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PROJECTIVE CONFIGURATIONS AND THE VARIANT OF CATHELINEAUS COMPLEX

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Abstract. In this paper we try to connect the Grassmannian subcomplex defined over the projective differential map d' and the variant of Cathelineau's complex. To do this we define some morphisms over the configuration space for both weight 2 and 3. we also prove the commutativity of corresponding diagrams.

Key words :Grassmannian complex; configuration; vector space; Infinitesimal; Cross Ratio.
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1. Introduction

Goncharov proves Zagier's conjecture, about polylogarithm and special L-values, for $n = 3$ see [5]. Some important ingredients of his work are, the introduction of motivic complexes (see [6]), morphism between Grassmannian subcomplexes and motivic complexes for trilogarithm, introduction of triple-ratio of six points and proof of the commutativity of corresponding diagrams.

Also in [6] he discussed the motivic complexes and Grassmannian complex by means of geometry of configurations and defines his maps f_0^3, f_1^3 and f_2^3 in configuration spaces. Cathelineau extended the work of Goncharov to *infinitesimal* and *tangential* settings by introducing F-vector spaces (F being a field of zero characteristic) $\beta_n(F)$ for infinitesimal case and a group $T\mathcal{B}_2(F)$ for tangential case (see [2] and [10]). Using the vector spaces $\beta_n(F)$, we can obtain another complex called Cathelineau's complex. To obtain vector spaces of the form $\beta_n^D(F)$, a derivation map D is defined over $\beta_n(F)$, for $n = 2$ and $n = 3$ (see [4] and [10]) and by setting this vector space into a complex, we obtain variant of Cathelineau's complex.

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In [10] the author discussed the geometry of configuration complexes for infinitesimal polylogarithms. He connected the Grassmannian subcomplex $(C_*(n), d)$, defined over the differential map "d"

$$d : (x_1, \ldots, x_m) \mapsto \sum_{i=0}^{m} (-1)^m (x_1, \ldots, \hat{x}_i, \ldots, x_m)$$

In order to connect Grassmannian subcomplex to variant of Cathelineau's complex, he introduced homomorphisms τ_0^2 and τ_1^2 for $n = 2$ and τ_0^3 , τ_1^3 , τ_2^3 for $n = 3$ and he also shown that the corresponding diagrams are commutative.

Since the Grassmannian complex is a bi-complex containing two differential maps d and d'. In [10] the Grassmannian subcomplex defined over the differential map d is connected to the variant of Cathelineau's complex. But in this work we have tried to relate the Grassmannian subcomplex, defined for the projective differential map d' such that

$$d' : (x_1, \ldots, x_m) \mapsto \sum_{i=0}^{m} (-1)^m (x_i | x_1, \ldots, \hat{x}_i, \ldots, x_m)$$

and the variant of Cathelineau complex for weight 2 weight 3. For weight 2, we define homomorphisms π_0^2 and π_1^2 (see §3.1) and also prove the commutativity of the corresponding diagram. The homomorphisms for weight 3 are defined as π_0^3 , π_1^3 and π_2^3 (see §3.2) and we prove that the diagrams are commutative.

2. Foundation and Basic concepts

In this section we explain the terminologies that will be used in our work and give a brief description of the prior work. We closely follow the work of ([6],[5],[7]) and ([10]) to define most of the terms.

2.1. **Grassmannain Complex.**

For any set X, let $C_m(X)$ be a free abelian group generated by the elements of G/X^m (which are called configurations of m points), where G is a group which acts on X, then we can define a differential map $d : C_m(X) \to C_{m-1}(X)$ as

$$d : (x_1, \ldots, x_m) \mapsto \sum_{i=1}^{m} (-1)^m (x_1, \ldots, \hat{x}_i, \ldots, x_m)$$

Let $C_m(n)$ be a free abelian group generated by the configurations of m-vectors of an n-dimensional vector space V_n. The configuration $(x_i | x_1, \ldots, \hat{x}_i, \ldots, x_m)$ is called a projective configuration of $m - 1$ vectors projected from x_i. Define a projective differential map d' as

$$d' : C_m(n) \to C_{(m-1)}(n - 1)$$

$$d' : (x_1, \dots, x_m) \mapsto \sum_{i=1}^{m} (-1)^i (x_i | x_1, \dots, \hat{x}_i, \dots, x_m)$$

then we have the following bi-complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & C_{n+5}(n+2) & \xrightarrow{d} & C_{n+4}(n+2) & \xrightarrow{d} & C_{n+3}(n+2) \\
 & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} \\
\cdots & \longrightarrow & C_{n+4}(n+1) & \xrightarrow{d} & C_{n+3}(n+1) & \xrightarrow{d} & C_{n+2}(n+1) \\
 & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} \\
\cdots & \longrightarrow & C_{n+3}(n) & \xrightarrow{d} & C_{n+2}(n) & \xrightarrow{d} & C_{n+1}(n)
\end{array}$$

called Grassmannian bi-complex.

From the above we will use the subcomplex $(C_*(n), d')$ in the remainder of the text.

2.2. **Bloch-Suslin.** Let $\mathbb{Z}[\mathbb{P}^1_F]$ be a free abelian group defined over a projective line $\mathbb{P}^1_F$ generated by the elements $[x]$ where $x \in \mathbb{P}^1_F$, then $\mathcal{B}_2(F)$ can be defined as the quotient of the $\mathbb{Z}[\mathbb{P}^1_F]$ and the subgroup $\mathcal{R}_2(F)$, i.e

$$\mathcal{B}_2(F) = \frac{\mathbb{Z}[\mathbb{P}^1_F \setminus \{0, 1, \infty\}]}{\mathcal{R}_2(F)}$$

where $\mathcal{R}_2(F)$ is a subgroup of $\mathbb{Z}[\mathbb{P}^1_F]$ generated by the five term relation

$$R_2(F) = \left\langle \sum_{i=0}^{4} (-1)^i [r(x_0, \dots, \hat{x}_i, \dots, x_4)], x_i \in \mathbb{P}^1_F \right\rangle$$

where $r(x_0, \dots, x_3) = \frac{(x_0 - x_3)(x_1 - x_2)}{(x_0 - x_2)(x_1 - x_3)}$ is the cross ratio of four points.

If we define a map

$$\delta_2 : \mathbb{Z}[\mathbb{P}^1_F] \to \wedge^2 F^\times, [x] \mapsto (1 - x) \wedge x$$

where $\wedge^2 F^\times$ represents the second exterior power defined as

$$\wedge^2 F^\times = F^\times \otimes_z F^\times / \langle x \otimes_z x / x \in F^\times \rangle$$

It is easy to verify that $\mathcal{R}_2(F) \subseteq \ker \delta_2$, so we can induce a map δ for which we have a complex $B_F(2)$ of the form

$$\delta : \mathcal{B}_2(F) \to \wedge^2 F^\times$$

called Bloch-Suslin complex.

2.3. **Goncharov's Complex.** Bloch group of weight 3 denoted by $\mathcal{B}_3(F)$ can be obtained by substituting n=3 in the Zagier's generelized Bloch group [4], i.e.

$$\mathcal{B}_3(F) = \frac{\mathbb{Z}[\mathbb{P}^1_F]}{\mathcal{R}_3(F)}$$

where the subgroup $\mathcal{R}_3(F) \subseteq \mathbb{Z}[\mathbb{P}^1_F]$ can be defined as

$$\mathcal{R}_3(F) = \left\langle \sum_{i=0}^{6} (-1)^i [r_3(l_0, \ldots, \hat{l}_i, \ldots, l_6)] | (l_0, \ldots, \hat{l}_i, \ldots, l_6) \in C_6(\mathbb{P}^2_F) \right\rangle$$

where $r_3(l_0, \ldots, , \ldots, l_5)$ represents the Goncharove's triple-ratio of six points $(l_0, \ldots, l_5) \in C_6(P^2(F))$ which can be written as

$$r_3(l_0, \ldots, l_5) = \text{Alt}_6 \frac{\Delta(l_0, l_1, l_3)\Delta(l_1, l_2, l_4)\Delta(l_2, l_0, l_5)}{\Delta(l_0, l_1, l_4)\Delta(l_1, l_2, l_5)\Delta(l_2, l_0, l_3)}$$

Now if we define a map

$$\delta_3 : \mathbb{Z}[P^1(F)] \to \wedge^3 F^\times$$

then by [5], we have $\delta_3(\mathcal{R}_3(F)) = 0$, so we can write a complex $B_F(3)$

$$\delta : \mathcal{B}_2(F) \to \wedge^2 F^\times$$

where δ is induced map of ∂_3.

2.4. **Cathelineau's Complex.** [2] introduces another version of Goncharov and Bloch groups by using infinitesimal procedure. He defines the infinitesimal vector spaces as

$$\beta_1(F) = F$$

where F is any field of characteristic 0. Also

$$\beta_2(F) = \frac{F[F^{\bullet\bullet}]}{r_2(F)}$$

where $F^{\bullet\bullet} = F - \{0, 1\}$ and $r_2(F)$ is the subspace of $F[F]$ generated by the following

$$[a] - [b] + a\left[\frac{b}{a}\right] + (1-a)\left[\frac{1-b}{1-a}\right]$$

and there is a map $\delta_2 : F[F^{\bullet\bullet}] \to F \otimes_{\mathbb{Z}} F^\times$ defined as

$$[a] \mapsto a \otimes_{\mathbb{Z}} a + (1-a) \otimes_F (1-a)$$

such that $r_2(F) \subseteq \ker \partial_2$. From the above setup we have have a complex

$$\beta_2(F) \xrightarrow{\delta} F \otimes_{\mathbb{Z}} F^\times$$

called infinitesimal complex of weight $n = 2$, where ∂ is induced by δ_2. Further, infinitesimal complex for weight $n = 3$ can be defined as

$$\beta_3(F) = \frac{F[F^{\bullet\bullet}]}{r_3(F)}$$

where $r_3(F)$ is the kernel of the map

$$\delta_3 : F[F^{\bullet\bullet}] \to \beta_2(F) \otimes F^{\times} \oplus F \otimes \mathcal{B}_2(F)$$
$$[a] \mapsto \langle a\rangle_2 \otimes a + (1-a) \otimes [a]_2$$

where $\langle a\rangle_2$ is a class of $[a]$ in $\beta_2(F)$, we can form a complex for the map δ as

$$\beta_3(F)) \xrightarrow{\delta} (\beta_2(F) \otimes_F F^{\times}) \oplus (F \otimes \mathcal{B}_2(F))) \xrightarrow{\delta} F \otimes \wedge^2 F^{\times}$$

where

$$\delta : \langle a\rangle_3 \mapsto \langle a\rangle_2 \otimes a + (1-a) \otimes [a]_2$$

and

$$\delta : \langle a\rangle_2 \otimes b + x \otimes [y]_2 \mapsto -(a \otimes a \wedge b + (1-a) \otimes (1-a) \wedge b) + x \otimes (1-y) \wedge y$$

where $\langle a\rangle_3$ is a class of $[a]$ in $\beta_3(F)$

2.5. **Variant of Cathelineau's Infinitesimal Complex.** Let D be the derivation map defined on the field F and denote $D(x)$ for the derivative of $x \in F$ over $\mathbb{Z}$, we define a vector space $\beta_2^D(F)$ generated by the elements $[\![a]\!]^D$ and satisfying the relation

$$[\![a]\!]_2^D - [\![b]\!]_2^D + \left[\!\!\left[\frac{b}{a}\right]\!\!\right]_2^D - \left[\!\!\left[\frac{1-b}{1-a}\right]\!\!\right]_2^D + \left[\!\!\left[\frac{1-b^{-1}}{1-a^{-1}}\right]\!\!\right]_2^D ; a \neq b, a \neq 1$$

where $[\![a]\!]^D := \frac{D(a)}{a(1-a)}[a] \in F[F^{\bullet\bullet}]$

Now we can apply this derivative map to all the constructions of infinitesimal vector spaces to form new vector spaces and to do this we will follow the work of [10]. First we define a map

$$\partial_2{}^D : F[F^{\bullet\bullet}] \to F \otimes F^{\times}$$
$$\partial_2{}^D : [\![a]\!]_2^D \mapsto -D\log(1-a) \otimes a + D\log(a) \otimes (1-a)$$

where $[\![a]\!]_2^D = \frac{D(a)}{a(1-a)}\langle a\rangle_2$

we define $\beta_3^D(F)$ as

$$\beta_3^D(F) = \frac{F(F^{\bullet\bullet})}{\rho_3^D(F)}$$

where $\rho_3^D(F)$ is the kernel of the map

$$\partial_3{}^D : [a]^D \mapsto [\![a]\!]_2^D \otimes a + Dlog(a) \otimes [a]_2$$

From which another map ∂^D can be induced which enables us to form the following complex

$$\beta_3^D(F) \xrightarrow{\partial^D} \beta_2^D(F)F^{\times} \oplus F \otimes \mathcal{B}_2(F) \xrightarrow{\partial^D} F \otimes \wedge^2 F$$

where

$$\partial^D\left([\![a]\!]_3^D\right) = [a]_2^D \otimes a + Dlog(a) \otimes [a]_2$$

2.5.1. *Functional equations of $\beta_n^D(F)$.* Functional equations of $\beta_n^D(F)$ for $n = 2$ and $n = 3$ can easily be computed from the functional equations of $\beta_n(F)$ by using the definition of derivation map see [10]

For the case $n = 2$

(1) Two term relation
$$[\![a]\!]_2^D = -[\![1-a]\!]_2^D$$
(2) Inversion relation
$$[\![a]\!]_2^D = -\left[\!\!\left[\frac{1}{a}\right]\!\!\right]_2^D$$
(3) Five term relation
$$[\![a]\!]_2^D - [\![b]\!]_2^D + \left[\!\!\left[\frac{b}{a}\right]\!\!\right]_2^D - \left[\!\!\left[\frac{1-b}{1-a}\right]\!\!\right]_2^D + \left[\!\!\left[\frac{1-b^{-1}}{1-a^{-1}}\right]\!\!\right]_2^D = 0$$

3. Main Results

In this section we are Considering Grassmannian subcomplex $(C_*(n), d')$ which requires slightly different morphisms from [10] between Grassmannian and infinitesimal complexes for weight $n = 2, 3$.

3.1. **Dilogarithmic complexes.** Consider the following

$$\begin{array}{ccc} C_4(2) & \xrightarrow{\pi_1^2} & \beta_2^D(F) \\ \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle \partial^D} \\ C_3(1) & \xrightarrow{\pi_0^2} & F \otimes F^\times \end{array} \qquad (3.1a)$$

then we can define
$$\pi_0^2 : (l_0, l_1, l_2) \mapsto \sum_{i=0}^{2} \frac{D\,(\Delta(l_i))}{\Delta(l_i)} \otimes \Delta(l_{i+2}) - \frac{D\,(\Delta(l_{i+1}))}{\Delta(l_{i+1})} \otimes \Delta(l_{i+2}), \quad i \mod 3$$
where $\Delta(l_i) = \langle \omega, l_i \rangle$, such that $\omega \in V_1^*$ is a volume element.
$$\pi_1^2 : (l_0, l_1, l_2, l_3) \mapsto [\![r(l_0, l_1, l_2, l_3)]\!]_2^D$$

$$\partial^D([a]_2^D) = -D\log(1-a) \otimes a + D\log(a) \otimes (1-a)$$
where $[\![a]\!]_2^D = \frac{D(a)}{a(a-1)}\langle a \rangle_2$ is the element of $\beta_2^D(F)$ and $r(l_0, \dots, l_3) = \frac{\Delta(l_0,l_3)\Delta(l_1,l_2)}{\Delta(l_0,l_2)\Delta(l_1,l_3)}$ is the cross ratio of four points.

Now it is easy to verify that maps π_0^2 and π_1^2 are well-defined.

Theorem 1. *The diagram (3.1a) is commutative. i.e.*
$$\pi_0^2 \circ d' = \partial^D \circ \pi_1^2$$

Proof. First we will compute $\pi_0^2 \circ d'(l_0, l_1, l_2, l_3)$

$$\pi_0^2 \circ d'(l_0, l_1, l_2, l_3) = \pi_0^2 \{(l_0|l_1, l_2, l_3) - (l_1|l_0, l_2, l_3) + (l_2|l_0, l_1, l_3) - (l_3|l_0, l_1, l_2)\} \quad (1)$$

We deal all four summands individually, take the last term and apply definition of π_0^2.

$$\begin{aligned}\pi_0^2(l_3|l_0, l_1, l_2) =& \frac{D(30)}{(30)} \otimes (32) - \frac{D(31)}{(31)} \otimes (32) - \frac{D(31)}{(31)} \otimes (30) \\ &+ \frac{D(32)}{(32)} \otimes (30) - \frac{D(32)}{(32)} \otimes (31) - \frac{D(30)}{(30)} \otimes (31)\end{aligned}$$

for the simplicity we use $\Delta(l_i, l_j) = (ij)$ where $\Delta(l_i, l_j) = \langle \omega, l_i, l_j \rangle$, such that $\omega \in V_2^*$ is a volume element. This is the expansion of last summand of (3.1) which contains six terms. Similarly the expansion of other three summands will give 18 more terms and by combining all of them we will get an expression of 24 terms. These 24 terms can be combined into two terms as

$$= D \log r(0321) \otimes r(0123) - D \log r(0123) \otimes r(0321)$$

This completes the calculation of left hand side.

Now we come to find the value of $\partial^D \circ \pi_1^2$ From the definition of ∂^D and π_1^2 we have

$$\partial^D \circ \pi_1^2(l_0, l_1, l_2, l_3) = \frac{D\{1 - r(0123)\}}{\{1 - r(0123)\}} \otimes r(0123) - \frac{D\{r(0123)\}}{r(0123)} \otimes \{1 - r(0123)\}$$

where $(l_p, l_q, l_r, l_s) = (pqrs)$, using the property of cross-ratio we can write

$$= D \log r(0321) \otimes r(0123) - D \log r(0123) \otimes r(0321)$$

□

3.2. **Trilogarithmic complexes.** Consider the diagram

$$\begin{array}{ccccc} C_6(3) & \xrightarrow{d'} & C_5(2) & \xrightarrow{d'} & C_4(1) \\ \downarrow{\scriptstyle \pi_2^3} & & \downarrow{\scriptstyle \pi_1^3} & & \downarrow{\scriptstyle \pi_0^3} \\ \beta_3^D(F) & \xrightarrow{\partial^D} & (\beta_2^D(F) \otimes F^\times) \oplus (F \otimes \mathcal{B}_2(F)) & \xrightarrow{\partial^D} & F \otimes \wedge^2 F^\times \end{array} \quad (3.2a)$$

where

$$\pi_0^3 : (l_0, \ldots, l_3) \mapsto \sum_{i=0}^{3} (-1)^i \frac{D\Delta(l_i)}{\Delta(l_i)} \otimes \frac{\Delta(l_{i+1})}{\Delta(l_{i+2})} \wedge \frac{\Delta(l_{i+3})}{\Delta(l_{i+2})}, \quad i \mod 4 \quad (2)$$

$$\pi_1^3 : (l_0, \ldots, l_4) \mapsto -\frac{1}{3}\sum_{i=0}^{4}(-1)^i \Bigg\{ [\![[r(l_0, \ldots, \hat{l}_i, \ldots, l_4)]]\!]_2^D \otimes \Pi_{i\neq j}\Delta(l_i, l_j) + \frac{D(\Pi_{i\neq j}\Delta(l_i, l_j))}{\Pi_{i\neq j}\Delta(l_i, l_j)} \otimes [r(l_0, \ldots, \hat{l}_i, \ldots, l_4)]_2 \Bigg\} \quad (3)$$

$$\pi_2^3 : (l_0, \ldots, l_5) \mapsto \frac{2}{45}\mathrm{Alt}_6 \left[\!\!\left[\frac{\Delta(l_0, l_1, l_3)\Delta(l_1, l_2, l_4)\Delta(l_2, l_0, l_5)}{\Delta(l_0, l_1, l_4)\Delta(l_1, l_2, l_5)\Delta(l_2, l_0, l_3)} \right]\!\!\right]_3^D \quad (4)$$

$$d' : (x_0, \ldots, x_4) \mapsto \sum_{i=0}^{4}(-1)^m (x_i | x_1, \ldots, \hat{x}_i, \ldots, x_4)$$

where

$$[\![[a]]\!]_3^D = \frac{D(a)}{a(a-1)}\langle a\rangle_3$$

and

$$\partial^D([\![[a]]\!]_2^D) = [a]_2^D \otimes a + \frac{D(a)}{a} \otimes [a]_2$$

The maps (3.2) , (3.3) and (3.4) are well defined and this can easily be varified by same procedure used in [10]. Here we only prove the commutativity of above diagram and will see that the diagram is still commutative by using projection maps in Grassmannian complex.

Theorem 2. *The diagram*

$$\begin{array}{ccc} C_5(2) & \xrightarrow{d'} & C_4(1) \\ \downarrow{\pi_1^3} & & \downarrow{\pi_0^3} \\ (\beta_2^D(F)\otimes F^\times)\oplus(F\otimes\mathcal{B}_2(F)) & \xrightarrow{\partial^D} & F\otimes\wedge^2 F^\times \end{array}$$

is commutative. i.e.

$$\partial^D \circ \pi_1^3 = \pi_0^3 \circ d'$$

Proof. Since we have already defined all the maps so here we do direct calculations

$$\begin{aligned} \pi_0^3 \circ d'(l_0, \ldots, l_4) =& \pi_0^3\left(\sum_{i=0}^{4}(-1)^i (l_i | l_0, \ldots, \hat{l}_i, \ldots, l_4)\right) \\ =& \pi_0^3(l_0|l_1, l_2, l_3, l_4) - \pi_0^3(l_1|l_0, l_2, l_3, l_4) + \pi_0^3(l_2|l_0, l_1, l_3, l_4) \\ &- \pi_0^3(l_3|l_0, l_1, l_2, l_4) + \pi_0^3(l_4|l_0, l_1, l_2, l_3) \end{aligned} \quad (5)$$

Now we solve each term individually

$$\pi_0^3(l_0|l_1,l_2,l_3,l_4)=\sum_{i=1}^{4}(-1)^i\left(\frac{D\,(\Delta(l_0,l_i))}{\Delta(l_0,l_i)}\otimes\frac{\Delta(l_0,l_{i+1})}{\Delta(l_0,l_{i+2})}\wedge\frac{\Delta(l_0,l_{i+3})}{\Delta(l_0,l_{i+2})}\right) \tag{6}$$

Form the expansion of this sum we get 12 terms of the form

$$\frac{D\,(\Delta(l_0,l_1))}{\Delta(l_0,l_1)}\otimes\Delta(l_0,l_2)\wedge\Delta(l_0,l_4)$$

and the terms with common $D\log$ factor can be combined. i.e. (6) can be written as

$$\begin{aligned}
=&\frac{D\,(\Delta(l_0,l_1))}{\Delta(l_0,l_1)}\otimes\{\Delta(l_0,l_2)\wedge\Delta(l_0,l_4)-\Delta(l_0,l_2)\wedge\Delta(l_0,l_3)-\Delta(l_0,l_3)\wedge\Delta(l_0,l_4)\}\\
-&\frac{D\,(\Delta(l_0,l_2))}{\Delta(l_0,l_2)}\otimes\{\Delta(l_0,l_3)\wedge\Delta(l_0,l_1)-\Delta(l_0,l_3)\wedge\Delta(l_0,l_4)-\Delta(l_0,l_4)\wedge\Delta(l_0,l_1)\}\\
+&\frac{D\,(\Delta(l_0,l_3))}{\Delta(l_0,l_3)}\otimes\{\Delta(l_0,l_4)\wedge\Delta(l_0,l_2)-\Delta(l_0,l_4)\wedge\Delta(l_0,l_1)-\Delta(l_0,l_1)\wedge\Delta(l_0,l_2)\}\\
-&\frac{D\,(\Delta(l_0,l_4))}{\Delta(l_0,l_4)}\otimes\{\Delta(l_0,l_1)\wedge\Delta(l_0,l_3)-\Delta(l_0,l_1)\wedge\Delta(l_0,l_2)-\Delta(l_0,l_2)\wedge\Delta(l_0,l_3)\}
\end{aligned} \tag{7}$$

Similarly we can expand the remaining four summands of (5) and will get 12 terms of each that can be converted into 4 terms as (7) . After setting all these terms of (5) we get a total of 20 terms.This completes the calculation of the right hand side.

Now we come to compute the other part. i.e. $\partial^D\circ\pi_1^3$

$$\begin{aligned}
\partial^D\circ\pi_1^3(l_0,\dots,l_4)=&\partial^D\Bigg(-\frac{1}{3}\sum_{i=0}^{4}(-1)^i\{[r(l_0,\dots,\hat{l}_i,\dots,l_4)]_2^D\}\otimes\Pi_{i\neq j}\Delta(l_i,l_j)\\
&+\frac{D(\Pi_{i\neq j}\Delta(l_i,l_j))}{\Pi_{i\neq j}\Delta(l_i,l_j)}\otimes[r(l_0,\dots,\hat{l}_i,\dots,l_4)]_2\Bigg)\\
=&-\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Bigg(\frac{D(1-r(l_0,\dots,\hat{l}_i,\dots,l_4))}{(1-r(l_0,\dots,\hat{l}_i,\dots,l_4))}\otimes[r(l_0,\dots,\hat{l}_i,\dots,l_4)]\wedge\Pi_{i\neq j}\Delta(l_i,l_j)\\
&-\frac{D(r(l_0,\dots,\hat{l}_i,\dots,l_4))}{(r(l_0,\dots,\hat{l}_i,\dots,l_4))}\otimes[(1-r(l_0,\dots,\hat{l}_i,\dots,l_4))]\wedge\Pi_{i\neq j}\Delta(l_i,l_j)\\
&+\frac{D\left(\Pi_{i\neq j}\Delta(l_i,l_j)\right)}{\Pi_{i\neq j}\Delta(l_i,l_j)}\otimes[1-r(l_0,\dots,\hat{l}_i,\dots,l_4)]\wedge[r(l_0,\dots,\hat{l}_i,\dots,l_4)]\Bigg)
\end{aligned} \tag{8}$$

For short hand we will use (ij) instead of $\Delta(l_i,l_j)$ and $(pqrs)$ instead of (l_p,l_q,l_r,l_s). From the above sum, for different values of i we get five different expressions. For example, if we put $i = 0$, we get

$$-\frac{1}{3}\Big(D\log(1-r(1234))\otimes r(1234)\wedge(01)(02)(03)(04) - D\log(r(1234))\otimes(1-r(1234))\wedge(01)(02)(03)(04)$$
$$+D\log((01)(02)(03)(04))\otimes(1-r(1234))\wedge r(1234)\Big)$$

or

$$-\frac{1}{3}\left(D\log\left(\frac{(12)(43)}{(13)(42)}\right)\otimes(\frac{(14)(23)}{(13)(24)})\wedge(01)(02)(03)(04)\right.$$
$$-\frac{D\left(\frac{(14)(23)}{(13)(24)}\right)}{\left(\frac{(14)(23)}{(13)(24)}\right)}\otimes(\frac{(12)(43)}{(13)(42)})\wedge(01)(02)(03)(04)$$
$$+\frac{D((01)(02)(03)(04))}{(01)(02)(03)(04)}\otimes\frac{(12)(43)}{(13)(42)}\wedge\frac{(14)(23)}{(13)(24)}$$

In a similar way we can find the other four expressions by putting $i = 1, 2, 3, 4$ in (8). After combining all five the expressions and simplifying we will get a large number of terms but most of them will cancel each other and remaining terms can be added to the similar ones . After this process only 60 terms will remains all with co-efficient 3 and if we arrange these terms in such a way that the terms whose first factor is same are combined then we get

$$-\frac{1}{3}\frac{D(12)}{(12)}\otimes\Big\{3(13)\wedge(14)+3(14)\wedge(10)-3(13)\wedge(10)$$
$$+3(23)\wedge(20)-3(23)\wedge(24)-3(24)\wedge(20)\Big\}$$
$$-\frac{1}{3}\frac{D(14)}{(14)}\otimes\Big\{3(10)\wedge(12)+3(12)\wedge(13)-3(10)\wedge(13)$$
$$+3(42)\wedge(40)-3(42)\wedge(43)-3(43)\wedge(40)\Big\}$$
$$\vdots$$

and so on.

At last we cancel the coefficient 3 with the factor $-\frac{1}{3}$ which gives the required result. □

Theorem 3. *The diagram*

$$\begin{array}{ccc} C_6(3) & \xrightarrow{\quad d' \quad} & C_5(2) \\ \downarrow{\scriptstyle \pi_2^3} & & \downarrow{\scriptstyle \pi_1^3} \\ \beta_3^D(F) & \xrightarrow{\partial^D} & (\beta_2^D(F)\otimes_F F^\times)\oplus(F\otimes\mathcal{B}_2(F)) \end{array}$$

is commutative. i.e.

$$\partial^D\circ\pi_2^3 = \pi_1^3\circ d'$$

Proof. Since there is no significant changes in map of π_2^3 and map defined in [10], so we are reffering Theorem 3.11 of [10] for the proof. □

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