Journal of Prime Research in Mathematics Vol. **12***(2016), 16-23*

g-NONCOMMUTING GRAPH OF SOME FINITE GROUPS

M. NASIRI[1], A. ERFANIAN[2], M. GANJALI[3], A. JAFARZADEH[4]

Abstract. Let G be a finite non-abelian group and g a fixed element of G. In 2014, Tolue et al. introduced the g-noncommuting graph of G, which was denoted by Γ_G^g with vertex set G and two distinct vertices x and y join by an edge if $[x, y] \neq g$ and g^{-1}. In this paper, we consider induced subgraph of Γ_G^g on $G \setminus Z(G)$ and survey some graph theoretical properties like connectivity, the chromatic and independence numbers of this graph associated to symmetric, alternating and dihedral groups.

Key words : non-commuting graph, g-noncommuting graph, connected graph.
AMS SUBJECT : Primary 05C25, Secondary 20P05.

1. Introduction

Studying algebraic structure via graphs is an interesting subject, which has been widely discussed in recent years. There are many graphs assigned to groups, semigroups and rings. We may refer to the works of Bertram et al. [4], Grunewald et al. [6], Moghadamfar et al. [8] and Williams [10] or recent papers on non-commuting graph, Engel graph and non-cyclic graph of groups given in [2, 1, 3]. Let $Z(G)$ be the center of a group G. Associate a graph Γ_G to G as follows: Take $G \setminus Z(G)$ as vertex set of Γ_G and join two distinct vertices x and y whenever $xy \neq yx$. This graph is called the non-commuting graph of G. Paul Erdös considered the non-commuting graph for the first time and asked if $\omega(\Gamma_G)$ is finite when Γ_G has no infinite clique. Abdollahi et al.

[1]Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran. Email: mahnasiri@yahoo.com
[2]Department of Pure Mathematics and Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, Mashhad, Iran. Email: erfanian@math.um.ac.ir
[3]Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran. Email: m.ganjali20@yahoo.com
[4]Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran. Email: jafarzadeh@um.ac.ir .

[2] tried to explore how the graph theoretical properties of Γ_G can affect the group theoretical properties of G. Recently, Tolue et al. [9] generalized the non-commuting graph as follows,

Definition 1. *The g-noncommuting graph of G is the graph with vertex set G and two distinct vertices x and y join by an edge if $[x, y] \neq g$ and g^{-1}. They denote this graph by Γ_G^g.*

In [9], Tolue et al. determined planarity, regularity, clique and dominating numbers. Moreover, they proved that if G and H are isoclinic groups with $|Z(G)| = |Z(H)|$ and (ϕ, ψ) is an isoclinism between G and H, then $\Gamma_G^g \cong \Gamma_H^{\psi(g)}$.

In this paper, we are going to consider the induced subgraph of g-noncommuting graph on $G \setminus Z(G)$. We denote it by Δ_G^g and will investigate connectivity, chromatic and independence numbers. One may note that if Γ_G^g is connected, then it is not necessary for the induced subgraph Δ_G^g to be connected. For instance, $\Gamma_{S_3}^{(1\ 2\ 3)}$ is connected, but $\Delta_{S_3}^{(1\ 2\ 3)}$ consists of one edge and three isolated vertices. It is clear that Δ_G^e coincides with the non-commuting graph.

Recall that $K(G) = \{[x, y] : x, y \in G\}$ is the set of commutators of G and $G' = \langle K(G) \rangle$ (see [7] for more details). If $g \notin K(G)$, then Δ_G^g is complete graph. So in the sequel, we always assume that $g \in K(G)$ and $g \neq e$.

In section 2 of this paper, we prove that for every $g \in A_n$ and $n \geq 4$, $\Delta_{S_n}^g$ and $\Delta_{A_n}^g$ are connected. Moreover, diam$(\Delta_{S_n}^g) = 2$, girth$(\Delta_{S_n}^g) = 3$ and $\Gamma_{S_n}^g$ is not complete graph.

Section 3 is devoted to a determination of the chromatic and independence numbers of $\Delta_{D_{2n}}^g$. Here our notations and terminologies are standard and can be found in [5].

2. Connectivity of g-noncommuting Graph of S_n and A_n

In this section, we will discuss connectivity, diameter and girth of g-noncommuting graph of the symmetric group S_n and the alternating group A_n, where $n \geq 4$. We start with the following interesting lemma.

Lemma 1. *$\Delta_{S_n}^g$ is connected for all non-identity elements $g \in A_n$ and all $n \geq 4$.*

Proof. Using the group theory package GAP and direct drawing the graph for S_4, we can see that $\Delta_{S_4}^g$ is connected for all cases that $g \in A_4$. So, it is enough to prove the lemma for $n \geq 5$. First, let us remind that for every element $\sigma \in S_n$, support of σ, denoted by supp(σ), is the set of all letters $k \in \{1, 2, \cdots, n\}$ for which $\sigma(k) \neq k$. If $|\text{supp}(g)| \leq n - 2$, then there are two letters i and j such that $g(i) = i$ and $g(j) = j$. The transposition $(i\ j)$ joins to all vertices $x \in S_n \setminus \{e\}$, because the equality $(i^x\ j^x) = (i\ j)g = g(i\ j)$ is impossible for such x. Note that $(i^x\ j^x)$ is a permutation which moves only two letters and fixes others, but $(i\ j)g = g(i\ j)$ moves more than two letters. Now, assume $|\text{supp}(g)| = n-1$, then $g(i) = i$ for some letter $\{1, \cdots, n\}$. We claim that the transposition $(i\ j)$, where $j \in \{1, 2, \cdots, n\}$ and $g(j) \neq j$, is adjacent to all other vertices. It is clear that $(i\ j)^x$ is a transposition but $|\text{supp}((i\ j)g)| \geq 3$. Therefore, the equality $(i\ j)^x = (i\ j)g$ is impossible. Similarly, $(i\ j)^x = g(i\ j)$ is not possible. Finally, suppose $|\text{supp}(g)| = n$. Then $g \in A_n$ can be written as a product of cycles of length 3. If $(\alpha_1\ \alpha_2\ \alpha_3)$ is one of the cycles which appear in this product, then $(\alpha_1\ \alpha_2)$ joins to all other vertices, because all the letters appear in g, and so $g(\alpha_1\ \alpha_2)$ and $(\alpha_1\ \alpha_2)g$ will fix at most one letter. Thus $(\alpha_1\ \alpha_2)^x \neq (\alpha_1\ \alpha_2)g$ and $(\alpha_1\ \alpha_2)^x \neq g(\alpha_1\ \alpha_2)$. Hence the result follows. □

Proposition 2. *$\Delta^g_{S_n}$ is not a complete graph for all $g \in A_n$. Moreover diam$(\Delta^g_{S_n}) = 2$ and girth$(\Delta^g_{S_n}) = 3$.*

Proof. For $e \neq g \in K(S_n) = A_n$, there exist permutations α and β such that $[\alpha, \beta] = g$. Thus $\Delta^g_{S_n}$ is not a complete graph. Proof of Lemma 1 implies that diam$(\Delta^g_{S_n}) = 2$. If $a \in S_n \setminus \{e\}$ with $a^2 \neq e$, then a joins a^{-1} and one can see that there is a transposition which is adjacent to all other vertices. Thus the girth of the graph is 3. □

In the next theorem, we prove that $\Delta^g_{A_n}$ is connected. The strategy of the proof is similar to the proof of Lemma 1.

Theorem 3. *$\Delta^g_{A_n}$ is connected for all $e \neq g \in A_n$ and $n \geq 4$.*

Proof. Connectivity of $\Delta^g_{A_4}$ obtains by direct drawing the graph. Now, we can consider that the case $n \geq 5$. Suppose g is an arbitrary element in A_n and $|\text{supp}(g)| \leq n-3$. Then there are letters i, j and k such that $g(i) = i$, $g(j) = j$ and $g(k) = k$. Since g is an even permutation, there exist at least three letters which appear in g and so $n \geq 6$. We claim that $(i\ j\ k)$ is adjacent to all other vertices, because equalities $(i\ j\ k)^x = (i\ j\ k)g$ and $(i\ k\ j)^x = g(i\ k\ j)$ are not valid here. Now, assume $|\text{supp}(g)| = n-2$. Then, we have $g(i) = i$ and $g(j) = j$ for some letters i and j. Consider the 3-cycle $(i\ j\ k)$, where k is a letter which is not fixed by g. Again we will have the following equality which is impossible $(i\ j\ k)^x = (i\ j\ k)g = (k\ i\ j\ k' \cdots)$ where $k' = g(k)$. Note that the left hand side moves three letters, but the right hand side moves at least four letters. Similarly, $(i\ k\ j)^x$ is a 3-cycle but $g(i\ k\ j)$ moves more than three letters. Hence $(i\ j\ k)$ joins to all other vertices in $\Delta^g_{A_n}$. If $|\text{supp}(g)| = n-1$, then $g(i) = i$. We may choose $(i\ j\ k)$ such that $g(j) = j' \neq k$ and $g(k) = k' \neq j$ for some letters j' and k'. Then, the equalities $(i\ j\ k)^x = (i\ j\ k)g$ and $(i\ k\ j)^x = g(i\ k\ j)$ are not possible and so $(i\ j\ k)$ is adjacent to all other vertices. Finally, in case

$|\mathrm{supp}(g)| = n$, we can choose $(i\ j\ k)$ such that $g(j) = j' \neq k$, $g(k) = k' \neq j$ and $g(i) = i'$ for some letters j', i' and k'. The following two cases arise:
Case 1. Transposition $(i\ j)$ appears in decomposition of g to disjoint cycles. In this case, permutation $(i\ j\ k)g$ fixes only letter i and moves at least four letters, so $(i\ j\ k)^x = (i\ j\ k)g$ is not possible.
Case 2. Transposition $(i\ k)$ appears in decomposition of g. Similar to the previous case $(i\ j\ k)g$ fixes only letter k and moves other letters then $(i\ j\ k)^x = (i\ j\ k)g$ does not hold. Hence again $(i\ k\ j)^x = g(i\ k\ j)$ is not possible and the proof is complete. □

Since $K(S_n) = K(A_n) = A_n$, so Theorem 3 implies to state the same result as in the Proposition 2 for $\Delta^g_{A_n}$.

3. Some Properties of g-noncommuting Graph of D_{2n}

This section is devoted to a determination of some graph theoretical properties for instance connectivity, Eulerian, Hamiltonian, chromatic and independence numbers of g-noncommuting graph of the dihedral group D_{2n} of order $2n$ where $D_{2n} = \langle a, b | a^n = b^2 = e, bab = a^{-1}\rangle$. We remind that D_{2n} has trivial center and $D'_{2n} = \langle a\rangle$ if n is odd and when n is even we have $Z(D_{2n}) = \{e, a^{n/2}\}$ and $D'_{2n} = \langle a^2\rangle$. First, we state the following result on the connectivity of $\Delta^g_{D_{2n}}$.

Theorem 4. $\Delta^g_{D_{2n}}$ *is connected if and only if* $n \neq 3, 4$ *and* 6.

Proof. It is easy to see that $\Delta^g_{D_6}$, $\Delta^g_{D_8}$ and $\Delta^g_{D_{12}}$ are not connected for all $g \in K(D_{2n})$. Let $n \neq 3, 4$ and 6. The following two cases occur.

(i) n is an even number. We have $g = a^{2i}$ for some integer i where $2 \leq i \leq (n-1)/2$. Suppose $g \neq a^{n/2}$. If $[a^r b, a^j] = g$ and $[b, a^j] = g$ for every integers r, j such that $1 \leq r, j \leq n-1$, then $j = i$ or $j = (n/2) + i$. If $[a^r b, a^j] = g^{-1}$ and $[b, a^j] = g^{-1}$, then $j = (n/2) - i$ or $j = n - i$. There exists an integer j, $1 \leq j \leq n-1$, such that $j \neq i, (n/2) - i, (n/2) + i$ and $n - i$. Thus a^j is adjacent to all other vertices. When $g = a^{n/2}$, then $g = g^{-1}$ and by the same method, a^j is adjacent to all others if and only if $j \neq n/4$ and $3n/4$. Thus $\Delta^g_{D_{2n}}$ is connected when $n \geq 8$ is an even number.

(ii) n is an odd number. In this case, again there exists an integer $1 \leq j \leq n-1$ such that a^j is adjacent to every other vertices.

□

Theorem 5. *Let D_{2n} be dihedral group of order $2n$, where n is an odd integer and $g = a^i$, $1 \leq i \leq n-1$. If $n = 3m$, then $\chi(\Delta^g_{D_{2n}}) = 4n/3 - 3$. Otherwise*

$$\chi(\Delta^g_{D_{2n}}) = \begin{cases} 2n - \frac{m}{2}[\dfrac{n}{m}] - 3 & \text{if } [\dfrac{n}{m}] \text{ is even} \\ n + \frac{m}{2}\left([\dfrac{n}{m}] + 1\right) - 3 & \text{if } [\dfrac{n}{m}] \text{ is odd,} \end{cases}$$

where $m = i/2$ or $(n-i)/2$ whenever i is an even or odd, respectively.

Proof. Assume that i is an even number and $m = i/2$. It is clear that $[a^r b, a^s b] = a^{2(r-s)}$, $0 \leq r, s \leq n-1$. If $[a^r b, a^s b] = g$ or g^{-1}, then $r - s = i/2$ or $r - s = n - (i/2)$. Thus, two vertices $a^r b$ and $a^s b$ are adjacent if and only if $r - s \neq i/2$ and $n - (i/2)$. Similarly, a^r is adjacent to $a^s b$ if and only if $r \neq i/2$ and $n - i/2$, where $r \neq 0$. To find the chromatic number of $\Delta^g_{D_{2n}}$, it is enough to count the number of vertices $a^r b$ with different colours. Consider the vertices $a^0 b, a^1 b, \cdots, a^{n-1} b$, since $n - i/2 > i/2$ we put them in the boxes with $i/2$ members.
Initially, assume that $n - i/2 \neq i$. In each box all vertices are adjacent together, so for colouring the first box, we need $i/2$ different colours. We denote this colours with $c_1, \cdots, c_{i/2}$. In the second box, vertices are not adjacent to some vertices of the first box. Thus, we can use all the colours $c_1, \cdots, c_{i/2}$ in the second box. In the third box, we can not use colours $c_1, \cdots, c_{i/2}$. Since, although the vertex $a^r b$ in the third box is not adjacent to $a^{r-i/2} b$ in the second box, but it is adjacent to $a^{r-i} b$ in the first box. Thus, we can not colour $a^r b$ the same as $a^{r-i/2} b$. Therefore we need new $i/2$ colours for vertices in the third box. Similarly, we can use these colours in the forth box. If $[n/(i/2)]$ is an even number, then we need $i/2([n/i])$ colours for colouring $[n/(i/2)]$ of boxes. Moreover, $n - [n/(i/2)](i/2)$ colours are needed for colouring the remain vertices. If $[n/(i/2)]$ is an odd number, then $i/2(([n/(i/2)] + 1)/2)$ colours are used for colouring $[n/(i/2)]$ of boxes. Also, in this case we can colour $n - [n/(i/2)](i/2)$ remain vertices with colours which are used in the final box. Now, since $a^{i/2}$ and $a^{n-i/2}$ are not adjacent to $a^r b$, then we can apply two colours which are used for colouring a^r to $a^r b$.
Secondly, suppose $n - i/2 = i$. We have exactly 3 boxes with $i/2$ vertices where can be coloured by $i/2$ colours. Hence the result holds.
Now, let i be an odd number and $m = (n-i)/2$. If $[a^r b, a^s b] = g$ or g^{-1}, then $r - s = (n+i)/2$ or $r - s = (n-i)/2$. It is enough to replace $i/2$ with $(n-i)/2$ in the previous case. □

Theorem 6. *Let $n \neq 6$ be an even number and $g = a^{2i} \notin Z(D_{2n})$, where $1 \leq 2i \leq n-1$ is an integer. Then*

$$\chi(\Delta^{g}_{D_{2n}}) = \begin{cases} 2n - 6 - kt[\dfrac{n}{(k+1)t}] & 0 \leq n - (k+1)t[\dfrac{n}{(k+1)t}] < t \\ \\ n - 6 + t([\dfrac{n}{(k+1)t}] + 1) & n - (k+1)t[\dfrac{n}{(k+1)t}] \geq t, \end{cases}$$

where $t = min\{i, n/2 - i, n/2 + i, n - i\}$, for some integer k.

Proof. We know that a^j is adjacent to $a^r b$, $0 \leq r \leq n-1$, if and only if $j \neq i, (n/2) - i, (n/2) + i$ and $n - i$. Furthermore $a^r b$ and $a^s b$ are adjacent if and only if $r - s \neq i, (n/2) - i, (n/2) + i$ and $n - i$. Thus, for computing chromatic number of $\Gamma^{g}_{D_{2n}}$, it is enough to count the number of vertices $a^r b$ with different colours. Suppose $A = \{i, (n/2) - i, (n/2) + i, n - i\}$ and $t := \min(A)$. Put every vertices $a^0 b, a^1 b, \cdots, a^{n-1} b$ in the boxes with t members. In each box all the vertices are adjacent. Thus for colouring of the first box we need t different colours. Every vertices in the second box is not adjacent to some vertices in the first box. Therefore we can apply the previous colours. If $2t \in A$, then we can use all the previous colours in the third box. Since vertices in the third box are not adjacent to some vertices of the first and the second box. But if $2t \notin A$, then in the third box, we can not use the previous colours. Assume $t, 2t, \cdots, kt \in A$ and $(k+1)t \notin A$ for some positive integer k. Count the number of boxes with t members such that each $(k+1)$ box have the same colours. Suppose that $n = q(k+1)t + r'$. If $r' < t$, then we need r' different colours. Otherwise t different colours are required. Thus, we can colour $a^0 b, a^1 b, \cdots, a^{n-1} b$ with $t[n/((k+1)t)] + r'$ colours if $r' < t$, otherwise with $t[n/((k+1)t)] + t$ colours. Now, since $a^i, a^{n/2-i}, a^{n/2+i}$ and a^{n-i} are not adjacent to $a^r b$ for each r, then we can use four different colours of a^r to colour $a^r b$. Hence the proof is complete. □

Proposition 7. *Let $n \neq 6$ be an even number, then*

$$\chi(\Delta^{a^{\frac{n}{2}}}_{D_{2n}}) = \frac{3n}{2} - 4$$

Proof. Similar to the proof of Theorem 6, all vertices can be classified into the boxes with $n/2$ members such that each box are coloured by $n/4$ colors. Thus, we will have $\chi(\Delta^{a^{n/2}}_{D_{2n}}) = 3n/2 - 4$. □

In Theorems 5 and 6, we compute the chromatic number for g-noncommuting graph of D_{2n}. Now, we are going to find an independence number of this graph. The method of the proof is very similar to the proof of Theorems 5 and 6 so we omit it here.

Proposition 8. *The independence number of $\Delta^{g}_{D_{2n}}$ is 3 or 4.*

Lemma 9. *For every $n \geq 1$, $\omega(\Delta^g_{D_{2n}}) = n-1$ if n is odd number and $\omega(\Delta^g_{D_{2n}}) = n-2$ if n is even number.*

Proof. The subset $< a > \setminus Z(D_{2n})$ is a maximal clique in $\Delta^g_{D_{2n}}$, because every element $a^r b$, $0 \leq r \leq n-1$, is not adjacent to at least one elemnet a^j, $1 \leq j \leq n-1$ and the proof follows. □

We know that the grpah is Eulerian if and only if degree of all vertices are even. Thus we have the following lemma.

Lemma 10. *The g-noncommuting graph of D_{2n} is not Eulerian graph.*

Proof. With computing the commutators of elements in D_{2n} and counting the all corresponding edges, we can get the degree of all vertices. The set of degrees of vertices of graph is denoted by $D(\Delta^g_{D_{2n}})$, then we have
$D_(\Delta^g_{D_{2n}}) = \{n-2, 2n-2, 2n-5\}$, when n is an odd number,
$D_(\Delta^g_{D_{2n}}) = \{n-3, 2n-3, 2n-7, 2n-11\}$, when n is an even number,
It is observed that there is at least one vertice of odd degree, therefore $\Delta^g_{D_{2n}}$ is not Eulerian graph. □

Remind that a graph that contains a Hamiltonian cycle is called a Hamiltonian graph.

Lemma 11. *The g-noncommuting graph of D_{2n} is Hamiltonian graph, unless $\Delta^{a^2}_{D_{16}}$.*

Proof. Let n be an odd number and $g = a^i$ where i is an even number. We know that vertices $a^r b$ and $a^s b$, $0 \leq r, s \leq n-1$ are adjacent if and only if $r - s \neq \frac{i}{2}, n - \frac{i}{2}$, Similarly a^j is adjacent to $a^r b$ if and only if $j \neq \frac{i}{2}, n - \frac{i}{2}$. Now, If $\frac{i}{2}, n - \frac{i}{2} \neq 1$, then $\{a, b, ab, a^2b, ..., a^{n-1}b, a^{n-1}, a^{n-2}, ..., a^2\}$ visits each vertices exactly once. If $\frac{i}{2}$ or $n - \frac{i}{2} = 1$, then there is the following cycle
$\{a^2, b, a^2b, a^4b, ..., a^{n-1}b, ab, a^3b, ..., a^{n-2}b, a^{n-2}, a^{n-1}, a^{n-3}, ..., a\}$. Thus, the graph is Hamiltonian. If i is an odd number, then the proof is very similar. Similarly, we can observe that $\Delta^g_{D_{2n}}$ is Hamiltonian when n is an even number, unless $\Delta^{a^2}_{D_{16}}$. □

Finally, we state the following conjecture for the g-noncommuting graph of S_n to be Hamiltonian. In spite of the fact that it is true for some small values of n, but we are not able to prove it as yet, for all $n \geq 3$.

Conjecture. The g-noncommuting graph of S_n is Hamiltonian.

References

[1] A. Abdollahi: *Engel graph associated with a group*, J. Algebra 318(2007), 680–691.

[2] A. Abdollahi, S. Akbari, H. R. Maimani: *Non-commuting graph of a group*, J. Algebra 298(2006), 468–492.

[3] A. Abdollahi, A. Mohammadi Hassanabadi: *Non-cyclic graph of a group*, Comm. Algebra 35(2007), 2057–2081.

[4] E. A. Bertram, M. Herzog, A. Mann: *On a graph related to conjugacy classes of groups*, Bull. London Math. Soc. 22(6)(1990), 569–575.

[5] J.A. Bondy, U.S.R. Murty: Graph theory, Graduate texts in Mathematics, 244, Springer-Verlag, New York, 2008.

[6] F. Grunewald, B. Kunyavskiĭ, D. Nikolova, E. Plotkin: *Two-variable identities in groups and Lie algebras*, J. Math. Sci. (N. Y.) 116(2003), 2972–2981.

[7] L. Kappe, R. Morse: On commutators in groups, http://faculty.evansville.edu/rm43/publications/commutatorsurvey.pdf.

[8] A. R. Moghaddamfar, W. J. Shi, W. Zhou, A. R. Zokayi: *On non-commuting graph associated with a finite group*, Siberian Math. J. 46(2)(2005), 325–332.

[9] B. Tolue, A. Erfanian, A. Jafarzadeh: *A kind of non-commuting graph of finite groups*, J. Sci. Islam. Repub. Iran 25(4)(2014), 379–384.

[10] J. S. Williams: *Prime graph components of finite groups*, J. Algebra 69(2)(1981), 487–513.