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THE t-PEBBLING NUMBER OF SOME WHEEL RELATED GRAPHS

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Abstract. Let G be a graph and some pebbles are distributed on its vertices. A pebbling move (step) consists of removing two pebbles from one vertex, throwing one pebble away, and moving the other pebble to an adjacent vertex. The t-pebbling number of a graph G is the least integer m such that from any distribution of m pebbles on the vertices of G, we can move t pebbles to any specified vertex by a sequence of pebbling moves. In this paper, we determine the t-pebbling number of some wheel related graphs.

Keywords : Pebbling number, Wheel graphs.
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1. Introduction

Pebbling in graphs was first considered by Chung [1]. Graph Pebbling is a network optimization model for the transportation of resources that are consumed in transit. The central problem in this model asks whether discrete pebbles from one set of vertices can be moved to another while pebbles are lost in the process. The graph pebbling model was born as a method for solving a combinatorial number theory conjecture of Erds and Lemke and has since been applied to problems in combinatorial group theory and p-adic diophantine equations. Here, the term graph refers to a simple graph. A *configuration* C of pebbles on a graph $G = (V, E)$ can be thought of as a function $C : V(G) \to N \cup \{0\}$. The value $C(v)$ equals the number of pebbles placed at vertex v, and

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the *size* of the configuration is the number $|C| = \sum_{v \in V(G)} C(v)$ of pebbles placed in total on G. Suppose C is a configuration of pebbles on a graph G. A *pebbling move (step)* consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. We say a pebble can be moved to a vertex v, the *target vertex*, if we can apply pebbling moves repeatedly (if necessary) so that in the resulting configuration the vertex v has at least one pebble.

Definition 1.1. ([2]) *The t-pebbling number of a vertex v in a graph G, $f_t(v, G)$, is the smallest positive integer m such that however m pebbles are placed on the vertices of the graph, t pebbles can be moved to v in finite number of pebbling moves, each move removes two pebbles of one vertex and placing one on an adjacent vertex. The t-pebbling number of G, $f_t(G)$, is defined to be the maximum of the pebbling numbers of its vertices.*

Thus the *t-pebbling number of a graph G*, $f_t(G)$, is the least m such that, for any configuration of m pebbles to the vertices of G, we can move t pebbles to any vertex by a sequence of moves, each move removes two pebbles of one vertex and placing one on an adjacent vertex. Clearly, $f_1(G) = f(G)$, the *pebbling number* of G.

7	1	0	1	0
5	2	0	1	0
3	3	0	1	0
1	4	0	1	0
1	2	1	1	0
1	0	2	1	0
1	0	0	2	0
1	0	0	0	1

Figure 1. An illustration of moving one pebble to the end vertex of the path P_5 from a configuration of size 9

Fact 1.2. ([8]) *For any vertex v of a graph G, $f(v, G) \geq n$ where $n = |V(G)|$.*

Fact 1.3. ([8]) *The pebbling number of a graph G satisfies*

$$f(G) \geq max\{2^{diam(G)}, |V(G)|\}.$$

With regard to t-pebbling number of graphs, we find the following theorems:

Theorem 1.4. *([3]). Let K_n be the complete graph on n vertices where $n \geq 2$. Then $f_t(K_n) = 2t + n - 2$.*

Theorem 1.5. *([6]). Let C_n denote a simple cycle with n vertices, where $n \geq 3$. Then $f_t(C_{2k}) = t2^k$ and $f_t(C_{2k+1}) = \frac{2^{k+2}-(-1)^{k+2}}{3} + (t-1)2^k$.*

Theorem 1.6. *([8]). Let P_n be a path on n vertices. Then $f_t(P_n) = t.2^{n-1}$.*

Theorem 1.7. *([6]). Let Q_n be the n-cube. Then $f_t(Q_n) = t.2^n$.*

Lourdusamy et al. proved the t-pebbling number of Jahangir graph $J_{2,m}$ and the t-pebbling number of Jahangir graph $J_{3,m}$ (for $m \geq 3$) in [7, 4]. And also they proved the t-pebbling number for squares of cycles ($t \geq 2$) in [5].

Notation 1.8. *Let $p(v)$ denote the number of pebbles on the vertex v and $p(A)$ denote the number of pebbles on the vertices of A, where $A \subseteq V(G)$. Let $< A >$ denote the subgraph induced by the vertices of A. Let $d(u, v)$ denote the distance between the vertices u and v in G.*

Remark 1.9. *Consider a graph G with n vertices and $f(G)$ pebbles are placed on its vertices. Suppose we choose a target vertex v from G to put a pebble on it. If $p(v) \geq 1$ or $p(u) \geq 2$ where $uv \in E(G)$, then we can move one pebble to v easily. So, we always assume that $p(v) = 0$ and $p(u) \leq 1$ for all $uv \in E(G)$ when v is the target vertex.*

2. The t-pebbling number of W_n

Definition 2.1. *The join $G+H$ of two graphs G and H is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.*

Definition 2.2. *Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ ($n \geq 3$) and $V(K_1) = \{v_0\}$, where C_n is the cycle graph of order n and K_1 is the trivial graph. Then the graph $C_n + K_1$ is called as wheel graph W_n of order $n+1$. We call the vertex v_0 as apex vertex and the vertex v_i ($i \neq 0$) as rim vertex of W_n.*

Theorem 2.3. [3] *The pebbling number of the wheel graph W_n is $f(W_n) = n + 1$ ($n \geq 3$).*

Theorem 2.4. [3] *The t-pebbling number of the wheel graph W_3 is $f_t(W_3) = f_t(K_4) = 2t + 2$.*

Theorem 2.5. *For the wheel graph W_n ($n \geq 4$), $f_2(W_n) = n + 4$.*

Proof. Consider the following distribution: $p(v_0) = 0$, $p(v_1) = 0$, $p(v_2) = 0$, $p(v_3) = 7$, $p(v_4) = 0$, $p(v_i) = 1$ for all $i \neq 0, 1, 2, 3, 4$. Then we cannot move two pebbles to the vertex v_1. Thus $f_2(W_n) \geq n + 4$.

Now, consider the distribution of $n+4$ pebbles on the vertices of W_n. Let v_i be the target vertex. Clearly, we are done if $p(v_i) \geq 2$. If $p(v_i) = 1$ then we can move another one pebble to the vertex v_i, since $p(W_n) - 1 \geq n+1$. So, we assume that $p(v_i) = 0$ when v_i is the target vertex.

Case 1: Let v_0 be the target vertex.
Since $f(W_n) = n+1$, we can move one pebble to the vertex v_0 using at most two pebbles. Then we can move another one pebble to v_0 easily, since $p(W_n) - 2 \geq n+1$.

Case 2: Let v_1 be the target vertex.
Since $f(W_n) = n+1$, we can move one pebble to the vertex v_1 using at most four pebbles. If we have used only two or three pebbles, to put a pebble on the vertex v_1, then we can move another one pebble to v_1 easily, since $p(W_n) - 3 \geq n+1$. Suppose we have used four pebbles to move a pebble to v_1, then on those distributions, we must have $p(v_0) = p(v_2) = p(v_n) = 0$. Clearly, at most $n-3$ vertices of $V(W_n) - \{v_0, v_1, v_2, v_n\}$ should have received the $n+4$ pebbles. By our assumption, first, we move one pebble to v_1 using exactly four pebbles. Then the remaining n pebbles are on the $n-3$ vertices or less vertices of $V(W_n) - \{v_0, v_1, v_2, v_n\}$ (i.e. we get at least three pebbles extra). Thus there exists a vertex v_i such that $p(v_i) \geq 4$ or there exists two vertices v_i and v_j such that $p(v_i) \geq 2$ and $p(v_j) \geq 2$, where $3 \leq i, j \leq n-1$. Clearly, we can move two pebbles to the vertex v_0 in both situations and hence we can move one pebble to v_1.

Thus $f_2(W_n) \leq n+4$. □

Theorem 2.6. *For the wheel graph W_n, $f_t(W_n) = 4t + n - 4$ ($t \geq 2$ and $n \geq 4$).*

Proof. Consider the following distribution: $p(v_0) = 0$, $p(v_1) = 0$, $p(v_2) = 0$, $p(v_3) = 4t - 1$, $p(v_4) = 0$, $p(v_i) = 1$ for all $i \neq 0, 1, 2, 3, 4$. Then we cannot move t pebbles to the vertex v_1. Thus $f_t(W_n) \geq 4t + n - 4$.

Next, we have to prove that $f_t(W_n) \leq 4t+n-4$. We prove this by induction on t. Clearly, it is true for $t = 2$ by Theorem 2.5. So, we assume the result is true for $3 \leq t' < t$. Now, consider the distribution of $4t + n - 4$ pebbles on the vertices of W_n. Since $4t + n - 4 \geq n + 8$ and $f(W_n) = n+1$, we can move one pebble to any target vertex v_i of W_n at a cost of at most four pebbles. Then the remaining number of pebbles on the vertices of W_n is at least $4(t-1) + n - 4$ and hence we can move the additional $t-1$ pebbles to the target vertex v_i by induction. Thus $f_t(W_n) \leq 4t + n - 4$. □

We introduce the following graph in this paper:

Definition 2.7. *Let $W_n * iP_m$ ($1 \leq i \leq n$) be the graph obtained by attaching a copy of P_m each to any of the i rim vertices of W_n.*

Note that $W_n * iP_m$ is a class of graphs depending on the choice of the i rim vertices of W_n. If $i = n$ then this class contains a unique graph.

3. The t-pebbling number of $W_n * nP_2$

For the graph $W_n * nP_2$, we label the pendant vertex as u_i which is adjacent to the rim vertex v_i $(1 \le i \le n)$ of W_n. Thus $V(W_n * nP_2) = V(W_n) \cup \{u_1, u_2, \cdots, u_n\}$.

Let $A = \{v_1, v_2, \cdots, v_n\}$, $B = \{u_1, u_2, \cdots, u_n\}$ and $C = A \cup \{v_0\}$.

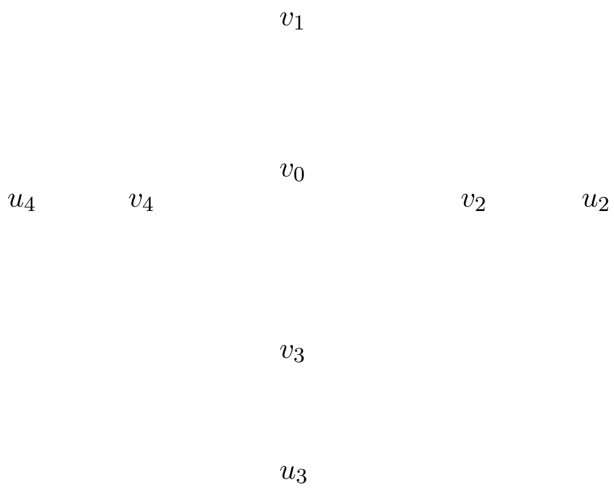

Figure 2. The graph $W_4 * 4P_2$

Theorem 3.1. *For the graph $W_3 * 3P_2$, $f(W_3 * 3P_2) = 12$.*

Proof. Consider the following distribution: $p(u_1) = 0$, $p(u_2) = 3$, $p(u_3) = 7$, $p(v_0) = 1$, $p(v_i) = 0$ for all $i \neq 0$. Then we cannot move one pebble to the vertex u_1. Thus $f(W_3 * 3P_2) \ge 12$.

Now, consider the distribution of 12 pebbles on the vertices of $W_3 * 3P_2$.

Case 1: Let v_i be the target vertex.

Since $< C > \cong W_3$ and $f(W_3) = 4$, we can move one pebble to the vertex v_i if $p(C) \ge 4$. So, we assume $p(C) \le 3$. This implies that, $p(B) \ge 9$ and note that at most three pebbles can be retained on the vertices of B. Hence we can move at least three pebbles to the vertices of A . Clearly we can move one pebble to v_i if $p(C) \ge 1$. Let $p(C) = 0$. Then $p(B) = 12$ and note that at most two pebbles can be retained on the vertices of B. Thus we can move at least four pebbles to the vertices of A and hence we can move one pebble to v_i easily.

Case 2: Let u_1 be the target vertex.

Since $< C > \cong W_3$ and $f_2(W_3) = 6$, we can move two pebbles to the vertex v_1 if $p(C) \ge 6$ and hence we can move one pebble to u_1. So, we assume

$p(C) = x \leq 5$. Thus $p(u_2) + p(u_3) = 12 - x$ and note that at most two pebbles can be retained on the vertices u_2 and u_3. Thus we can move at least $5 - \lfloor \frac{x}{2} \rfloor$ pebbles to the vertices of A. Now, the number of pebbles on the vertices of C is at least $x + 5 - \lfloor \frac{x}{2} \rfloor \geq 5 + \lceil \frac{x}{2} \rceil$. Clearly we can move two pebbles to v_1, if $x \geq 1$ and hence we can move one pebble to u_1 easily. Let $p(C) = 0$. We have $p(u_2) + p(u_3) = 12$. Without loss of generality, let $p(u_2) \geq 6$. Since $d(u_1, u_2) = 3$, we can move one pebble to u_1, if $p(u_2) \geq 8$. Assume $p(u_2) = 6$ or 7. We get $p(u_3) \geq 4$ and hence we move one pebble each to v_1 from the vertices u_2 and u_3. So, one pebble can be moved to the vertex u_1.

Thus $f(W_3 * 3P_2) \leq 12$. □

Theorem 3.2. *For the graph $W_n * nP_2$, $f(W_n * nP_2) = 3n + 6$ $(n \geq 4)$.*

Proof. Consider the following distribution: $p(u_1) = 0$, $p(u_2) = 1$, $p(u_3) = 15$, $p(u_n) = 1$, $p(u_j) = 3$ for all $j \neq 1, 2, 3, n$, and $p(v_i) = 0$ for all i. Then we cannot move one pebble to the vertex u_1. Thus $f(W_n * nP_2) \geq 3n + 6$.

Now, consider the distribution of $3n+6$ pebbles on the vertices of $W_n * nP_2$.

Case 1: Let v_i be the target vertex.

Case 1.1: Let $p(C) \geq n + 1$.

Clearly, we can move one pebble to v_i, since $< C > \cong W_n$ and $f(W_n) = n + 1$.

Case 1.2: Let $0 \leq p(C) = x \leq n$.

We have $p(B) = 3n + 6 - x$ and note that at most n pebbles can be retained on the vertices of B. Thus we can move at least $n + 3 - \lfloor \frac{x}{2} \rfloor$ pebbles to the vertices of A. Now, the number of pebbles on the vertices of C is at least $x + n + 3 - \lfloor \frac{x}{2} \rfloor \geq n + 1$ and hence we can move one pebble to v_i.

Case 2: Let u_1 be the target vertex.

Case 2.1: Let $p(C) \geq n + 4$.

Clearly, we can move two pebbles to v_1, since $< C > \cong W_n$ and $f_2(W_n) = n+4$. Hence we can move one pebble to u_1 easily.

Case 2.2: Let $0 \leq p(C) = x \leq n + 3$.

We have $p(B - \{u_1\}) = 3n + 6 - x$ and note that at most $n - 2$ pebbles can be retained on the vertices of B. Thus we can move at least $n + 4 - \lfloor \frac{x}{2} \rfloor$ pebbles to the vertices of A. Now, the number of pebbles on the vertices of C is at least $x + n + 4 - \lfloor \frac{x}{2} \rfloor \geq n + 4$ and hence we can move one pebble to u_1 through v_1.

Thus $f(W_n * nP_2) \leq 3n + 6$. □

Theorem 3.3. *For the graph $W_3 * 3P_2$, $f_t(W_3 * 3P_2) = 8t + 4$.*

Proof. Consider the following distribution: $p(u_3) = 8t - 1$, $p(u_2) = 3$, $p(v_0) = 1$, $p(u_1) = 0$, $p(v_i) = 0$ for all $i \neq 0$. Then we cannot move t pebbles to the vertex u_1. Thus $f_t(W_3 * 3P_2) \geq 8t + 4$.

Next, we have to prove that $f_t(W_3 * 3P_2) \leq 8t + 4$. We prove this by induction on t. Clearly, it is true for $t = 1$ by Theorem 3.1. So, we assume the result is true for $2 \leq t' < t$. Now, consider the distribution of $8t+4$ pebbles on the vertices of $W_3 * 3P_2$. Since $8t+4 \geq 20$ and $f(W_3 * 3P_2) = 12$, we can move one pebble to any target vertex of $W_3 * 3P_2$ at a cost of at most eight pebbles. Then the remaining number of pebbles on the vertices of $W_3 * 3P_2$ is at least $8(t-1)+4$ and hence we can move the additional $t-1$ pebbles to the target vertex by induction. Thus $f_t(W_3 * 3P_2) \leq 8t+4$. □

Theorem 3.4. *For the graph $W_n * nP_2$, $f_t(W_n * nP_2) = 16t + 3n - 10$ $(n \geq 4)$.*

Proof. Consider the following distribution: $p(u_1) = 0$, $p(u_2) = 1$, $p(u_3) = 16t-1$, $p(u_n) = 1$, $p(u_j) = 3$ for all $j \neq 1, 2, 3, n$, and $p(v_i) = 0$ for all i. Then we cannot move t pebbles to the vertex u_1. Thus $f_t(W_n * nP_2) \geq 16t + 3n - 10$.

Next, we have to prove that $f_t(W_n * nP_2) \leq 16t + 3n - 10$. We prove this by induction on t. Clearly, it is true for $t = 1$ by Theorem 3.2. So, we assume the result is true for $2 \leq t' < t$. Now, consider the distribution of $16t+3n-10$ pebbles on the vertices of $W_n * nP_2$. Since $16t+3n-10 \geq 3n+22$ and $f(W_n * nP_2) = 3n+6$, we can move one pebble to any target vertex of $W_n * nP_2$ at a cost of at most sixteen pebbles. Then the remaining number of pebbles on the vertices of $W_n * nP_2$ is at least $16(t-1) + 3n - 10$ and hence we can move the additional $t-1$ pebbles to the target vertex by induction. Thus $f_t(W_n * nP_2) \leq 16t + 3n - 10$. □

4. The t-pebbling number of $W_n * P_m$

Let $V(P_m) = \{u_1, u_2, \cdots, u_m\}$ where P_m is the path on $m \geq 2$ vertices. Without loss of generality, we attach a copy of P_m to the rim vertex v_1 of W_n. Thus $V(W_n * P_m) = V(W_n) \cup (V(P_m) - \{u_1\})$. Let $D = \{v_0, v_1, v_2, \cdots, v_n\}$, $E = \{u_2, \cdots, u_m\}$ and $F = \{v_1\} \cup E$. Note that $< D > \cong W_n$, $< E > \cong P_{m-1}$ and $< F > \cong P_m$.

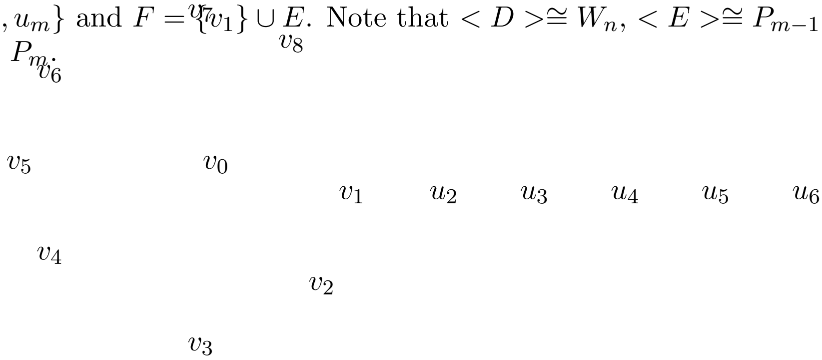

Figure 3. The graph $W_8 * P_6$

Theorem 4.1. *For the graph $W_3 * P_m$ $(m \geq 2)$, $f(W_3 * P_m) = 2^m + 2$.*

Proof. Consider the following distribution: $p(v_0) = 0, p(v_1) = 0, p(v_2) = 1, p(v_3) = 1, p(u_m) = 2^m - 1$ and $p(u_j) = 0$ where $2 \leq j \leq m-1$. Then we cannot move one pebble to the vertex v_0. Thus, $f(W_3 * P_m) \geq 2^m + 2$.

Now, consider the distribution of 2^m+2 pebbles on the vertices of $W_3 * P_m$.
Case 1: Let v_i be the target vertex. $(0 \le i \le 3)$
If $p(D) \ge 4$ then clearly we can move one pebble to v_i, since $f(W_3) = 4$.
So assume that $p(D) \le 3$. This implies that $p(E) \ge 2^m - 1 \ge 2^{m-1}$ and hence we can move one pebble to v_1 (since $f(P_m) = 2^{m-1}$). Clearly we are done if v_1 is our target vertex. Let v_i $(i \ne 1)$ be our target.
If $p(D) = 3$ then $p(E) = 2^m - 1$ and hence we can move one pebble to v_1, since $f(P_m) = 2^{m-1}$. Now $p(D) + 1 = 4$ and thus we can move one pebble to v_i easily.
If $0 \le p(D) \le 2$ then we can move two pebbles to v_1 easily (since $p(E) \ge 2^m$). Also, note that, $d(v_i, v_1) = 1$ $(i \ne 1)$. Thus we can move one pebble to v_i easily, from v_1.
Case 2: Let u_j be the target vertex. $(2 \le j \le m)$
If $p(E) \ge 2^{m-2}$ then clearly we can move one pebble to u_j, since $f(P_{m-1}) = 2^{m-2}$.
If $p(E) = y \le 2^{m-2} - 1$ then $p(D) = 2^m + 2 - y$ and hence we can move $\frac{2^m+2-y-3}{2} = \frac{2^m-1-y}{2} \ge 2^{m-1} - \left\lfloor \frac{y+1}{2} \right\rfloor$ pebbles to the vertex v_1 from the vertices of D. Thus we have $2^{m-1} + y - \left\lfloor \frac{y+1}{2} \right\rfloor \ge 2^{m-1}$ pebbles on the vertices of F. Hence, we can move one pebble to u_j, since $f(P_m) = 2^{m-1}$.
Thus $f(W_3 * P_m) \le 2^m + 2$. □

Theorem 4.2. *For the graph $W_n * P_m$ ($n \ge 4$ and $m \ge 2$), $f(W_n * P_m) = 2^{m+1} + n - 4$.*

Proof. Consider the following distribution: $p(v_0) = 0, p(v_1) = 0, p(v_2) = 0, p(v_3) = 0, p(v_n) = 0, p(v_i) = 1$(for all $i \ne 0, 1, 2, 3, n$), $p(u_m) = 2^{m+1} - 1$ and $p(u_j) = 0$ where $2 \le j \le m-1$. Then we cannot move one pebble to the vertex v_3. Thus, $f(W_n * P_m) \ge 2^{m+1} + n - 4$.

Now, consider the distribution of $2^{m+1} + n - 4$ pebbles on the vertices of $W_n * P_m$.
Case 1: Let v_i be the target vertex $(0 \le i \le n)$.
If $p(D) \ge n+1$ then clearly we can move one pebble to v_i, since $f(W_n) = n+1$.
So assume that $p(D) \le n$. This implies that $p(E) \ge 2^{m+1} - 4 \ge 2^m$ and hence we can move at least two pebbles to v_1 (since $f(P_m) = 2^{m-1}$). Clearly we are done if v_k is our target vertex, where $k = 0, 1, 2, n$. Let v_i $(i \ne 0, 1, 2, n)$ be our target. If $p(v_0) = 1$ then we can move one pebble to v_i easily. Assume $p(v_0) = 0$. If there exists a vertex v_l such that $p(v_l) \ge 2$ $(1 \le l \le n)$, then also we can move two pebbles to v_0 and hence we can move one pebble to v_i. So, we assume $p(v_l) \le 1$ for all $l = 1$ to n. Clearly, $p(D) \ne n$ by Remark 1.9. Let $p(D) = n-1$ or $n-2$. We can see that there exists a path $v_1 v_2 \cdots v_{i-1}$ or the path $v_1 v_n \cdots v_{i+1}$ such that each vertex has exactly one pebble on it. Thus we can move one pebble to v_i easily (we use the two pebbles of v_1 which

are received from the vertices of E). If $p(D) = n - 3$ then we move at least three pebbles to the vertex v_1 from the vertices of E. Clearly, we are done if $p(v_1) = 1$. So, we assume that $p(v_1) = 0$. Now, there exists a path $v_2v_3 \cdots v_{i-1}$ or the path $v_nv_{n-1} \cdots v_{i+1}$ such that each vertex has exactly one pebble on it. Thus we can move one pebble to v_i easily. If $0 \leq p(D) \leq n - 4$ then we can move four pebbles to v_1 easily (since $p(E) \geq 2^{m+1}$). Also, note that, $d(v_i, v_1) \leq 2$. Thus we can move one pebble to v_i easily.

Case 2: Let u_j be the target vertex. $(2 \leq j \leq m)$

If $p(E) \geq 2^{m-2}$ then clearly we can move one pebble to u_j, since $f(P_{m-1}) = 2^{m-2}$.

If $p(E) = y \leq 2^{m-2} - 1$ then $p(D) = 2^{m+1} + n - 4 - y$ and hence we can move $\frac{2^{m+1}+n-4-y-n}{4} = \frac{2^{m+1}-4-y}{4} \geq 2^{m-1} - \left\lfloor \frac{y+4}{4} \right\rfloor$ pebbles to the vertex v_1 from the vertices of D. Thus we have $2^{m-1} + y - \left\lfloor \frac{y+4}{4} \right\rfloor \geq 2^{m-1}$ (only for $y \geq 1$) pebbles on the vertices of F. Hence, we can move one pebble to u_j if $y \geq 1$, since $f(P_m) = 2^{m-1}$. If $y = 0$ then $p(D) = 4(2^{m-1}) + n - 4$. Clearly, we can move 2^{m-1} pebbles to the vertex v_1 (by Theorem 2.6) and hence we can move one pebble to the vertex u_j.

Thus $f(W_n * P_m) \leq 2^{m+1} + n - 4$. □

Theorem 4.3. *For the graph $W_3 * P_m$, $f_t(W_3 * P_m) = t.2^m + 2$.*

Proof. Consider the following distribution: $p(v_0) = 0, p(v_1) = 0, p(v_2) = 1, p(v_3) = 1, p(u_m) = t.2^m - 1$ and $p(u_j) = 0$ where $2 \leq j \leq m - 1$. Then we cannot move t pebbles to the vertex v_0. Thus, $f_t(W_3 * P_m) \geq t.2^m + 2$.

Next, we have to prove that $f_t(W_3 * P_m) \leq t.2^m + 2$. We prove this by induction on t. Clearly, it is true for $t = 1$ by Theorem 4.1. So, we assume the result is true for $2 \leq t' < t$. Now, consider the distribution of $t.2^m + 2$ pebbles on the vertices of $W_3 * P_m$. Since $t.2^m + 2 \geq 2.2^m + 2$ and $f(W_3 * P_m) = 2^m + 2$, we can move one pebble to any target vertex of $W_3 * P_m$ at a cost of at most 2^m pebbles. Then the remaining number of pebbles on the vertices of $W_3 * P_m$ is at least $(t - 1)2^m + 2$ and hence we can move the additional $t - 1$ pebbles to the target vertex by induction. Thus $f_t(W_3 * P_m) \leq t.2^m + 2$. □

Theorem 4.4. *For the graph $W_n * P_m$ $(n \geq 4)$, $f_t(W_n * P_m) = t.2^{m+1} + n - 4$.*

Proof. Consider the following distribution: $p(v_0) = 0, p(v_1) = 0, p(v_2) = 0, p(v_3) = 0, p(v_n) = 0, p(v_i) = 1$(for all $i \neq 0, 1, 2, 3, n$)$, p(u_m) = t.2^{m+1} - 1$ and $p(u_j) = 0$ where $2 \leq j \leq m - 1$. Then we cannot move t pebbles to the vertex v_3. Thus, $f_t(W_n * P_m) \geq t.2^{m+1} + n - 4$.

Next, we have to prove that $f_t(W_n * P_m) \leq t.2^{m+1} + n - 4$. We prove this by induction on t. Clearly, it is true for $t = 1$ by Theorem 4.2. So, we assume the result is true for $2 \leq t' < t$. Now, consider the distribution of $t.2^{m+1} + n - 4$ pebbles on the vertices of $W_n * P_m$. Since $t.2^{m+1} + n - 4 \geq 4.2^m + n - 4$ and

$f(W_n * P_m) = 2^{m+1} + n - 4$, we can move one pebble to any target vertex of $W_n * P_m$ at a cost of at most 2^{m+1} pebbles. Then the remaining number of pebbles on the vertices of $W_n * P_m$ is at least $(t-1)2^{m+1} + n - 4$ and hence we can move the additional $t - 1$ pebbles to the target vertex by induction. Thus $f_t(W_n * P_m) \leq t.2^{m+1} + n - 4$. □

References

[1] F.R.K. Chung, Pebbling in hypercubes, SIAM J. Disc. Math., 2 (4) (1989), 467-472.

[2] D.S. Herscovici, A.W. Higgins, The pebbling number of $C_5 \times C_5$, Discrete Mathematics 187 (1998) 123-135.

[3] A. Lourdusamy, t-pebbling the graphs of diameter two, Acta Ciencia Indica, XXLX (M. No.3) (2003), 465-470.

[4] A. Lourdusamy, and T. Mathivanan, The t-pebbling number of Jahangir graph $J_{3,m}$, Proyecciones Journal of Mathematics, Vol. 34, No. 2 (2015), 161-174.

[5] A. Lourdusamy, and T. Mathivanan, The t-pebbling number of squares of cycles, Journal of Prime Research in Mathematics, Vol. 11 (2015), 61-76.

[6] A. Lourdusamy and A. Punitha Tharani, On t-pebbling graphs, Utilitas Mathematica, Vol. 87 (March 2012), 331-342.

[7] A. Lourdusamy, S. Samuel Jayaseelan and T. Mathivanan, The t-pebbling number of Jahangir graph, International Journal of Mathematical Combinatorics, Vol. 1 (2012), 92-95.

[8] L. Pachter, H.S. Snevily and B. Voxman, On pebbling graphs, Congressus Numerantium, 107 (1995), 65-80.