Journal of Prime Research in Mathematics Vol. **12***(2016), 01-15*

A NOTE ON SELF-DUAL AG-GROUPOIDS

AZIZ-UL-HAKIM[1], I. AHMAD[2,*] AND M. SHAH[3]

Abstract. In this paper, we enumerate self-dual AG-groupoids up to order 6, and classify them on the basis of commutativity and associativity. A self-dual AG-groupoid-test is introduced to check an arbitrary AG-groupoid for a self-dual AG-groupoid. We also respond to an open problem regarding cancellativity of an element in an AG-groupoid. Some features of ideals in self-dual AG-groupoids are explored. Some desired algebraic structures are constructed from the known ones subject to certain conditions and some subclasses of self-dual AG-groupoids are introduced.

Key words :AG-groupoid, enumeration, connected set, ideal, anti-rectangular AG-groupoid.
AMS SUBJECT : Primary 20N02, 20M12, 20N05.

1. Introduction

Kazim and Naseeruddin introduced AG-groupoid in 1972 and various properties of the structure were investigated in [9]. An AG-groupoid S is an algebraic structure satisfying the left invertive law $(ab)c = (cb)a$ for all a, b, c in S. The same structure has also been called left almost semigroup in [9]. A groupoid G is called a right AG-groupoid if it satisfies the right invertive law [19] $a(bc) = c(ba)$ for all a, b, c in G. The same structure has been called right almost semigroup in [9]. Every AG-groupoid S satisfies the medial law $(ab)(cd) = (ac)(bd)$ for all a, b, c, d in S [19]. An AG-groupoid S that satisfies the paramedial law $(ab)(cd) = (db)(ca)$ for all a, b, c, d in S, is called paramedial AG-groupoid [7, 5]. An AG-groupoid S that satisfies the identity $a(bc) = b(ac)$ for all a, b, c in S, is called AG**-groupoid [16]. If AG-groupoid S has left identity, then S is an AG**-groupoid [12]. An AG-groupoid S that satisfies any

[1,2]Department of Mathematics, University of Malakand, Chakdara Dir(L), Pakistan.
[1]Email: aziz.hakeem@hotmail.com, *Corresponding author's email: iahmaad@hotmail.com
[3]Department of Mathematics, Government Post Graduate College Mardan, Pakistan.
shahmaths_problem@hotmail.com.

of the weak associative laws $(ab)c = b(ac)$ or $(ab)c = b(ca)$ for all a, b, c in S, is called AG*-groupoid [5]. An AG*-groupoid with left identity becomes a commutative semigroup [17]. An element a of an AG-groupoid S is called left cancellative (right cancellative) if $ax = ay \Rightarrow x = y$ ($xa = ya \Rightarrow x = y$) for all $x, y \in S$ [5]. It is easy to prove that every right cancellative element is left cancellative [19]. An AG-groupoid S that satisfies the identity $(ba)b = a$ for all a, b in S, is called anti-rectangular AG-groupoid [15]. An AG-groupoid S is called Type-1 AG-groupoid (T^1-AG-groupoid) if $ab = cd \Rightarrow ba = dc$ for all a, b, c, d in S [5]. An AG- groupoid S is called a Type-2 AG-groupoid (T^2-AG-groupoid) if $ab = cd \Rightarrow ac = bd$ for all a, b, c, d in S [19]. An AG-groupoid S is called a forward Type-4 AG-groupoid (T^4_f-AG-groupoid) if $ab = cd \Rightarrow ad = cb$ for all a, b, c, d in S [21]. An AG-groupoid S is called a backward Type-4 AG-groupoid (T^4_b-AG-groupoid) if $ab = cd \Rightarrow da = bc$ for all a, b, c, d in S [21]. An AG-groupoid S is called a Type-4 AG-groupoid (T^4-AG-groupoid) if it is both a T^4_f-AG-groupoid and a T^4_b-AG-groupoid.

After the exposure of the structure of AG-groupoid, researchers took a keen interest in the field and as a result many notions such as AG-group, ideals, LA-ring, almost field, Γ -AG-groupoid, inflations, zeroids, and idempoids were introduced [1, 10, 11, 13, 22, 23]. Recently, the notions of fuzzy AG-subgroup, and modulo AG-group were introduced and some usefull results were established in [2, 3, 4]. Also, a variety of classes of AG-groupoids were introduced in [19]. These classes were studied with various angles and many interesting results were derived [18, 20, 21]. It is worth mentioning here that the structure of AG-groupoid has its applications in various fields like theory of flocks [9], geometry [19], Matrix theory [3], and Topology [15].

The notion of almost semigroup (a groupoid G in which both left and right invertive laws hold) was first inducted in [9]. The same structure was called self-dual AG-groupoid in [19], but not studied in detail. We study and investigate some features of the structure in this very note.

The article contains various sections. In section 2, self-dual AG-groupoids are enumerated and classified. A self-dual AG-groupoid-test is introduced in section 3. We devote section 4 to respond to an open problem in connection with cancellativity of an element in an AG-groupoid, proposed in [19]. We discuss ideals in self-dual AG-groupoids to some extent, hence section 5 is reserved for the purpose. In section 6, we discuss connected sets in self-dual AG-groupoids. Some algebraic structures are constructed from the known ones in section 7. We introduce some subclasses of self-dual AG-groupoid in section 8.

To begin, we recall the definition of a self-dual AG-groupoid and give its example in the form of a Cayley's table.

Definition. [5] An AG-groupoid S that satisfies right invertive law is called self-dual AG-groupoid.

Example. $(S, \cdot)$ is a self-dual AG-groupoid with the following Cayley table.

$\cdot$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

2. Enumeration and classification of self-dual AG-groupoids

Enumeration and classification of algebraic structures is a well worked area of research in pure mathematics and indeed it plays a vital role in the development of an algebraic structure. Various classes of AG-groupoids have been enumerated up to order 6 [8]. In the same fashion, we also enumerate and classify self-dual AG-groupoids up to order 6, using GAP package AGGROUPOIDS as below.

Table 2.1 shows the enumeration of self-dual AG-groupoids up to order 6 and their classification on the basis of associativity and commutativity.

Order	3	4	5	6
Total AG-groupoids	20	331	31913	40104513
Total self-dual AG-groupoids	12	70	579	17273
Associative & commutative	12	58	325	7510
Associative & non-commutative	0	4	121	5367
Non-associative & non-commutative	0	8	133	4396

Table 2.1: Enumeration, classification of self-dual AG-groupoids

3. Self-dual AG-groupoid-Test

In this section, we provide a method that helps check an arbitrary AG-groupoid for self-dual AG-groupoid. We also illustrate the method with examples.

Let $(S, \cdot)$ be an arbitrary finite AG-groupoid with a Cayley table. In order to test whether $(S, \cdot)$ is a self-dual AG-groupoid, for all $a, b \in S$, we define the binary operations:

$$\begin{aligned} a \otimes b &= a(bx), \\ a \times b &= x(ba), \end{aligned}$$

for some fixed $x \in S$. To test whether $(S, \cdot)$ is a self-dual AG-groupoid, it is enough to check if the operations $\otimes$ and $\times$ coincide for all $x \in S$. We construct the table of the operation $\otimes$ for any fixed $x \in S$ by rewriting x-column of the "$\cdot$" table as an index row of the new table and by multiplying its elements with the index column of the "$\cdot$" table. The table of the operation $\times$ for any fixed $x \in S$ can be obtained by multiplying elements of the "$\cdot$" table row wise from the left by x to write the columns of the new table for each x. It is convenient to write the $\times$ tables under the $\otimes$ tables. Then it is checked whether the upper table coincide with the lower one.

Example 1. *Let $(S, \cdot)$ be an AG-groupoid given by the following Cayley table:*

$\cdot$	a	b	c	d
a	a	c	d	b
b	d	b	a	c
c	b	d	c	a
d	c	a	b	d

Table 3.1.

To check whether $(S, \cdot)$ is a self-dual AG-groupoid, we extend Table 3.1 in the way explained earlier to get the following table:

$\cdot$	a	b	c	d	a	d	b	c	c	b	d	a	d	a	c	b	b	c	a	d
a	a	c	d	b	a	b	c	d	d	c	b	a	b	a	d	c	c	d	a	b
b	d	b	a	c	d	c	b	a	a	b	c	d	c	d	a	b	b	a	d	c
c	b	d	c	a	b	a	d	c	c	d	a	b	a	b	c	d	d	c	b	a
d	c	a	b	d	c	d	a	b	b	a	d	c	d	c	b	a	a	b	c	d
					a	b	c	d	d	c	b	a	b	a	d	c	c	d	a	b
					d	c	b	a	a	b	c	d	c	d	a	b	b	a	d	c
					b	a	d	c	c	d	a	b	a	b	c	d	d	c	b	a
					c	d	a	b	b	a	d	c	d	c	b	a	a	b	c	d

Table 3.2.

It is obvious that $\otimes$ and $\times$ tables coincide for every $x \in S$. Thus $(S, \cdot)$ is a self-dual AG-groupoid.

Example 2. *Let $(S, \cdot)$ be an AG-groupoid with the following Cayley table:*

$\cdot$	a	b	c
a	b	c	a
b	a	b	c
c	c	a	b

Table 3.3.

To test whether AG-groupoid S shown by Table 3.3 is a self-dual AG-groupoid, we extend the table in the way explained earlier and obtain, for the binary operations $\otimes$ and $\times$, the following table.

·	a	b	c	b	a	c	c	b	a	a	c	b
a	b	c	a	c	b	a	a	c	b	b	a	c
b	a	b	c	b	a	c	c	b	a	a	c	b
c	c	a	b	a	c	b	b	a	c	c	b	a
				c	b	a	b	a	c	a	c	b
				a	c	b	c	b	a	b	a	c
				b	a	c	a	c	b	c	b	a

It is evident that the $\times$ table doesn't coincide with the $\otimes$ table, hence S is not a self-dual AG-groupoid.

4. A partial solution to an open problem

In this section, we respond to an open problem in connection with cancellativity of an element in an AG-groupoid. We find out a partial solution to the problem and prove that every left cancellative element of a self-dual AG-groupoid is right cancellative. To begin, we restate the following theorem.

Theorem 1. [19] *Every right cancellative element of an AG-groupoid S is left cancellative.*

Generally, the converse of the above theorem is not true. Thus, in his Ph.D thesis, M. Shah proposed the open problem: prove or disprove that in an AG-groupoid S, without left identity, every left cancellative element is right cancellative. In [19], the problem has partially been resolved: (a) In an AG-groupoid S, with left identity e, every left cancellative element is right cancellative, (b) In an AG-groupoid S, a left cancellative element a is right cancellative if any of the following holds; (i) If a^2 is left cancellative, (ii) If a is idempotent, (iii) If there exists a left nuclear left cancellative element in S. The converse of the problem has been proved for AG*-groupoid, i.e., every left cancellative element of an AG*-groupoid is right cancellative [19]. For AG**-groupoid, the converse has also been proved, i.e., every left cancellative element of an AG**-groupoid is right cancellative [19]. We claim that the converse of Theorem 1 holds for a self-dual AG-groupoid as well. Thus, we have the following theorem.

Theorem 2. *Every left cancellative element of a self-dual AG-groupoid is right cancellative.*

Proof. Let S be a self-dual AG-groupoid and a an arbitrary left cancellative element of S. To show that a is right cancellative, let $xa = ya$ for any $x, y \in S$. Then, using left invertive, right invertive and medial laws, we have

$$\begin{aligned} a(a(ax)) &= (ax)(aa) = (aa)(xa) \\ &= (aa)(ya) = (ay)(aa) = a(a(ay)), \end{aligned}$$

which by repeated use of left cancellativity of a implies that $x = y$. Hence a is right cancellative. The proof is complete. □

Theorem 3. *Every cancellative element of a self-dual AG-groupoid is the product of two cancellative elements.*

Proof. Let S be a self-dual AG-groupoid and a an arbitrary left cancellative element of S. Let $a = a_1a_2$, where a_1 and a_2 are any arbitrary elements of S. We show that a_1 and a_2 are cancellative. Let $xa_2 = ya_2$ for any x, y in S. Then, we have

$$\begin{aligned} ax &= (a_1a_2)x = (xa_2)a_1 \\ &= (ya_2)a_1 = (a_1a_2)y = ay, \end{aligned}$$

which by left cancellativity of a implies that $x = y$. Thus a_2 is right cancellative and hence cancellative by Theorem 1 . Now suppose $a_1x = a_1y$ for any x, y in S. Then, we have

$$\begin{aligned} a(xa_2) &= (a_1a_2)(xa_2) = (a_1x)(a_2a_2) \\ &= (a_1y)(a_2a_2) = (a_1a_2)(ya_2) = a(ya_2), \end{aligned}$$

which by left cancellativity of a and right cancellativity of a_2 implies that $x = y$. This shows that a_1 is left cancellative, hence it is cancellative by Theorem 2. Thus a_1 and a_2 are cancellative elements. The proof is complete. □

Theorem 4. *Let S be a self-dual AG-groupoid and k any fixed element of S such that $ak = ka$ and $bk = kb$ for some a, b in S. If k is cancellative element, then a, b commute.*

Proof. Let $a, b \in S$ such that $ak = ka$ and $bk = kb$. First, assume that k is left cancellative. Then, we have $k(ab) = b(ak) = b(ka) = a(kb) = a(bk) = k(ba)$, which by left cancellativity of k implies that $ab = ba$.

Now, suppose that k is right cancellative. Then, we have $(ab)k = (kb)a = (bk)a = (ak)b = (ka)b = (ba)k$, which by right cancellativity of k implies that $ab = ba$. Hence the theorem follows. □

5. Ideals in self-dual AG-groupoids

Here, ideals are dealt with in self-dual AG-groupoids.

Definition 1. [14] *A subset I of an AG-groupoid S is called a left ideal (right ideal), if $SI \subseteq I$ ($IS \subseteq I$). A subset I of an AG-groupoid S is called ideal if it is both a left and a right ideal of S. .*

In [6], the authors have defined left commutative, right commutative and bi-commutative groupoids. We induct the same notions here.

Definition 2. *An AG-groupoid S is said to be left commutative AG-groupoid if and only if $(xy)z = (yx)z$ for any $x, y, z \in S$. Similarly, an AG-groupoid S is said to be right commutative AG-groupoid if and only if $x(yz) = x(zy)$ for any $x, y, z \in S$. An AG-groupoid is said to be bi-commutative if it is both left and right commutative.*

Theorem 5. *Any subset of a self-dual AG-groupoid S with a cancellative element is left ideal if and only if it is right ideal.*

Proof. Let I be a left ideal of S. Then, using right invertive law and the facts $SI \subseteq I$, $SS = S$, we have $IS = I(SS) = S(SI) \subseteq SI \subseteq I$. Thus I is a right ideal of S. Conversely, suppose that I be a right ideal of S. Then, using left invertive law and the facts $IS \subseteq I$, $SS = S$, we have $SI = (SS)I = (IS)S \subseteq IS \subseteq I$. Thus I is a left ideal of S. The proof is complete. □

Theorem 6. *If I is a right ideal of a self-dual-AG-groupoid S , then I^2 is a left ideal of S.*

Proof. Let I be a right ideal of S. Then, we have $SI^2 = S(II) = I(IS) \subseteq II = I^2$. This implies that I^2 is a left ideal of S. Hence, the theorem follows. □

Theorem 7. *If A is an ideal of a self-dual AG-groupoid S, then A^2 is an ideal of S.*

Proof. Let A be an ideal of S. Then A^2 is a left ideal of S by Theorem 6. Next, we have $(AA)S = (SA)A \subseteq AA = A^2$. This implies that A^2 is a right ideal of S. Hence A^2 is an ideal of S, being a left and a right ideal of S. □

Theorem 8. *Let S be a self-dual-bi-commutative AG-groupoid. Then I^2 is a left ideal of S if any of the following assertions holds:*

(1) I is a left ideals of S.
(2) I is a right ideals of S.

Proof. Straightforward. □

Theorem 9. *Let S be a self-dual-AG-groupoid and B an ideal of S. If A is a subset of S such that $SA = AS$, then AB is a left ideal of S.*

Proof. If A be a subset of S and B an ideal. Then, we have $S(AB) = B(AS) = B(SA) = A(SB) \subseteq AB$. Thus AB is a left ideal of S. The proof is complete. □

Theorem 10. *Let S be a self-dual-right commutative AG-groupoid. Then AB is an ideal of S if any of the following assertions holds:*

(1) A and B are left ideals of S.
(2) A and B are right ideals of S.

Proof. Straightforward. ☐

Theorem 11. *Let A be a subset of a self-dual-AG^*-groupoid S. Then AB is an ideal of S if B is an ideal of S.*

Proof. Easy. ☐

Theorem 12. *Let S be a self-dual-T^1-AG-groupoid and B a non-empty subset of S. Then AB is a right ideal of S if A is a left ideal of S.*

Proof. Let A be a left ideal of S. Then, we have

$$\begin{aligned} S(AB) &= B(AS) \Rightarrow (AB)S = (AS)B = (BS)A \\ \Rightarrow S(AB) &= A(BS) = S(BA) \\ \Rightarrow (AB)S &= (BA)S = (SA)B \subseteq AB. \end{aligned}$$

Thus AB is a right ideal of S. The proof is complete. ☐

Theorem 13. *Let S be a self-dual-AG^{**}-groupoid and A a non-empty subset of S. Then AB is a left ideal of S if B is a right ideal of S.*

Proof. Let B is a right ideal of S. Then, we have $S(AB) = B(AS) = A(BS) \subseteq AB$. Hence AB is a left ideal of S. The proof is complete. ☐

Theorem 14. *If I is a right ideal of a self-dual-AG^{**}-groupoid S, then aI is a left ideal of S for any $a \in S$.*

Proof. Let I be a right ideal of S. Then, for an arbitrary element $a \in S$, we have $S(aI) = I(aS) = a(IS) \subseteq aI$. This implies that aI is a left ideal of S for any $a \in S$. The proof of the theorem is complete. ☐

Theorem 15. *If I is a right ideal of a self-dual-AG^*-groupoid S, then Ia is a right ideal of S for any $a \in S$.*

Proof. Let I be a right ideal of S. Then, for an arbitrary element $a \in S$, we have $(Ia)S = a(IS) = S(Ia) = (IS)a \subseteq Ia$. This implies that Ia is a right ideal of S for any $a \in S$. The proof is complete. ☐

Theorem 16. *Let A be a right ideal of a self-dual-AG^{**}-groupoid S. If B is an ideal of S such that $SB = BS$, then AB is an ideal of S.*

Proof. Let B be an ideal of S such that $SB = BS$. Then, we have

$$S(AB) = B(AS) = A(BS) \subseteq AB,$$

which implies that AB is a left ideal of S. Now, we have

$$(AB)S = (SB)A = (BS)A = (AS)B \subseteq AB,$$

which implies that AB is a right ideal of S. Hence AB is an ideal of S, being a left and a right ideal of S. ☐

Theorem 17. *Let A be a subset of a self-dual-AG^*-groupoid S. Then AB is an ideal of S if B is a left ideal of S.*

Proof. Straightforward. □

Theorem 18. *Let S be a self-dual-T^1-AG-groupoid and A a subset of S. Then AB is a left ideal of S if B is a right ideal of S.*

Proof. Let A be a subset of S and B a right ideal ideal. Then, we have $S(AB) = B(AS) \Rightarrow (AB)S = (AS)B = (BS)A \Rightarrow S(AB) = A(BS) \subseteq AB$. Thus AB is a left ideal of S. The proof is complete. □

Theorem 19. *Let S be a self-dual-T^1-AG-groupoid. If A be a left and B a right ideal of S, then AB is an ideal of S.*

Proof. Let A be a left and B a right ideal of S. Then, we have

$$\begin{aligned} (AB)S &= (SB)A \Rightarrow S(AB) = A(SB) = B(SA) \\ \Rightarrow (AB)S &= (SA)B \subseteq AB. \end{aligned}$$

Thus AB is a right ideal of S. Similarly, we have

$$\begin{aligned} S(AB) &= B(AS) \Rightarrow (AB)S = (AS)B = (BS)A \\ \Rightarrow S(AB) &= A(BS) \subseteq AB. \end{aligned}$$

Thus AB is a left ideal of S. Hence AB is an ideal of S, being a left and a right ideal of S. The proof is complete. □

Theorem 20. *Let S be a self-dual-T^1-AG-groupoid. If A be a right and B a left ideal of S, then AB is an ideal of S.*

Proof. Let A be a right and B a left ideal of S. Then, we have

$$(AB)S = (SB)A \Rightarrow S(AB) = A(SB) \subseteq AB.$$

Thus AB is a left ideal of S. Similarly, we have

$$S(AB) = B(AS) \Rightarrow (AB)S = (AS)B \subseteq AB.$$

Thus AB is a right ideal of S. Hence AB is an ideal of S, being a left and a right ideal of S. The proof is complete. □

Theorem 21. *Let S be a self-dual-T^1-AG-groupoid. Then AB is an ideal of S if any of the following assertions holds:*

(1) A and B are left ideals of S.
(2) A and B are right ideals of S.

Proof. Easy. □

6. Connected sets in self-dual AG-groupoids

In this section, we discuss connected sets in self-dual AG-groupoids. To begin, we recall the following definition.

Definition 3. [14] *Two subsets A and B of an AG-groupoid S are called right (left) connected, if $AS \subseteq B$ and $BS \subseteq A$ (resp. $SA \subseteq B$, $SB \subseteq A$). A and B are connected if they are both left and right connected.*

Theorem 22. *If A and B are ideals of a self-dual AG-groupoid, then AB and BA are connected sets.*

Proof. Easy. □

Theorem 23. *If A and B are left ideals of a self-dual-right commutative AG-groupoid S, then AB and BA are connected sets.*

Proof. Let A, B be left ideals of S. Then, we have

$$(AB)S = (SB)A \subseteq BA.$$

Similarly, we have

$$(BA)S = (SA)B \subseteq AB.$$

Thus AB and BA are left connected. Next, we have

$$S(AB) = B(AS) = B(SA) \subseteq BA.$$

Similarly, we have

$$S(BA) = A(BS) = A(SB) \subseteq AB.$$

Thus AB and BA are right connected. Hence AB and BA are connected, being left and right connected. □

Theorem 24. *If A and B are right ideals of a self-dual-left commutative AG-groupoid S, then AB and BA are connected sets.*

Proof. Similar to Theorem 23 □

Theorem 25. *For any ideals A and B of a self-dual AG-groupoid S, the following assertions hold.*

(1) A^2B^2 and B^2A^2 are connected.
(2) $A^2(A^2B^2)$ and $(A^2A^2)B^2$ are connected.

Proof. Easy. □

Theorem 26. *Let B be a subset of a self-dual-AG*-groupoid S. If A is an ideal of S, then AB and BA are connected.*

Proof. Let A be an ideal and B a subset of S. Then, we have

$$S(AB) = B(AS) \subseteq BA.$$

Similarly, we have

$$S(BA) = A(BS) = (BA)S = (SA)B \subseteq AB.$$

This implies that AB and BA are left connected.

Next, we have

$$(AB)S = B(AS) \subseteq BA.$$

Similarly, we have

$$(BA)S = (SA)B \subseteq AB.$$

This implies that AB and BA are right connected. Hence AB and BA are connected, being left and right connected. □

7. Self-dual AG-groupoids and construction of some desired algebraic structures

Construction of Algebraic structure has been always important as well as a challenging task. Here, we show that how and under what conditions some specific structures can be obtained from the known ones.

Theorem 27. *Let a be a fixed idempotent element of a self-dual AG-groupoid S.Then the set $Q = \{ x \epsilon S : ax = x \}$ is a subsemigroup of S.*

Proof. Since $a = aa$, we have $a \in Q$. This implies that Q is non-empty. For arbitrary elements $x, y \in Q$, we have $xy = (ax)(ay) = (aa)(xy) = a(xy)$. This implies that Q is closed. Also left invertive law holds in Q as it holds in S. Thus Q is an AG-subgroupoid of S. Now, it remains to show that Q is a semigroup. For this, let $x, y \in Q$. Then, we have $xy = x(ay) = y(ax)$ $= yx$. Thus Q is commutative, so it is associative as commutativity of an AG-groupoid implies associativity. Hence Q is a subsemigroup of S. □

Theorem 28. *Let $(S, \cdot)$ be a self-dual AG-groupoid. Define $(*)$ as $x * y = y(px)$ for a fixed $p \in S$. Then $(S, *)$ is commutative medial.*

Proof. Let $a, b, c, d \in S$. Then, we have $a * b = b(pa) = a(pb) = b * a$. Next, we have

$$\begin{aligned}(a * b) * (c * d) &= (d \cdot pc)\,[p(b \cdot pa)] = (d \cdot pc)(pa \cdot bp) \\ &= (d \cdot pc)(pb \cdot ap) = (d \cdot pb)(pc \cdot ap) \\ &= (d \cdot pb)(pa \cdot cp) = (d \cdot pb)\,[p(c \cdot pa)] \\ &= (a * c) * (b * d).\end{aligned}$$

Thus $(S, *)$ is commutative medial. The proof is complete. □

Theorem 29. *Let $(S, \cdot)$ be a self-dual-right commutative AG-groupoid. Define $(*)$ as $x * y = y(px)$ for a fixed $p \in S$. Then $(S, *)$ is a commutative semigroup.*

Proof. Let $x, y, z \in S$. Then, by definition of $(*)$, we have $x * y = y(px) = x(py) = y * x$. Thus $(S, *)$ is commutative. Next, we have

$$\begin{aligned}(x * y) * z &= z\,[p(y \cdot px)] = z\,[p(x \cdot py)] \\ &= (x \cdot py)(pz) = (x \cdot py)(zp) \\ &= p\,[z(x \cdot py)] = p\,[z(py \cdot x)] \\ &= p\,[x(py \cdot z)] = p\,[x(z \cdot py)] \\ &= (z \cdot py)(xp) = (z \cdot py)(px) \\ &= x * (y * z).\end{aligned}$$

Thus $(S, *)$ is a commutative semigroup. The proof is complete. □

Theorem 30. *Let $(S, \cdot)$ be a self-dual-paramedial AG-groupoid. Define $(*)$ on S as $x * y = x(py)$ for a fixed $p \in S$. Then $(S, *)$ is commutative semigroup.*

Proof. Let $x, y, z \in S$. Then, we have

$$\begin{aligned}(x * y) * z &= (x \cdot py)(pz) = (z \cdot py)(px) \\ &= x\,[p(z \cdot py)] = x(py \cdot zp) \\ &= x(pz \cdot yp) = x\,[p(y \cdot pz)] \\ &= x * (y * z).\end{aligned}$$

Thus $(S, *)$ is semigroup. Now, using definition of $(*)$ and right invertive law, we have $x * y = x \cdot py = y \cdot px = y * x$. Hence $(S, *)$ is commutative semigroup. The proof is complete. □

8. Discovery of subclasses of self-dual AG-groupoids

Self-dual AG-groupoids possess a dual nature, satisfying both left and right invertive laws, which is its beauty. Hence, it is highly desired and interesting to discover subclasses of self-dual AG-groupoids.

Theorem 31. *Every anti-rectangular AG-groupoid is a self-dual AG-groupoid.*

Proof. Let S be an anti-rectangular AG-groupoid and $a, b, c \in S$. Then, using definition of anti-rectangular AG-groupoid and left invertive law, we have $a(bc) = (ba \cdot b)(bc) = (bc \cdot b)(ba) = c(ba)$. This shows that S satisfies right invertive law. Thus S is a self-dual AG-groupoid. The proof is complete. □

However, the converse of the above theorem is not true. The following counterexample confirms the claim.

Example 3. *A self-dual AG-groupoid S has been shown by the following table which is not an anti-rectangular AG-groupoid.*

$\cdot$	1	2	3	4
1	2	4	3	1
2	3	1	2	4
3	1	3	4	2
4	4	2	1	3

Here, we have $(3 \cdot 4)3 \neq 4$, hence S is not anti-rectangular.

Theorem 32. *An AG-groupoid S is a self-dual AG-groupoid if, for all a, b, c, d in S, the following identity holds:*

$$(ab)(cd) \quad = \quad a(bc) \tag{1}$$

Proof. Let $a, b, c, d \in S$. Then, we have

$$\begin{aligned} a(bc) &= (ab)(ca) && \text{(by identity 1)} \\ &= (ab \cdot c)(aa) && \text{(by identity 1)} \\ &= (cb \cdot a)(aa) && \text{by left invertive law} \\ &= (cb)(aa) && \text{(by identity 1)} \\ &= c(ba) && \text{(by identity 1)} \\ \Rightarrow a(bc) &= c(ba). \end{aligned}$$

Thus S satisfies right invertive law. Hence S is a self-dual AG-groupoid. □

Theorem 33. *Every right commutative AG^{**}-groupoid is self-dual AG-groupoid.*

Proof. Let S be a right commutative AG**-groupoid and $a, b, c \in S$. Then, using right commutative and AG**-properties, we have $a(bc) = a(cb) = c(ab) = c(ba)$. Thus S satisfies right invertive law. Hence S is a self-dual AG-groupoid. □

Corollary 1. *Every right commutative T^1 -AG-groupoid is self-dual AG-groupoid.*

Corollary 2. *Every right commutative T^2 -AG-groupoid is self-dual AG-groupoid.*

Corollary 3. *Every right commutative T^4 -AG-groupoid is self-dual AG-groupoid.*

Conclusion

In this article, self-dual AG-groupoids have been enumerated and classified on the basis of commutativity and associativity. A self-dual AG-groupoid-test has been introduced to check an arbitrary AG-groupoid for a self-dual AG-groupoid. A partial solution to an open problem has been searched out. It has been proved that every left cancellative element of a self-dual AG-groupoid is right cancellative. Some features of ideals have also been discussed in self-dual

AG-groupoids. It was revealed that if A is an ideal of a self-dual AG-groupoid S, then A^2 is an ideal of S. Also, connected sets in self-dual AG-groupoids have been studied to some extent. It has been proved that if A, B are ideals of a self-dual AG-groupoid S, then A^2B^2 and B^2A^2 are connected sets. Subject to certain conditions, some desired algebraic structures have been constructed from the known ones. Moreover, some subclasses of self-dual AG-groupoid have been discovered that enjoy both left and right invertive laws, hence they are more innovative. The readers are motivated to study these new subclasses in more detail.

References

[1] I. Ahmad, Amanullah and M. Shah, *Fuzzy AG-subgroups*, Life Science Journal 9(4), (2012) 3931-3936.

[2] Amanullah, I. Ahmad and M. Shah, *On the Equal-height Elements of Fuzzy AG-subgroups*, Life Science Journal 10(4),(2013) 3143-3146.

[3] Amanullah, M. Rashad, I. Ahmad, *Abel-Grassmann's Groupoids of Modulo Matrices*, Mehran University Research Journal Vol. 35(1), (2016) 63-70.

[4] Amanullah, M. Rashad, I. Ahmad, M. Shah, *On Modulo AG-groupoids*, Journal of Advances in Mathematics 8(3), (2014) 1606-1613.

[5] Aziz-ul-Hakim, *Relationship between self-dual AG-groupoid and AG-groupoids*, M.Phil thesis, Malakand University, (2014).

[6] V. Celakoska-Jordanova, B. Janeva, *Free left commutative groupoids*, Annual Review of European University, Republic of Macedonia, (2009) 687-694.

[7] J. R. Cho, Pusan, Jezek J. and Kepka T., Praha, *Paramedial Groupoids*, Czechoslovak Mathematical Journal, 49(124), (1996).

[8] A. Distler, M. Shah and V. Sorge, *Enumeration of AG-groupoids*, Lecture Notes in Computer Science, Volume 6824 (2011), 1-14.

[9] M. A. Kazim and M. Naseerudin, *On almost semigroups*, Portugaliae Mathematica, 36(1), (1977).

[10] M. Khan, Q. Mushtaq, *Ideals in left almost semigroups.* In proceedings of 4th International Pure Mathematics Conference, (2003) 65-77.

[11] Q. Mushtaq, *Zeroids and Idempoids in AG-groupoids*, Quasigroups and related Systems 11, (2004) 79-84.

[12] Q. Mushtaq and Q. Iqbal, *Decomposition of a locally associative LA-semigroup*, Semigroup Forum 41 (1990), 155-164.

[13] Q. Mushtaq, M. Khan, *On left almost ring*, In proceedings of 7th International Pure Mathematics Conference, (2006).

[14] Q. Mushtaq and M. Khan, *A note on an Abel-Grassmann's 3-band*, Quasigroups and Related Systems 15 (2007), 295-301.

[15] Q. Mushtaq, M. Khan and Kar Ping Shum, *Topological structures on LA-Semigroups*, Bull. Malays. Math. Sci. Soc. (2) 36(4), (2013) 901-906.

[16] P. V. Protic and M. Bozinnovic, *Some congruences on AG^{**}-groupoid*, Algebra Logic and Discrete Mathematics, Nis, 3, 9(1995), 879-886.

[17] P. V. Protic and N. Stevanovic, *AG-test and some general properties of Abel-Grassmann's groupoids*, PU. M. A. Vol. 6 (1995), No. 4, pp. 371-383.

[18] M. Rashad, I. Ahmad, and M. Shah, *Some general properties of Stein AG-groupoids and Stein AG-test*, Sindh Univ. Res. Jour. (Sci. Ser) Vol. 48(3), (2016) 679-684.

[19] M. Shah, *A theoretical and computational investigation of AG-groups*, PhD thesis, Quaid-i-Azam University Islamabad, (2012).

[20] M. Shah, I. Ahmad and A. Ali, *Discovery of new classes of AG-groupoids*, Research Journal of Recent Sciences Vol.1(11), (2012) 47-49.

[21] M. Shah, I. Ahmad and M. Rashad, *On Introduction of New Classes of AG-groupoids*, Research Journal of Recent Sciences Vol. 2(1), (2013), 67-70.

[22] T. Shah and I. Rehman, *On gamma-ideals and gamma-bi-ideals in gamma- AG-groupoids*, International Journal of Algebra, 4(6),(2010) 267-276.

[23] N. Stevanovic and P.V. Protic, *Inflations of the AG-groupoids*, Novi Sad J. Math., 29(1),(1999) 19-26.