

ON THE MIXED HODGE STRUCTURE ASSOCIATED HYPERSURFACE SINGULARITIES

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ABSTRACT. Let $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be a germ of hypersurface with isolated singularity. One can associate to f a polarized variation of mixed Hodge structure \mathcal{H} over the punctured disc, where the Hodge filtration is the limit Hodge filtration of W. Schmid and J. Steenbrink. By the work of M. Saito and P. Deligne the VMHS associated to cohomologies of the fibers of f can be extended over the degenerate point 0 of disc. The new fiber obtained in this way is isomorphic to the module of relative differentials of f denoted Ω_f . A mixed Hodge structure can be defined on Ω_f in this way. The polarization on \mathcal{H} deforms to Grothendieck residue pairing modified by a varying sign on the Hodge graded pieces in this process. This also proves the existence of a Riemann-Hodge bilinear relation for Grothendieck pairing and allow to calculate the Hodge signature of Grothendieck pairing.

Key words : plane curves, polar curves, discriminant.

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INTRODUCTION

One of the important subjects of study in Hodge theory and D-modules is the asymptotic behaviour of the underlying variation of (mixed) Hodge structures and also the possibility to extend such structures over the degeneracy. If the VMHS is coming from an isolated hypersurface singularity, the mixed Hodge structure we are interested in is the Steenbrink limit mixed Hodge structure (W. Schmid LMHS in projective fibrations). Classically there are two equivalent ways to define Steenbrink LMHS. The first, due to J. Steenbrink is by a spectral sequence argument applied to the resolution of singularities in

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a projective fibration. In the affine case one needs to embed the original family in a projective one and then use the Invariant cycle theorem. Another method is by the structure of lattices in the Gauss-Manin system associated to VMHS on the punctured disc. It concerns the definition of Brieskorn lattices. By applying the powers of the Gauss-Manin connection to the Brieskorn lattice one can define a decreasing filtration, that is equivalent to the limit Hodge filtration of J. Steenbrink. Our strategy is to extend such local system of MHS over the degeneracy. We explain this by gluing vector bundles. It mainly involves the fact that, V -lattices can be glued with the Brieskorn lattices in different charts. An observation made by P. Deligne and M. Saito is that, one can define a new filtration on the vanishing cohomology, that is opposite to the original Hodge filtration in this way.

Throughout the whole text assume $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a holomorphic germ with isolated singularity at $0 \in \mathbb{C}^{n+1}$ and $f : X \rightarrow T$ a representative for the Milnor fibration. One can associate a local system to this data by

$$\mathcal{H} = R^n f_* \mathbb{C} = \bigcup_{t \in T'} H^n(X_t, \mathbb{C})$$

Denote by M the monodromy, and $N = \log(M_u) \otimes 1/2\pi i$ the logarithm of the unipotent part of M . By the monodromy theorem M is quasi-unipotent. We will use the notation $X_\infty = H \times_{T'} X_t$, where H is the upper half plane for the canonical fiber, and

$$\Omega_f \cong \frac{\Omega_{X,0}^{n+1}}{df \wedge \Omega_{X,0}^n}$$

Let \mathcal{G} be the associated Gauss-Manin system via the Riemann-Hilbert correspondence. We shall extend the Gauss-Manin system \mathcal{G} as well as \mathcal{H} over the puncture. The stack on 0 of the extended Gauss-Manin module can be naturally identified with the module of relative differentials Ω_f . This is a direct consequence of isomorphisms between V -lattices and the Brieskorn lattices, which also pairs the Hodge filtration and the (actually a re-indexed) V -filtration in an opposite way. A classical example of this situation is when the variation of MHS is Hodge-Tate. We show a modification of Grothendieck residue defines a polarization for a mixed Hodge structure on Ω_f .

Theorem 1. *Assume $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a holomorphic hypersurface germ with isolated singularity at $0 \in \mathbb{C}^{n+1}$. Then the variation of mixed Hodge structure, cf. [4]. This VMHS can be extended to the puncture with the extended fiber isomorphic to Ω_f in the sense of sec. 4 and 5, and it is polarized by Theorem 6. The Hodge filtration on the new fiber Ω_f corresponds to an opposite Hodge filtration on $H^n(X_\infty, \mathbb{C})$ in the way explained in sec. 4.*

The way to reach to this result is by a standard gluing argument on the structure theory of Gauss-Manin system explained in [21]. We equip Ω_f with a mixed Hodge structure compatible with this gluing. The definition is based on a linear isomorphism Φ between the two vector spaces $(H^n(X_\infty, \mathbb{C}), M)$ and Ω_f , which already have been used by A. Varchenko in its graded form. However we take a lift of the map employed by Varchenko. The main theorem (Theorem 6) express that this isomorphism pairs the polarization on the vanishing cohomology and a modification of the Grothendieck residue of f . The computation of a twist in Grothendieck residue which goes back to the inter-relation with higher residue pairing with the linking form of f makes the contribution non-trivial. It also allows to formulate a Riemann-Hodge bilinear relation for Grothendieck residue. We present several applications of the above procedure.

1. STEENBRINK LIMIT MIXED HODGE STRUCTURE

Suppose we have an isolated singularity holomorphic germ $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$. By the Milnor fibration Theorem we can always associate to f a C^∞ -fiber bundle over a small punctured disc T' . The corresponding cohomology bundle \mathcal{H} , constructed from the middle cohomologies of the fibers defines a variation of mixed Hodge structure on T' . Recall the Brieskorn lattice is defined by,

$$(\mathcal{H}^{(0)} =)H'' = f_* \frac{\Omega_{X,0}^{n+1}}{df \wedge d\Omega_{X,0}^{n-1}}$$

The Brieskorn lattice is the stack at 0 of a locally free \mathcal{O}_T -module \mathcal{H}'' of rank μ with $\mathcal{H}_{T'}'' \cong \mathcal{H}$, and hence $H'' \subset (i_*\mathcal{H})_0$, with $i : T' \hookrightarrow T$. The regularity of the Gauss-Manin connection proved by Brieskorn and Malgrange implies that $H'' \subset \mathcal{G}$. It is a theorem by Malgrange (see [21] 1.4.10) $H'' \subset V^{-1}$. The Leray residue formula can be used to express the action of ∂_t in terms of differential forms by $\partial_t^{-1}s[d\omega] = s[df \wedge \omega]$, where, $\omega \in \Omega_X^n$ and $s(\eta)(t) = \text{Res}_{f=t}(\eta/(f-t))$. In particular, $\partial_t^{-1}.H'' \subset H''$, and

$$\frac{H''}{\partial_t^{-1}.H''} \cong \frac{\Omega_{X,0}^{n+1}}{df \wedge \Omega_{X,0}^n} \cong \frac{\mathbb{C}\{z\}}{(\partial(f))}. \quad (1)$$

for the module of relative differentials of the map f . The Hodge filtration on $H^n(X_\infty, \mathbb{C})$ is defined by

$$F^p H^n(X_\infty)_\lambda = \psi_\alpha^{-1} \partial_t^{n-p} Gr_V^{\alpha+n-p} \mathcal{H}^{(0)}. \quad (2)$$

where $\psi_\alpha : H^n(X_\infty)_\lambda \rightarrow C^\alpha \subset \mathcal{G}$ is

$$\psi_\alpha(A_\alpha) = t^\alpha \exp(\log(t)N/2\pi i)A_\alpha, \quad \lambda = \exp(2\pi i\alpha), \quad -1 \leq \alpha < 0$$

Therefore, $Gr_F^p H^n(X_\infty, \mathbb{C})_\lambda = Gr_V^{\alpha+n-p} \Omega_f$. The V -filtration on Ω_f is defined by

$$V^\alpha \Omega_f = pr(V^\alpha \cap H'') \quad (3)$$

Clearly $V^\alpha \Omega_f = \bigoplus_{\beta \geq \alpha} \Omega_f^\beta$ and $\Omega_f \cong \bigoplus Gr_V^\alpha \Omega_f$ hold.

A theorem of A. Varchenko, relates the operator N , on vanishing cohomology and multiplication by f on Ω_f . More specifically, the maps $Gr(f)$ and $N = \log M_u \in End(H^n(X_\infty, \mathbb{C}))$ have the same Jordan normal forms. We have a canonical isomorphism

$$Gr_F H^n(X_\infty, \mathbb{C}) = \bigoplus_{-1 < \alpha \leq 0} Gr_F C^\alpha$$

and the corresponding endomorphism

$$\begin{aligned} N_{p,\alpha} : Gr_F^p C^\alpha &\rightarrow Gr_F^{p-1} C^\alpha \\ N_{p,\alpha}(x) &= -2\pi i(t\partial_t - \alpha)x \cong -2\pi i.t\partial_t x \quad (\text{mod } F^p). \end{aligned}$$

If $\beta \in \mathbb{Q}$, $\beta = n - p + \alpha$ with $p \in \mathbb{Z}$ and $-1 < \alpha \leq 0$, the map

$$\partial_t^{n-p} : V^\beta \cap F^n \mathcal{H}_{X,0} \rightarrow V^\alpha / V^{>\alpha} = C^\alpha$$

induces an isomorphism from $Gr_\beta^V \Omega_f \rightarrow Gr_F^p C^\alpha$, and the diagram

$$\begin{array}{ccc} Gr_\beta^V \Omega_f & \xrightarrow{Gr(f)} & Gr_{\beta+1}^V \Omega_f \\ \downarrow & & \downarrow \\ Gr_F^p C^\alpha & \xrightarrow{N_{p,\alpha}} & Gr_F^{p-1} C^\alpha \end{array} .$$

commutes up to a factor of $-2\pi i$. Hence $Gr(f)$ and $Gr_F N$ have the same Jordan normal form.

2. INTEGRALS ALONG LEFSCHETZ THIMBLES

The materials in this section are all well known, however we added this for the convenience in the notations used for the main theorem in sec. 5. Consider the function $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ with isolated singularity at 0, and a holomorphic

differential $(n + 1)$ -form ω given in a neighborhood of the critical point. We shall study the asymptotic behaviour of the integral,

$$\int_{\Gamma} e^{\tau f} \omega \quad (4)$$

for large values of the parameter τ , namely a complex oscillatory integral. Lets propose the same notation as sec. 1. In the long exact homology sequence of the pair (X, X_t) where X is a tubular neighborhood of the singular fiber X_0 in the Milnor ball,

$$\dots \rightarrow H_{n+1}(X) \rightarrow H_{n+1}(X, X_t) \xrightarrow{\partial_t} H_n(X_t) \rightarrow H_n(X) \rightarrow \dots \quad (5)$$

X is contractible. Therefore, we get an isomorphism $\partial_t : H_{n+1}(X, X_t) \cong H_n(X_t)$, and similar in cohomologies, i.e. $H^{n+1}(X, X_t) \cong H^n(X_t)$. If ω is a holomorphic differential $(n + 1)$ -form on X , and $\Gamma \in H_{n+1}(X, X_t)$, we have the following;

Proposition 2. (cf. [24] Theorems 8.6, 8.7, 8.8, 11.2) Assume $\omega \in \Omega^{n+1}$, and let $\Gamma \in H_n(X, X_t)$. Then

$$\int_{\Gamma} e^{-\tau f} \omega = \int_0^{\infty} e^{-t\tau} \int_{\Gamma \cap \{f=t\}} \frac{\omega}{df} \Big|_{X_t} dt = e^{\tau \cdot f(0)} \int_{\Gamma \cap \{f=t\}} \frac{\omega}{df} \Big|_{X_t} \quad (6)$$

for $Re(\tau)$ large, and in this way can also be expressed as $\sum \tau^{\alpha} \log \tau^k A_{\alpha,k}$ in that range.

By Proposition 2, we identify the cohomology classes $\int_{\Gamma} e^{-\tau f} \omega$ and $\int_{\Gamma \cap \{f=t\}} \frac{\omega}{df} \Big|_{X_t}$ via integration on the corresponding homology cycles. We can also choose Γ such that its intersection with each Milnor fiber has compact support, and its image under f is the positive real line, [12].

The asymptotic integral

$$I(\tau) = \int e^{\tau f} \phi dx_0 \dots dx_n, \quad \tau \rightarrow \pm \infty$$

satisfies

$$\frac{d^p}{d\tau^p} I = \int e^{\tau f} f^p \phi dx_0 \dots dx_n.$$

In case f is analytic then it has an asymptotic expansion

$$I(\tau) = \sum_{\alpha, p, q} c_{\alpha, p, q}(f) \tau^{\alpha - p} (\log \tau)^q, \quad \tau \rightarrow +\infty$$

for finite number of rational numbers $\alpha < 0$, $p \in \mathbb{N}$, $0 \leq q \leq n - 1$. Then $\phi \rightarrow c_{\alpha, p, q}(\phi)$ is a distribution with support contained in the support of f , [3].

Remark 1. (see [12] page 27) We have the formula;

$$I(\tau) = (2\pi)^{n/2} (\text{Hess} f)^{-1/2} f(0) \tau^{-n/2} [1 + O(1/\tau)].$$

It follows that the form $e^{-\tau f} \omega$ (for τ large enough) and the form $\frac{\omega}{df} |_{X_t}$, define the same cohomology class via integration on cycles.

Proposition 3. ([24] lemma 11.4, 12.2, and its corollary) *There exists a basis $\omega_1, \dots, \omega_\mu$ of Ω_f such that the corresponding Leray residues $\omega_1/df, \dots, \omega_\mu/df$ define a basis for the sections of vanishing cohomology.*

3. EXTENSION OF THE GAUSS-MANIN SYSTEM

As mentioned in the introduction the Gauss-Manin system of an isolated hypersurface singularity can be extended over the puncture by a process of gluing vector bundles. The gluing is done by some comparison between V -lattices and the Brieskorn lattices in \mathcal{G} . We are interested to understand the fiber of \mathcal{G} on 0 after the extension. Let $\Omega^{n+1}[\tau, \tau^{-1}]$ be the space of Laurent polynomials with coefficients in Ω^{n+1} . According to its very definition ([5] sec. 10.4, [20] lemma 2.4), the Gauss-Manin System is given by;

$$\mathcal{G} = \frac{\Omega^{n+1}[\tau, \tau^{-1}]}{(d - \tau df \wedge) \Omega^{n+1}[\tau, \tau^{-1}]},$$

$$(d - \tau df \wedge) \sum_k \eta_k \tau^k = \sum_k (d\eta_k - df \wedge \eta_{k-1}) \tau^k.$$

We put $\tau = t^{-1}$, and consider (τ, t) as coordinates on \mathbb{P}^1 . Then \mathcal{G} is a $\mathbb{C}[t, t^{-1}]$ -module with connection and $\partial_\tau = -t^2 \partial_t$. In the chart t , the Brieskorn lattice

$$\mathcal{H}^{(0)} := \text{image}(\Omega^{n+1}[\tau^{-1}] \rightarrow \mathcal{G}) = \frac{\Omega^{n+1}[t]}{(td - df \wedge) \Omega^{n+1}[t]}$$

is a free $\mathbb{C}[t]$ module of rank μ . It is stable by the action of $\partial_\tau = -t^2\partial_t$. Therefore ∂_t is a connection on \mathcal{G} with a pole of order 2. We consider the increasing exhaustive filtration $\mathcal{G}_p := \tau^p\mathcal{H}^{(0)}$ of \mathcal{G} .

In the chart τ , there are various natural lattices indexed by \mathbb{Q} , we denote them by V^α , with $V^{\alpha-1} = \tau V^\alpha$. On the quotient space $C_\alpha = V^\alpha/V^{>\alpha}$ there exists a nilpotent endomorphism $(\tau\partial_\tau - \alpha)$. The space $\bigoplus_{\alpha \in [0,1[} C_\alpha$ is isomorphic to $H^n(X_\infty, \mathbb{C})$, and $\bigoplus_{\alpha \in [0,1[} F^p C_\alpha$ is the limit MHS on $H^n(X_\infty, \mathbb{C})$. A basic isomorphism can be constructed, as

$$\begin{array}{c} \frac{\mathcal{G}_p \cap V^\alpha}{\mathcal{G}_{p-1} \cap V^\alpha + \mathcal{G}_p \cap V^{>\alpha}} = Gr_F^{n-p}(C_\alpha) \\ \downarrow \times t^p \\ \frac{V^{\alpha+p} \cap \mathcal{G}_0}{V^\alpha \cap \mathcal{G}_{-1} + V^{>\alpha} \cap \mathcal{G}_0} = Gr_{\alpha+p}^V(\mathcal{G}_0/\mathcal{G}_{-1}) \end{array} .$$

Thus, the gluing is done via the isomorphisms, $Gr_F^{n-p}(H_\lambda) \cong Gr_{\alpha+p}^V(\mathcal{H}^{(0)}/\tau^{-1}\mathcal{H}^{(0)})$ where $\lambda = \exp(2\pi i\alpha)$ and we have chosen $-1 \leq \alpha < 0$ (cf. [6], [7], [8]). We have

$$\frac{\mathcal{H}^{(0)}}{\tau^{-1}\mathcal{H}^{(0)}} = \frac{\Omega^{n+1}}{df \wedge \Omega^n} = \Omega_f \quad (7)$$

canonically. The identity (7) defines the extension fiber of the Gauss-Manin system of the isolated singularity $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$. The same conclusion can be obtained when f is a holomorphic germ. However one needs to consider the completions of the modules involved, (see [3] page 422 or [18]). In this way for $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ we have the same equation as (7). By identifying the sections of vanishing cohomology with sections of relative cohomology via Proposition 2, this formula is a direct consequence of the formula

$$\int_\Gamma e^{-\tau f} d\omega = \tau \int_\Gamma e^{-\tau f} df \wedge \omega, \quad \omega \in \Omega_X^n.$$

We refer to [3] page 422 for details.

4. MHS ON THE EXTENDED FIBER

Assume $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a germ of isolated singularity. Up to now we extended the local system of vanishing cohomology over the degeneracy point 0. Now our task is how to set a mixed Hodge structure on that. We build an isomorphism

$$\Phi : H^n(X_\infty, \mathbb{C}) \rightarrow \Omega_f$$

which allows us to equip a mixed Hodge structure on Ω_f , [4], [20]. This also motivates the definition of opposite filtrations of M. Saito. It is based on the following theorem.

Proposition 4. ([4] prop. 5.1) *Assume $\{(\alpha_i, d_i)\}$ is the spectrum of a germ of isolated singularity $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$. There exists elements $s_i \in C^{\alpha_i}$ with the properties*

- (1) s_1, \dots, s_μ project onto a \mathbb{C} -basis of $\bigoplus_{-1 < \alpha < n} Gr_V^\alpha H'' / Gr_V^\alpha \partial_t^{-1} H''$.
- (2) $s_{\mu+1} := 0$; there exists a map $\nu : \{1, \dots, \mu\} \rightarrow \{1, \dots, \mu, \mu+1\}$ with $(t - (\alpha_i + 1)\partial_t^{-1})s_i = s_{\nu(i)}$
- (3) There exists an involution $\kappa : \{1, \dots, \mu\} \rightarrow \{1, \dots, \mu\}$ with $\kappa(i) = \mu+1-i$ if $\alpha_i \neq \frac{1}{2}(n-1)$ and $\kappa(i) = \mu+1-i$ or $\kappa(i) = i$ if $\alpha_i = \frac{1}{2}(n-1)$, and

$$P_S(s_i, s_j) = \pm \delta_{(\mu+1-i)j} \cdot \partial_t^{-1-n}$$

where P_S is the K. Saito higher residue pairing.

Suppose,

$$H^n(X_\infty, \mathbb{C}) = \bigoplus_{p,q,\lambda} I_\lambda^{p,q}$$

is the Deligne-Hodge bigrading, and generalized eigen-spaces of vanishing cohomology, and also $\lambda = \exp(-2\pi i\alpha)$ with $\alpha \in (-1, 0]$. Consider the isomorphism obtained by composing the three maps,

$$\Phi_\lambda^{p,q} : I_\lambda^{p,q} \xrightarrow{\hat{\Phi}_\lambda} Gr_V^{\alpha+n-p} H'' \xrightarrow{pr} Gr_V^\bullet H'' / \partial_t^{-1} H'' \xrightarrow{\cong} \Omega_f \quad (8)$$

where

$$\Phi = \bigoplus_{p,q,\lambda} \Phi_\lambda^{p,q}, \quad \Phi_\lambda^{p,q} = pr \circ \hat{\Phi}_\lambda^{p,q}, \quad \hat{\Phi}_\lambda^{p,q} := \partial_t^{p-n} \circ \psi_\alpha | I_\lambda^{p,q}$$

Lemma: The map Φ is a well-defined \mathbb{C} -linear isomorphism.

We list some of the properties of the map ϕ as follows;

- $\hat{\Phi}_\lambda^{p,q}$ takes values in $C^{\alpha+n-p}$. By the formula $F^p = \bigoplus_{r \geq p} I^{r,s}$, any cohomology class in $I_\lambda^{p,q}$, is of the form $\psi_\alpha^{-1}[\partial_t^{n-p} h'' + V^{>\alpha}] = \psi_\alpha^{-1} \partial_t^{n-p} [h'' + V^{>\alpha+n-p}]$, for $h'' \in H''$, cf. def. 6.3.3. By substituting in the formula it explains the image of $\hat{\Phi}_\lambda^{p,q}$.

- Taking two different representatives $\omega_1, \omega_2 \in \Omega_X^{n+1}$ for h'' does not effect on the class $h'' + V^{>\alpha+n-p}$. Because by identifying H'' with its image in V^{-1} , the difference $\omega_1 - \omega_2$ belongs to $V^{>\alpha+n-p}$.
- The map Φ is obviously a \mathbb{C} -linear isomorphism because both of the ψ_α and ∂_t^{-1} are \mathbb{C} -linear isomorphisms on the appropriate domains, and

$$\Phi(I_\lambda^{p,q}) \subset \Phi(F^p H^n(X_\infty)_\lambda) \subset V^\alpha \cap H'' / V^{\alpha+1} \hookrightarrow Gr_V^\alpha H'' / \partial_t^{-1} H''$$

- The definition of Φ fixes an isomorphism $Gr_V^\bullet \Omega_f \cong \Omega_f$. On the eigen space H_λ this corresponds to a choice of sections of $Gr_V^\alpha(V^\alpha \cap H'') \rightarrow Gr_V^\alpha H'' / \partial_t^{-1} H''$ for $-1 \leq \alpha < 0$.

The mixed Hodge structure on Ω_f is defined by using the isomorphism Φ . This means that

$$W_k(\Omega_f) = \Phi W_k H^n(X_\infty, \mathbb{Q}), \quad F^p(\Omega_f) = \Phi F^p H^n(X_\infty, \mathbb{C})$$

and all the data of the Steenbrink MHS on $H^n(X_\infty, \mathbb{C})$ such as the \mathbb{Q} or \mathbb{R} -structure is transformed via the isomorphism Φ to that of Ω_f . Specifically; in this way we also obtain a conjugation map

$$\bar{\cdot} : \Omega_{f,\mathbb{Q}} \otimes \mathbb{C} \rightarrow \Omega_{f,\mathbb{Q}} \otimes \mathbb{C}, \quad \Omega_{f,\mathbb{Q}} := \Phi H^n(X_\infty, \mathbb{Q}) \tag{9}$$

defined from the conjugation on $H^n(X_\infty, \mathbb{C})$ via this isomorphism.

The basis discussed in 4 is usually called a good basis. The condition (1) correspond to the notion of opposite filtrations. Two filtrations F and U on H are called opposite (cf. [20] sec. 3) if

$$Gr_p^F Gr_q^U H = 0, \quad \text{for } p \neq q.$$

When one of the filtrations is decreasing say $\{F^p\}$ and the other increasing say $\{U_q\}$ then this is equivalent to

$$H = F^p \oplus U_{p-1}, \quad \forall p. \tag{10}$$

Similarly, two decreasing filtrations F and U are said to be opposite if F is opposite to the increasing filtration $U'_q := U^{k-q}$, [14]. Note that $V^\alpha H''$ is the submodule generated by $s(V^\alpha \Omega_f)$.

Proposition 5. ([20] prop. 3.5) *The filtration*

$$U^p C^\alpha := C^\alpha \cap V^{\alpha+p} H''$$

is opposite to the filtration Hodge filtration F on \mathcal{G} .

By the Proposition 5 the two filtrations F^p and

$$U'_q := U^{n-q} = \psi^{-1}\{\oplus_\alpha C^\alpha \cap V^{\alpha+n-q} H''\} = \psi^{-1}\{\oplus_\alpha Gr_V^\alpha[V^{\alpha+n-q} H'']\}$$

are two opposite filtrations on $H^n(X_\infty, \mathbb{C})$. We also have

$$F^p H^n(X_\infty, \mathbb{C})_\lambda \cong U'_p H^n(X_\infty, \mathbb{C}).$$

A standard example of such a situation is when the variation of MHS namely \mathcal{H} is mixed Tate (also called Hodge-Tate). By definition a mixed Tate Hodge structure H is when $Gr_{2l-1}^W H = 0$ and $Gr_{2l}^W H = \oplus_i \mathbb{Q}(-n_i)$. In that case one easily shows the Deligne-Hodge decomposition becomes

$$\bigoplus_p (W_{2p} \cap F^p) H = H_{\mathbb{C}}$$

and the two filtrations F and W are opposite.

In general

$$H'' \cap V^\alpha \mathcal{G} \neq (H'' \cap V^\alpha) + (H'' \cap V^{>\alpha})$$

This is why we have to take grading Gr_V^α .

The complex structure defined on Ω_f via $\Phi : H^n(X_\infty) \cong \Omega_f$ is not unique, and it depends to the good basis chosen. However it does not affect the polarization, discussed in the next section.

5. POLARIZATION FORM ON EXTENSION

This section concerns the main contribution. Assume $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a germ of isolated singularity. In Sec. 4 we described the extension of MHS associated to f over the degenerate point 0. In Sec. 5 we defined a MHS on the new fiber appearing over 0, namely Ω_f . The MHS was defined by the specific isomorphism Φ in (9). We use the isomorphism $\Phi : H^n(X_\infty, \mathbb{C}) \rightarrow \Omega_f$ introduced in the previous section to express a correspondence between polarization form on vanishing cohomology and the Grothendieck pairing on Ω_f .

Theorem 6. *Assume $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$, is a holomorphic germ with isolated singularity at 0. Then, the isomorphism Φ makes the following diagram commutative up to a complex constant;*

$$\begin{array}{ccccc} \widehat{Res}_{f,0} : & \Omega_f & \times & \Omega_f & \longrightarrow \mathbb{C} \\ & \downarrow & & \downarrow & \downarrow \times * \\ S : & H^n(X_\infty) & \times & H^n(X_\infty) & \longrightarrow \mathbb{C} \end{array} \quad * \neq 0. \quad (11)$$

where,

$$\widehat{Res}_{f,0} = res_{f,0} (\bullet, \tilde{C} \bullet)$$

and \tilde{C} is defined relative to the Deligne decomposition of Ω_f , via the isomorphism Φ . If $J^{p,q} = \Phi^{-1}I^{p,q}$ is the corresponding subspace of Ω_f , then

$$\Omega_f = \bigoplus_{p,q} J^{p,q} \quad \tilde{C}|_{J^{p,q}} = (-1)^p \quad (12)$$

In other words;

$$S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times res_{f,0}(\omega, \tilde{C}.\eta), \quad 0 \neq * \in \mathbb{C} \quad (13)$$

Part of this proof is given in [1] for homogeneous fibrations in the context of mirror symmetry, see also [12].

Proof. Before starting the proof lets mention that the map Φ is classically used to correspond the mixed Hodge structure on $H^n(X_\infty, \mathbb{C})$ and Ω_f . We only prove the correspondence on polarizations.

Step 1: Choose a \mathbb{C} -basis of the module $\Omega(f)$, namely $\{\phi_1, \dots, \phi_\mu\}$, where $\phi_i = f_i.d\mathbf{x}$. We identify the class $[e^{-f/t}\phi_i]$ with a cohomology class in $H(X_t)$. We may also choose the basis $\{\phi_i\}$ so that the forms $\{\eta_i = e^{(-f/t)}\phi_i\}$ correspond to a basis of vanishing cohomology, by the formula

$$\int_\Gamma e^{-\tau f} \omega = \int_0^\infty e^{-t\tau} \int_{\Gamma \cap X_t} \frac{\omega}{df} |_{X_t} \quad (14)$$

Step 2: In this step, we assume the Poincarè product is non-degenerate. By this assumption we may also assume f is homogeneous of degree d and ϕ_i 's are chosen by homogeneous basis of Ω_f . Consider the deformation

$$f_s = f + \sum_{i=0}^n s_i x_i$$

and set

$$S_{ij}(s, z) := \langle [e^{-f_s/z} \phi_i], [e^{+f_s/z} \phi_j] \rangle.$$

The cup product is the one on the relative cohomology, and we may consider it in the projective space \mathbb{P}^{n+1} . The perturbation f_s and also the Saito form S_{ij} are weighted homogeneous. This can be seen by choosing new weights, $\deg(x_i) = 1/d$, $\deg(s_i) = 1 - 1/d$, and $\deg(z) = 1$ then the invariance of the product with respect to the change of variable $x \rightarrow \lambda^{1/d}x$, $z \rightarrow \lambda z$, shows that $S_{ij}(s, z)$ is weighted homogeneous. We show that S_{ij} is some multiple of $\widehat{Res}_{f,0}$.

$$S_{ij}(s, z) := (-1)^{n(n+1)/2} (2\pi iz)^{n+1} (Res_f(\phi_i, \phi_j) + O(z)).$$

Suppose that s is generic so that $x \rightarrow Re(f_s/z)$ is a Morse function. Let $\Gamma_1^+, \dots, \Gamma_\mu^+$, (resp. $\Gamma_1^-, \dots, \Gamma_\mu^-$) denote the Lefschetz thimbles emanating from the critical points $\sigma_1, \dots, \sigma_\mu$ of $Re(f_s/t)$ given by the upward gradient flow (resp. downward). Choose an orientation so that $\Gamma_r^+ \cdot \Gamma_s^- = \delta_{rs}$. We have

$$S_{ij}(s, z) = \sum_{r=1}^{\mu} \left(\int_{\Gamma_r^+} e^{-f_s/z} \phi_i \right) \left(\int_{\Gamma_r^-} e^{f_s/z} \phi_j \right).$$

For a fixed argument of z we have the stationary phase expansion as $z \rightarrow 0$.

$$\left(\int_{\Gamma_r^+} e^{-f_s/z} \phi_i \right) \cong \pm \frac{(2\pi z)^{(n+1)/2}}{\sqrt{Hess f_s(\sigma_r)}} (f_i(\sigma_r) + O(z))$$

where $\phi_i = f_i(x) d\mathbf{x}$. Therefore,

$$S_{ij}(s, z) = (-1)^{n(n+1)/2} (2\pi iz)^{n+1} \sum_{r=1}^{\mu} \left(\frac{f_i(\sigma_r) f_j(\sigma_r)}{Hess(f_s)(\sigma_r)} + O(z) \right)$$

where the lowest order term in the right hand side equals the Grothendieck residue. As this holds for an arbitrary argument of z , and S_{ij} is holomorphic for $z \in \mathbb{C}^*$; the conclusion follows for generic s . By analytic continuation the same holds for all s . By homogeneity we get,

$$S_{ij}(0, z) = (-1)^{n(n+1)/2} (2\pi iz)^{n+1} \text{Res}_f(\phi_i, \phi_j). \quad (15)$$

Note that there appears a sign according to the orientations chosen for the integrals; however this only modifies the constant in the theorem. Thus, we have;

$$S_{ij}(0, 1) = (-1)^{n(n+1)/2} (2\pi i)^{n+1} \text{Res}_f(\phi_i, \phi_j). \quad (16)$$

Step 3: The sign appearing in residue pairing is caused by comparing the two products

$$(e^{-f} \phi_i, e^{-f} \phi_j), \quad (e^{-f} \phi_i, e^{+f} \phi_j). \quad (17)$$

Assume we embed the fibration in a projective one as before, replacing f with a homogeneous polynomial germ of degree d . We can consider a change of variable as $I : z \rightarrow e^{\pi i/d} z$ which changes f by $-f$. Thus this map is an involution on the value of f . The cohomology bases in $Gr_F^p Gr_{n+1}^W H^n \subset I^{p,q}$ can be characterized by the degree of forms, i.e. by the $\phi_i = f_i dz$, with f_i homogeneous, and $p \leq \text{deg}(\phi_i) = l(\phi_i) < p+1$. It follows that the cohomology class $e^{-f} \phi_j$ after the this change of variable is replaced by $c_p \cdot e^{+f} \phi_j$ where $c_p \in \mathbb{C}$ only depends to the Hodge filtration (defined by degree of forms). By the change of variable we obtain;

$$(e^{-f} \phi_i, I^* e^{-f} \phi_j) = (e^{-f} \phi_i, (-1)^{\text{deg} \phi_j/d} e^{+f} \phi_j) \quad (18)$$

because $I^d = id$, if we iterate I^* , d times we obtain;

$$(e^{-f} \phi_i, e^{-f} \phi_j) = \text{res}_{f,0}(a, (-1)^{(d-1) \text{deg} \phi_j/d} b)$$

The Riemann-Hodge bilinear relations in $H_{\neq 1}$ implies that, the products of the forms under consideration is non-zero except when the degrees of ϕ_i and ϕ_j sum to n . This explains the formula in $H_{\neq 1}$. The above argument will still hold when the form is replaced by $(\bullet, N_Y \bullet)$, by the linearity of N_Y . Thus, we still have the same result on $H_{\neq 1}$.

Step 4: In case the Poincarè product is degenerate, we still assume f is homogeneous but we change the cup product by applying N_Y on one component. The same relation can be proved between the level form $(\bullet, N_Y \bullet)$, and the corresponded local residue, i.e.

$$S_Y(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \widehat{\text{Res}}(a, \tilde{C}.b), \quad * \neq 0, \quad a, b \in \Omega_f$$

implies

$$S_Y(\Phi^{-1}(\omega), N_Y \cdot \Phi^{-1}(\eta)) = * \cdot \widehat{Res}(a, \mathfrak{f} \cdot \tilde{C} \cdot b), \quad * \neq 0, \quad a, b \in \Omega_f$$

where \mathfrak{f} is the nilpotent transformation corresponded to N_Y via Φ .

Step 5: By a standard theorem of J. Scherk, [2] page 38, [15], (see also the corollary in [2] sec. 3 page 37), continuity of Grothendieck residue, [22] page 657, after embedding of the Milnor fibration of f into that of f_Y , the Grothendieck pairing for f_Y is the prolongation of that of f .

Step 6: Until now we have proved the relation

$$S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times \text{res}_{f_Y, 0}(\omega, (-1)^{p(d-1)/d} \cdot \eta), \quad 0 \neq * \in \mathbb{C} \quad (19)$$

For some d and moreover, d can be as large as we like. Because the left hand side is independent of d , if we let $d \rightarrow \infty$ then by Step 5 we obtain (13). \square

Example 1. According to the above description the isomorphism Φ is as follows,

$$\Phi^{-1} : [z^i dz] \mapsto c_i \cdot [\text{res}_{f=1}(z^i dz / (f-1)^{l(i)})]$$

with $c_i \in \mathbb{C}$, and z^i in the basis mentioned above (see [1], Appendix A). For instance by taking $f = x^3 + y^4$, then as basis for Jacobi ring, we choose

$$z^i : 1, y, x, y^2, xy, xy^2$$

which correspond to top forms with degrees

$$l(i) : 7/12, 10/12, 11/12, 13/12, 14/12, 17/12$$

respectively. The above basis projects onto a basis

$$\bigoplus_{-1 < \alpha = l(i) - 1 < n} Gr_{\alpha}^V H'' \rightarrow Gr_V \Omega_f$$

as in Theorem 5.1. The Hodge filtration is explained as follows. First we have $h^{1,0} = h^{0,1} = 3$. Therefore, because Φ is an isomorphism.

$$\langle 1 \cdot \omega, y \cdot \omega, x \cdot \omega \rangle = \Omega_f^{0,1}, \quad \langle y^2 \cdot \omega, xy \cdot \omega, xy^2 \cdot \omega \rangle = \Omega_f^{1,0}$$

where $\omega = dx \wedge dy$, and the Hodge structure is pure, because $Gr_2^W H^n(X_{\infty}) = 0$.

$$\overline{\langle 1.dx \wedge dy, y.dx \wedge dy, x.dx \wedge dy \rangle} = \langle c_1.xy^2.dx \wedge dy, xy.dx \wedge dy, y^2.dx \wedge dy \rangle$$

Theorem 7. *Assume $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a holomorphic isolated singularity germ. The modified Grothendieck residue provides a polarization for the extended fiber Ω_f , via the aforementioned isomorphism Φ . Moreover, there exists a unique set of forms $\{\widehat{Res}_k\}$ polarizing the primitive subspaces of $Gr_k^W \Omega_f$ providing a graded polarization for Ω_f .*

Proof. Because $H^n(X_\infty)$ is graded polarized, hence using Theorem 6 Ω_f is also graded polarized via the isomorphism Φ . By the Mixed Hodge Metric theorem, the Deligne-Hodge decomposition;

$$\Omega_f = \bigoplus_{p,q} J^{p,q} \quad (20)$$

is graded polarized and there exists a unique Hermitian form; \mathcal{R} with,

$$i^{p-q} \mathcal{R}(v, \bar{v}) > 0, \quad v \in J^{p,q} \quad (21)$$

and the decomposition is orthogonal with respect to \mathcal{R} . Here the conjugation is that in (9). This shows that the polarization forms $\{\widehat{Res}_l\}$ are unique.

Let $N := \log M_u$ be the logarithm of the unipotent part of the monodromy for the Milnor fibration defined by f . We have

$$H^n(X_\infty) = \bigoplus_r N^r P_{l-2r}, \quad P_l := \ker N^{l+1} : Gr_l^W H^n \rightarrow Gr_{-l-2}^W H^n$$

and the level forms

$$S_l : P_l \otimes P_l \rightarrow \mathbb{C}, \quad S_l(u, v) := S(u, N^l v)$$

polarize the primitive subspaces P_l . By using the isomorphism Φ , similar type of decomposition exists for Ω_f . That is the isomorphic image $P'_l := \Phi^{-1} P_l$ satisfies

$$Gr_l^W \Omega_f = \bigoplus_r N^r P'_{l-2r}, \quad P'_l := \ker \mathfrak{f}^{l+1} : Gr_l^W \Omega_f \rightarrow Gr_{-l-2}^W \Omega_f$$

and the level forms

$$\widehat{Res}_l : P'_l \otimes P'_l \rightarrow \mathbb{C}, \quad \widehat{Res}_l := \widehat{Res}(u, \mathfrak{f}^l v)$$

polarize the primitive subspaces P'_l , where \mathfrak{f} is the map induced from multiplication by f on $Gr_l^W \Omega_f$. Specifically, this shows

- $\widehat{Res}_l(x, y) = 0, \quad x \in P'_r, y \in P'_s, r \neq s$
- $Const \times \widehat{Res}_l(C_l x, \mathfrak{f}^l \bar{x}) > 0, \quad 0 \neq x \in P'_l$

where C_l is the corresponding Weil operator.

□

Using this corollary and summing up all the material in sec. 4 and sec. 5, we can give the following picture for the extension of PVMHS associated to isolated hypersurface singularity.

Theorem 8. *Assume $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a holomorphic hypersurface germ with isolated singularity at $0 \in \mathbb{C}^{n+1}$. Then the variation of mixed Hodge structure, cf. [4]. This VMHS can be extended to the puncture with the extended fiber isomorphic to Ω_f in the sense of sec. 4 and 5, and it is polarized by Theorems 6 and 7. The Hodge filtration on the new fiber Ω_f corresponds to an opposite Hodge filtration on $H^n(X_\infty, \mathbb{C})$ in the way explained in Proposition 5.*

The Riemann-Hodge bilinear relations for the MHS on Ω_f and its polarization \widehat{Res} can be deduced from that of $(H^n(X_\infty), S)$.

Corollary 9. *(Riemann-Hodge bilinear relations for modified Grothendieck) Assume $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a holomorphic germ with isolated singularity. Suppose \mathfrak{f} is the corresponding map to N on $H^n(X_\infty)$, via the isomorphism Φ . Define*

$$P_l = PGr_l^W := \ker(\mathfrak{f}^{l+1} : Gr_l^W \Omega_f \rightarrow Gr_{-l-2}^W \Omega_f)$$

Going to W -graded pieces;

$$\widehat{Res}_l : PGr_l^W \Omega_f \otimes_{\mathbb{C}} PGr_l^W \Omega_f \rightarrow \mathbb{C} \quad (22)$$

is non-degenerate and according to Lefschetz decomposition

$$Gr_l^W \Omega_f = \bigoplus_r \mathfrak{f}^r P_{l-2r}$$

we will obtain a set of non-degenerate bilinear forms,

$$\widehat{Res}_l \circ (id \otimes \mathfrak{f}^l) : PGr_l^W \Omega_f \otimes_{\mathbb{C}} PGr_l^W \Omega_f \rightarrow \mathbb{C}, \quad (23)$$

$$\widehat{Res}_l = res_{f,0} (id \otimes \tilde{C}. \mathfrak{f}^l) \quad (24)$$

where \tilde{C} is as in Theorem 6, such that the corresponding Hermitian form associated to these bilinear forms is positive definite. In other words,

- $\widehat{Res}_l(x, y) = 0$, $x \in P_r, y \in P_s, r \neq s$
- If $x \neq 0$ in P_l ,

$$Const \times res_{f,0} (C_l x, \tilde{C}. \mathfrak{f}^l \bar{x}) > 0$$

where C_l is the corresponding Weil operator.

Proof. This follows directly from Theorems 6 and 7. □

Example 2. According to Example 1, if $f = x^3 + y^4$ for instance $\widehat{Res}(x.\omega, xy.\omega) = 0$, and

$$* \times \widehat{Res}(1.\omega, \overline{1.\omega}) > 0$$

$$* \times \widehat{Res}(x.\omega, \overline{x.\omega}) > 0$$

$$* \times \widehat{Res}(y.\omega, \overline{y.\omega}) > 0$$

simultaneously for one $* \in \mathbb{C}$.

The conjugation on the elementary sections of the Deligne extension is done by the following formulas

$$\psi_\alpha^{-1}(\overline{t^\alpha (\log t)^l A_{\alpha,l}}) = \begin{cases} \psi_{1-\alpha}^{-1}(t^{1-\alpha} (\log t)^{\nu-l} \bar{A}_{1-\alpha, \nu-l}), & \alpha \in]0, 1[\\ \psi_0^{-1}((\log t)^l \bar{A}_{0, \nu-l}), & \alpha = 0 \end{cases}$$

where ν is the size of the corresponding Jordan block. Regarding the map Φ defined in Sec. 4, the conjugation on Ω_f must satisfy similar relations. That is the conjugate of an element in $Gr_V^\alpha Gr_l^W \Omega_f$ is either in $Gr_V^{1-\alpha} Gr_{\nu-l}^W \Omega_f$ or $Gr_V^0 Gr_{\nu-l}^W \Omega_f$, in respective cases, such that the corresponding sections of vanishing cohomology satisfy the above.

Consider the map

$$F : \Omega_X^{n+1} \rightarrow i_* \bigcup_z Hom(H_n(X, f^{-1}(\eta. \frac{z}{|z|}), \mathbb{Z}) \cong \oplus_i \mathbb{Z} \Gamma_i, \mathbb{C}), \quad \mathcal{H} := Im(F)$$

$$\omega \mapsto [z \rightarrow (\Gamma_i \rightarrow \int_{\Gamma_i} e^{-t/z} \omega)],$$

where Γ_i are the classes of Lefschetz thimbles. The vector bundle \mathcal{H} is exactly the Fourier-Laplace transform of the cohomology bundle $R^n f_* \mathbb{C}_{S^*} = \cup_t H^n(X_t, \mathbb{C})$, equipped with a connection with poles of order at most two at ∞ .

$$(\cup_t H^n(X_t, \mathbb{C}), \nabla) \cong (\mathcal{H}, \nabla')$$

The modified Grothendieck residue

$$\widehat{Res}_{f,0} = res_{f,0}(\bullet, \tilde{C}\bullet)$$

with \tilde{C} as in Theorem 6, is the Fourier-Laplace transform of the polarization S on $H^n(X_\infty, \mathbb{C})$ ([23] page 53, 54, prop. 2.6 - [9] sec. 5).

6. ASYMPTOTIC HODGE THEORY

A variation of graded polarized \mathbb{C} -mixed Hodge structure is defined analogously having horizontality for F , and a collection of $(Gr_k^{\mathcal{W}}, \mathcal{F}Gr_k^{\mathcal{W}}, Q_k)$ of pure polarized \mathbb{C} -Hodge structures. One of the important facts in asymptotic Hodge theory and also Mirror symmetry is the following.

Theorem 10. (*P. Deligne*) *Let $\mathcal{V} \rightarrow \Delta^{*n}$ be a variation of pure polarized Hodge structure of weight k , for which the associated limiting mixed Hodge structure is Hodge-Tate. Then the Hodge filtration \mathcal{F} pairs with the shifted monodromy weight filtration $\mathcal{W}[-k]$, of \mathcal{V} , to define a Hodge-Tate variation over a neighborhood of 0 in Δ^{*n} .*

Theorem 11. ([14] *Theorem 3.28*) *Let \mathcal{V} be a variation of mixed Hodge structure, and*

$$\mathcal{V} = \bigoplus_{p,q} I^{p,q}$$

denotes the C^∞ -decomposition of \mathcal{V} to the sum of C^∞ -subbundles, defined by point-wise application of Deligne theorem. Then the Hodge filtration \mathcal{F} of \mathcal{V} pairs with the increasing filtration

$$\bar{U}_q = \sum_k \bar{\mathcal{F}}^{k-q} \cap \mathcal{W}_k \tag{25}$$

to define an un-polarized $\mathbb{C}VHS$.

Given a pair of increasing filtrations A and B of a vector space V one can define the convolution $A*B$ to be the increasing filtration $A*B = \sum_{r+s=q} A_r \cap B_s$.

Theorem 12. (G. Pearlstein-J. Fernandez)[14] *Let \mathcal{V} be an admissible variation of graded polarized mixed Hodge structures with quasi-unipotent monodromy, and $\mathcal{V} = \bigoplus I^{p,q}$ the decomposition relative to the limiting mixed Hodge structure. Define*

$$U'_p = \bigoplus_{a \leq p} I^{p,q} \tag{26}$$

and $\mathfrak{g}_- = \{\alpha \in \mathfrak{g}_{\mathbb{C}} \mid \alpha(U'_p) \subset U'_{p-1}\}$, then;

(a) U' is opposite to F_∞ . Moreover, relative to the decomposition

$$\mathfrak{g} = \bigoplus_{r,s} \mathfrak{g}^{r,s}. \tag{27}$$

(b) If $\psi(s) : \Delta^{*n} \rightarrow \check{D}$ is the associated untwisted period map, then in a neighborhood of the origin it admits a unique representation of the form

$$\psi(s) = e^{\Gamma(s)}.F_\infty \tag{28}$$

where $\Gamma(s)$ is a \mathfrak{g}_- -valued function.

(c) U' is independent of the coordinate chosen for F_∞ . Moreover,

$$U' = \overline{F_{nilp}^\vee} * W = \overline{F_\infty^\vee} * W. \tag{29}$$

Here above F_{nilp} is an arbitrary element in the nilpotent orbit of the limit Hodge filtration corresponded to the nilpotent cone (i.e. positive linear combination) of the logarithms of the generators of the monodromy group, i.e. $F_{nilp} = \exp(z_1 N_1 + \dots + z_r N_r) F_\infty$ where N_k are logarithms of different local monodromies, cf. [14].

Theorem 13. *Let \mathcal{V} be an admissible variation of polarized mixed Hodge structure associated to a holomorphic germ of an isolated hyper-surface singularity. Set*

$$U' = \overline{F_\infty^\vee} * W. \tag{30}$$

Then U' extends to a filtration \underline{U}' of \mathcal{V} by flat sub-bundles, which pairs with the limit Hodge filtration \mathcal{F} of \mathcal{V} , to define a polarized \mathbb{C} -variation of Hodge structure, on a neighborhood of the origin.

The polarization of a complex variation of Hodge structure will probably be interpreted to mean a parallel hermitian form which makes the system of Hodge bundles \mathcal{V}^p orthogonal, and becomes positive definite on multiplying the form by $(-1)^p$ on \mathcal{V}^p . Suppose that in the situation of Theorem 8.6 there is any such hermitian form \mathcal{R} . Then, on the one hand since \mathcal{R} and U' are flat, so is the orthogonal complement of U'_{p-1} in U'_p . On the other hand, the way things have been setup, the orthogonal complement of U'_{p-1} in U'_p is exactly

$$\mathcal{V}^p = U'_p \cap F^p. \quad (31)$$

But this is the system of Hodge bundles, and so the Hodge filtration is also flat. The above discussion also proves the following,

Corollary 14. *The mixed Hodge structure on the extended fiber Ω_f defined in sec. 4, can be identified with*

$$\Phi(U' = \overline{F_\infty^\vee} * W)$$

where Φ is as in (8).

Proof. The corollary is a consequence of Theorems 11, 12(c) and 13. \square

We end this section by a remark related to SL_2 -orbit Theorem. By a theorem of Kattani-Caplan-Schmid, [11], there exists a unique nilpotent transformation δ in the Lie algebra of the linear endomorphisms $End_{\mathbb{R}}(\Omega_f)^{-1,-1}$ preserving the weight filtration, s.t. $(W, e^{-2i\delta}.F)$ is a mixed Hodge structure splits over \mathbb{R} on Ω_f . Then,

$$\tilde{F} := e^{i\delta}.F.$$

Since $\delta \in \mathfrak{g}_{\mathbb{R}}^{-1,-1} \subset W_{-2}^{\mathfrak{gl}}$, this element leaves W invariant and acts trivially on the quotient Gr_l^W . Therefore both F, \tilde{F} induce the same filtrations on $Gr_l^W H$. The bigrading $J_1^{p,q}$ defined by $J_1^{p,q} := e^{-i\delta}.J^{p,q}$ is split over \mathbb{R} . The operator $\tilde{C}_1 = Ad(e^{-i\delta}).\tilde{C} : \Omega_f \rightarrow \Omega_f$ defines a real structure on Ω_f . If $\Omega_{f,1} = \bigoplus_{p < q} J_1^{p,q}$ then

$$\Omega_f = \Omega_{f,1} \oplus \overline{\Omega_{f,1}} \oplus \bigoplus_p J_1^{p,p}, \quad \overline{J_1^{p,p}} = J_1^{p,p}.$$

The statement of Theorem 6 is valid when the operator \tilde{C} is replaced with \tilde{C}_1 ;

$$S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times \text{res}_{f,0}(\omega, \tilde{C}_1.\eta), \quad 0 \neq * \in \mathbb{C} \quad (32)$$

and this equality is defined over \mathbb{R} . The real splitting $I_1^{q,p} = \overline{I_1^{p,q}}$ corresponds to a semisimple transformation $Y_1.v = (p + q - k).v$ for $v \in I_1^{p,q}$. Then the pair $\{Y_1, N\}$ can be completed to an \mathfrak{sl}_2 -triple $\{N_1^+, Y_1, N\}$. N_1^+ is real and $\delta, Y_1, N_1^+ \in \mathfrak{g}_{\mathbb{R}}$ are infinitesimal isometries of the polarization [11] page 477. This shows that Ω_f can be equipped with a MHS that is real split and is a sum of pure Hodge structures.

$$\Omega_f = \bigoplus_k \bigoplus_{p+q=k} J_1^{p,q}$$

and also an \mathfrak{sl}_2 -triple $\{f_1^+, Y_1', f\}$ as infinitesimal isometries of the bilinear form $\widehat{\text{Res}}_{f,0}$ which are morphisms of the MHS (F_1', W) explained above and are of types $(1, 1), (0, 0), (-1, -1)$ respectively.

7. APPLICATIONS

We provide two applications of the procedure explained in sec. 4, sec. 5, and sec. 6.

(1) Let C be a smooth projective curve over the field \mathbb{C} , and $J(C)$ its Jacobian. Then it is well known fact that the Poincarè duality of $H^1(C, \mathbb{C})$ can be identified with the polarization of $J(C)$, given by the Θ -divisor. We want to consider this situation in a family C_s parametrized by a 1-dimensional variety S . Suppose that

$$J^1(H_s^1) = H_{s,\mathbb{Z}}^1 \setminus H_{s,\mathbb{C}}^1 / F^0 H_{s,\mathbb{C}}^1$$

$$J(\mathcal{H}) = \bigcup_{s \in S^*} J^1(H_s)$$

is the family of Jacobians associated to the variation of Hodge structure in a projective degenerate family of algebraic curves (here we have assumed the Hodge structures have weight -1), and $\dim(S) = 1$. Then the fibers of this model are principally polarized abelian varieties. The polarization of each fiber is given by the Poincarè product of the middle cohomology of the curves, via a holomorphic family of Θ -divisors. We are going to apply the construction in sec. 4 and 5 to the variation of Jacobians.

Theorem 15. *Consider a projective family of smooth curves parametrized by the manifold S where $\dim(S) = 1$ and degenerate over an isolated point in S . Let the Jacobian bundle $J(\mathcal{H})$ be defined as above. Then $J(\mathcal{H})$ extends over the degeneracy point as the Jacobian of the Hodge structure on Ω_f , where f is the local holomorphic map defining the fibration.*

Proof. To extend $J(\mathcal{H})$ to a space over S , we let \mathcal{G} be the Gauss-Manin system on S^* , obtained from the variation \mathcal{H} . On S^* we have an extension of integral local classes

$$0 \rightarrow \mathcal{H}_s \rightarrow \mathcal{J}_s \rightarrow \mathbb{Z}_s \rightarrow 0. \quad (33)$$

On the Gauss-Manin systems we get

$$0 \rightarrow M \rightarrow N \rightarrow \mathbb{Q}_{S^*}^H[n] \rightarrow 0 \quad (34)$$

with $\mathbb{Q}_S^H[n]$ is the trivial module with sheaf of sections \mathcal{O}_{S^*} . The first and the last objects in the short exact sequences (33) and (34) extend to the punctures in a way that the extended fiber is polarized by modified Grothendieck pairing. The extended fiber of the Jacobian bundle is the Jacobian of the opposite Hodge filtration. In this way the extended fiber is an abelian variety and principally polarized, with some Θ -divisor. The extended Jacobian simply is

$$X_0 = J^1(\Omega_f) = \Omega_{f,\mathbb{Z}} \setminus \Omega_f / F^0 \Omega_f.$$

At the level of local systems (Hodge structures) we have the diagram of flat pairings,

$$\begin{array}{ccccccc} \kappa : & \mathcal{H} & \otimes & \mathcal{H} & \rightarrow & \mathbb{C} & \\ & \downarrow & & \downarrow & & & \\ \kappa_J : & J & \otimes & J & \rightarrow & \mathbb{C} & . \\ & \downarrow & & \downarrow & & & \\ \times : & \mathbb{Q} & \otimes & \mathbb{Q} & \rightarrow & \mathbb{C} & \end{array} \quad (35)$$

The extension of the first and the last provides an extension of the middle line. Similar non-degenerate bilinear forms can be defined on the Gauss-Manin modules, where the above diagram is its reduction on fibers;

$$\begin{array}{ccccccc} K : & \mathcal{G} & \otimes & \mathcal{G} & \rightarrow & \mathbb{C}[t, t^{-1}] & \\ & \downarrow & & \downarrow & & & \\ K_J : & N & \otimes & N & \rightarrow & \mathbb{C}[t, t^{-1}] & \\ & \downarrow & & \downarrow & & & \\ \times : & \mathbb{Q}_S^H & \otimes & \mathbb{Q}_S^H & \rightarrow & \mathbb{C}[t, t^{-1}] & \end{array} \quad (36)$$

where the map in the first line is the K. Saito higher residue pairing. □

(2) Hodge theory assigns to any polarized Hodge structure (H, F, S) a signature which is the signature of the hermitian form $S(C\bullet, \bar{\bullet})$, where C is the Weil operator. In case of a polarized mixed Hodge structure $(H, F, W(N), S)$, where N is a nilpotent operator this signature is defined to be the sum of the signatures of the hermitian forms associated to the graded polarizations $S_l : PGr_l^W H \times PGr_l^W H \rightarrow \mathbb{C}$, i.e signatures of $h_l := S_l(C_l\bullet, N^l\bar{\bullet})$ for all l . A basic example of this is the signature associated to mixed Hodge structure on the total cohomology of a compact Kahler manifold, namely Hodge index theorem. In this case the MHS is polarized by

$$S(u, v) = (-1)^{m(m-1)/2} \int_X u \wedge v \wedge \omega^{n-m}, \quad u, v \in H^m.$$

where ω is the Kahler class. The signature associated to the polarization S is calculated by W. Hodge in this case.

Theorem 16. (*W. Hodge*) *The signature associated to the polarized mixed Hodge structure of an even dimensional compact Kahler manifold is $\sum_{p,q} (-1)^q h^{p,q}$, where the sum runs over all the Hodge numbers, $h^{p,q}$. This signature is 0 when the dimension is odd.*

Similar definitions can be applied to polarized variation of mixed Hodge structure, according to the invariance of Hodge numbers in a variation of MHS. In the special case of isolated hypersurface singularities, the polarization form is given by $S_{\neq 1} \oplus S_1$ where

$$\begin{aligned} S_{\neq 1}(a, b) &= S_Y(i^*a, i^*b), & a, b \in H^n(X_\infty)_{\neq 1} \\ S_1(a, b) &= S_Y(i^*a, i^*N_Y b), & a, b \in H^n(X_\infty)_1. \end{aligned}$$

A repeating application of theorem 16 gives the following,

Theorem 17. [16] *The signature associated to the polarized variation of mixed Hodge structure of an isolated hypersurface singularity with even dimensional fibers is given by*

$$\sigma = \sum_{p+q=n+2} (-1)^q h_1^{p,q} + 2 \sum_{p+q \geq n+3} (-1)^q h_1^{p,q} + \sum_{\neq 1} (-1)^q h_{\neq 1}^{p,q} \quad (37)$$

where $h_1 = \dim H^n(X_\infty)_1$, $h_{\neq 1} = \dim H^n(X_\infty)_{\neq 1}$ are the corresponding Hodge numbers. This signature is 0 when the fibers have odd dimensions.

Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a germ of analytic function having an isolated singularity at the origin. Consider

$$A_f = \frac{\mathbb{C}[[x_0, \dots, x_n]]}{(\partial_0 f, \dots, \partial_n f)}.$$

By Jacobson-Morosov theorem, there exists a unique increasing filtration W_l on A_f (or A) such that

$$\times \bar{f} : Gr_l^W A \rightarrow Gr_{l-2}^W A, \quad \times \bar{f}^l : Gr_l^W A \cong Gr_{-l}^W A.$$

Define the primitive components $P_l = PGr_l^W A_f := \ker \bar{f}^{l+1} : Gr_l^W A \rightarrow Gr_{-l}^W A$. Then, we obtain a set of non-degenerate forms

$$Q_m : PGr_m^W A \times PGr_m^W A \rightarrow \mathbb{C}.$$

The mixed Hodge structure defined on Ω_f . We defined a MHS on Ω_f in sec. 4 and saw in sec. 5 and sec. 6 that it is polarized by the form \widehat{Res}_f in a way that the map Φ is an isomorphism of polarizations.

Theorem 18. *The signature associated to the modified Grothendieck pairing $\widehat{Res}_{f,0}$ associated to an isolated hypersurface singularity germ f ; is equal to the signature of the polarization form associated to the MHS of the vanishing cohomology, given by (37).*

Proof: Trivial by Theorems 6 and 7 and Theorem 17.

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