & & & & f(u_{2t+i}) & = & 2 \quad 1 \leq i \leq t \\
 & & & & f(v_{2t+i}) & = & 2 \quad 1 \leq i \leq t-1 \\
 & & & & f(w_{t+i}) & = & 1 \quad 1 \leq i \leq t-1 \\
 & & & & f(w_{2t-1+i}) & = & 2 \quad 1 \leq i \leq t.
\end{array}
$$

The following table 8 shows that f is a mean cordial labeling.

i	0	1	2
$v_f(i)$	$3t$	$3t-1$	$3t-1$
$e_f(i)$	$4t-2$	$4t-1$	$4t-1$

TABLE 8

Case 2. $n \equiv 1$ (*mod* 3).

Put $n = 3t + 1$. Define a map $f : V(Q_n) \to \{0, 1, 2\}$ by

$$\begin{array}{ccccccccc} & & & & f(u_i) & = & 0 & 1 \le i \le t+1 \\ & & f(w_i) & = & f(v_i) & = & 0 & 1 \le i \le t \\ f(w_{t+i}) & = & f(v_{t+i}) & = & f(u_{t+1+i}) & = & 1 & 1 \le i \le t \\ f(w_{2t+i}) & = & f(v_{2t+i}) & = & f(u_{2t+1+i}) & = & 2 & 1 \le i \le t. \end{array}$$

From 9, one can easily check that f is a mean cordial labeling.

i	0	1	2
$v_f(i)$	$3t+1$	$3t$	$3t$
$e_f(i)$	$4t$	$4t$	$4t$

TABLE 9

Case 3. $n \equiv 2\ (mod\ 3)$.
Put $n = 3t + 1$. Define a map f from $V(Q_n)$ to the set $\{0, 1, 2\}$ as follows.

$$f(u_i) = \begin{cases} 0 & \text{if } 1 \le i \le t+1 \\ 1 & \text{if } t+2 \le i \le 2t+1 \\ 2 & \text{if } 2t+2 \le i \le 3t+2 \end{cases} \qquad f(v_i) = \begin{cases} 0 & \text{if } 1 \le i \le t+1 \\ 1 & \text{if } t+2 \le i \le 2t+1 \\ 2 & \text{if } 2t+2 \le i \le 3t+1 \end{cases}$$

$$f(w_i) = \begin{cases} 0 & \text{if } 1 \le i \le t \\ 1 & \text{if } t+1 \le i \le 2t+1 \\ 2 & \text{if } 2t+2 \le i \le 3t+1 \end{cases}$$

The following table 10 shows that f is a mean cordial labeling.

i	0	1	2
$v_f(i)$	$3t+2$	$3t+1$	$3t+1$
$e_f(i)$	$4t+1$	$4t+1$	$4t+2$

TABLE 10

□

Final investigation is about alternate quadrilateral snake. Every alternate edge of a path $P_n : u_1, u_2, \ldots, u_n$ is replaced by the square C_4 is called the *alternate quadrilateral snake* $A(Q_n)$. That is this graph is a chain consisting of alternating four cycles and edges. It has the following three types.

(1) The squares start from u_2 and end with u_{n-1}.
Clearly here n is even. We define the vertex set of $A(Q_n)$ by $V(P_n) \cup \{v_i, w_i : 1 \le i \le \frac{n-2}{2}\}$ and the edge set by $E(P_n) \cup \{v_i u_{2i}, w_i u_{2i+1} : 1 \le i \le \frac{n-2}{2}\}$.

(2) The squares start from u_1 and end with u_n.
Note that n is even here. The vertex set is $V(P_n) \cup \{v_i, w_i : 1 \le i \le \frac{n}{2}\}$ and the edge set by $E(P_n) \cup \{v_i u_{2i-1}, w_i u_{2i} : 1 \le i \le \frac{n}{2}\}$.

(3) The squares start from u_2 and end with u_n.
In this case n is odd. The vertex set is $V(P_n) \cup \{v_i, w_i : 1 \leq i \leq \frac{n-1}{2}\}$ and the edge set by $E(P_n) \cup \{v_i u_{2i}, w_i u_{2i+1} : 1 \leq i \leq \frac{n-1}{2}\}$.

Theorem 12. The alternate quadrilateral snake $A(Q_n)$ is mean cordial iff
The square starts from u_2, ends with u_{n-1} and $n \equiv 0, 2 \pmod 3$. (or)
The square starts from u_1, ends with u_n and $n \equiv 0, 2 \pmod 3$. (or)
The square starts from u_2, ends with u_n and $n \equiv 0, 1 \pmod 3$.

Proof. **Case 1.** The square starts from u_2, ends with u_{n-1} and $n \equiv 0, 2 \pmod 3$.
In this case $p = 2n - 2$ and $q = \frac{5n-8}{2}$. Suppose $n \equiv 0 \pmod 3$. Let $n = 3t$. Consider the path vertices $u_1, u_2, \ldots, u_n$. Put the label 2 to the vertex u_1. Then we label the next t vertices with the number 0 and then $t - 1$ successive vertices receives the label 1. The remaining path vertices are labeled by 2. Now consider the vertices v_i $(1 \leq i \leq n)$. The first $\frac{t}{2}$ vertices of v_i receives the label 0. Then we label the vertices $v_{\frac{t}{2}+1}, v_{\frac{t}{2}+2}, \ldots, v_t$ by 1. Finally the vertices $v_{t+1}, \ldots, v_{\frac{3t-2}{2}}$ are labeled with 2. The vertices w_i are labeled as in $v_i (1 \leq i \leq \frac{n-2}{2})$. Let f be the labeling defined above then f satisfies the vertex and edge conditions of mean cordial labeling. Here ws display the values of $v_f(i), e_f(i)$ $(i = 0, 1, 2)$ in Table 11.

i	0	1	2
$v_f(i)$	$\frac{2n}{3}$	$\frac{2n-3}{3}$	$\frac{2n-3}{3}$
$e_f(i)$	$\frac{5n-6}{6}$	$\frac{5n-6}{6}$	$\frac{5n-12}{6}$

TABLE 11

Suppose $n \equiv 2 \pmod 3$. Then $n = 3t + 2$. Define a map f from $V(A(Q_n))$ to the set $\{0, 1, 2\}$ as follows.

$$\begin{array}{llllll} f(w_i) & = & f(v_i) & = & 0 & 1 \leq i \leq \frac{t}{2} \\ f(w_{\frac{t}{2}+i}) & = & f(v_{\frac{t}{2}+i}) & = & 1 & 1 \leq i \leq \frac{t}{2} \\ f(w_{t+i}) & = & f(v_{t+i}) & = & 2 & 1 \leq i \leq \frac{t}{2} \\ & & f(u_i) & = & 0 & 1 \leq i \leq t+1 \\ & & f(u_{t+1+i}) & = & 1 & 1 \leq i \leq t \\ & & f(u_{2t+1+i}) & = & 2 & 1 \leq i \leq t+1. \end{array}$$

The table 12 shows that f is a mean cordial labeling.
Case 2. The square starts from u_1, ends with u_n and $n \equiv 0, 2 \pmod 3$.
In this case $p = 2n$ and $q = \frac{5n-2}{2}$. Suppose $n \equiv 0 \pmod 3$. Let $n = 3t$. Here, the given graph consisits of $\frac{3t}{2}$ squares. Now assign the label 0 to the vertices of the first $\frac{t}{2}$ squares. The vertices of the next $\frac{t}{2}$ squares are labeled with 1. Finally, the vertices of the last $\frac{t}{2}$ squares receives the label 2. If f denotes the

i	0	1	2
$v_f(i)$	$\frac{2n-1}{3}$	$\frac{2n-4}{3}$	$\frac{2n-1}{3}$
$e_f(i)$	$\frac{5n-10}{6}$	$\frac{5n-10}{6}$	$\frac{5n-4}{6}$

TABLE 12

above labeling then one can easily check that f satisfies the vertex and edge conditions of mean cordial labeling (see table 13).

i	0	1	2
$v_f(i)$	$\frac{2n}{3}$	$\frac{2n}{3}$	$\frac{2n}{3}$
$e_f(i)$	$\frac{5n-6}{6}$	$\frac{5n}{6}$	$\frac{5n}{6}$

TABLE 13

If $n \equiv 2 \pmod 3$ then $n = 3t + 2$. Define a map $f : V(A(Q_n)) \to \{0, 1, 2\}$ as follows.

$$f(u_i) = \begin{cases} 0 & \text{if} \quad 1 \le i \le t+1 \\ 1 & \text{if} \quad t+2 \le i \le 2t+1 \\ 2 & \text{if} \quad 2t+2 \le i \le 3t+2 \end{cases} \qquad f(v_i) = \begin{cases} 0 & \text{if} \quad 1 \le i \le \frac{t+2}{2} \\ 1 & \text{if} \quad \frac{t+4}{2} \le i \le t+1 \\ 2 & \text{if} \quad t+2 \le i \le \frac{3t+2}{2} \end{cases}$$

$$f(w_i) = \begin{cases} 0 & \text{if} \quad 1 \le i \le \frac{t}{2} \\ 1 & \text{if} \quad \frac{t+2}{2} \le i \le t+1 \\ 2 & \text{if} \quad t+2 \le i \le \frac{3t+2}{2} \end{cases}$$

The table 14 shows that f satisfies the requirements of a mean cordial labeling.

i	0	1	2
$v_f(i)$	$\frac{2n+2}{3}$	$\frac{2n-1}{3}$	$\frac{2n-1}{3}$
$e_f(i)$	$\frac{5n-4}{6}$	$\frac{5n-4}{6}$	$\frac{5n+2}{6}$

TABLE 14

Case 3. The square starts from u_2, ends with u_n and $n \equiv 0, 1 \pmod 3$.
In this case $p = 2n - 1$ and $q = \frac{5n-5}{2}$. Suppose $n \equiv 0 \pmod 3$ then $n = 3t$. The figure 2 shows that $A(Q_3)$ is a mean cordial graph.

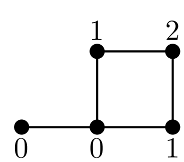

Figure 2

Let $t > 1$. Define a labeling $f : V(A(Q_n)) \to \{0, 1, 2\}$ by

$$\begin{array}{lllll} f(w_i) & = & f(v_i) & = & 0 \quad 1 \le i \le \frac{t-1}{2} \\ f(w_{\frac{t-1}{2}+i}) & = & f(v_{\frac{t-1}{2}+i}) & = & 1 \quad 1 \le i \le \frac{t+1}{2} \\ f(w_{t+i}) & = & f(v_{t+i}) & = & 2 \quad 1 \le i \le \frac{t-1}{2} \\ & & f(u_i) & = & 0 \quad 1 \le i \le t+1 \\ & & f(u_{t+1+i}) & = & 1 \quad 1 \le i \le t-1 \\ & & f(u_{2t+i}) & = & 2 \quad 1 \le i \le t. \end{array}$$

From table 15, it is easy to see that f is a mean cordial labeling.

i	0	1	2
$v_f(i)$	$\frac{2n}{3}$	$\frac{2n}{3}$	$\frac{2n-3}{3}$
$e_f(i)$	$\frac{5n-9}{6}$	$\frac{5n-3}{6}$	$\frac{5n-3}{6}$

Table 15

If $n \equiv 1 \pmod 3$ then $n = 3t + 1$. The labeling f defined in subcase 2 of case 2 inherits the property of a mean cordial labeling together with an edge uu_1 where $f(u) = 0$. In this case the vertex and edge conditions are given in table 16.

i	0	1	2
$v_f(i)$	$\frac{2n+1}{3}$	$\frac{2n-2}{3}$	$\frac{2n-2}{3}$
$e_f(i)$	$\frac{5n-5}{6}$	$\frac{5n-5}{6}$	$\frac{5n-5}{6}$

Table 16

Case 4. The square starts from u_2, ends with u_{n-1} and $n \equiv 1 \pmod 3$.
Suppose f is a mean cordial labeling. Here $v_f(0) = v_f(1) = v_f(2) = \frac{2n-2}{3}$. But $e_f(0) \le \frac{5n-14}{6}$, a contradiction.
Case 5. The square starts from u_1, ends with u_n and $n \equiv 1 \pmod 3$.
Here $v_f(0) = \frac{2n+1}{3}$ or $\frac{2n-2}{3}$. In either case $e_f(0) \le \frac{5n-8}{6}$. It follows that f can not be a mean cordial labeling.
Case 6. The square starts from u_2, ends with u_n and $n \equiv 2 \pmod 3$.
Then $v_f(0) = v_f(1) = v_f(2) = \frac{2n-1}{3}$ or $\frac{2n-2}{3}$. This implies $e_f(0) \le \lfloor \frac{5n-8}{6} \rfloor - 1$. This shows that there does not exists a mean cordial labeling f. □

Acknowledgment. *The authors thank to the referee's for their valuable suggestion.*

References

[1] Albert William, Indra Rajasingh and Roy, Mean cordial labeling of certain graphs, *J. Comp & Math. Sci.*, **4** (2013), 274-281.
[2] F. Harary, Graph theory, *Addision wesley*, New Delhi.
[3] J. A. Gallian, A Dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics,* **18** (2011), # DS6.
[4] R. Ponraj, M. Sivkumar and M. Sundaram, Mean cordial labeling of graphs, *Open Journal of Discreate Mathematics,* Vol. 2, **No. 4**, 2012, 145-148.
[5] R. Ponraj and M. Sivakumar, On Mean cordial graphs, *International Journal of Mathematical Combinatorics,* **3** (2013), 78-84.
[6] R. Ponraj and S. Sathish Narayanan, Mean cordiality of some snake graphs, *Palestine Journal of Mathematics* **4(2)**(2015), 439 - 445.