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VERTEX-TO-CLIQUE DETOUR DISTANCE IN GRAPHS

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ABSTRACT. Let v be a vertex and C a clique in a connected graph G. A vertex-to-clique $u - C$ path P is a $u - v$ path, where v is a vertex in C such that P contains no vertices of C other than v. The vertex-to-clique distance, $d(u, C)$ is the length of a smallest $u-C$ path in G. A $u-C$ path of length $d(u, C)$ is called a $u - C$ geodesic. The vertex-to-clique eccentricity $e_1(u)$ of a vertex u in G is the maximum vertex-to-clique distance from u to a clique $C \in \zeta$, where ζ is the set of all cliques in G. The vertex-to-clique radius r_1 of G is the minimum vertex-to-clique eccentricity among the vertices of G, while the vertex-to-clique diameter d_1 of G is the maximum vertex-to-clique eccentricity among the vertices of G. Also the vertex-to-clique detour distance, $D(u, C)$ is the length of a longest $u-C$ path in G. A $u-C$ path of length $D(u, C)$ is called a $u-C$ detour. The vertex-to-clique detour eccentricity $e_{D1}(u)$ of a vertex u in G is the maximum vertex-to-clique detour distance from u to a clique $C \in \zeta$ in G. The vertex-to-clique detour radius R_1 of G is the minimum vertex-to-clique detour eccentricity among the vertices of G, while the vertex-to-clique detour diameter D_1 of G is the maximum vertex-to-clique detour eccentricity among the vertices of G. It is shown that $R_1 \leq D_1$ for every connected graph G and that every two positive integers a and b with $2 \leq a \leq b$ are realizable as the vertex-to-clique detour radius and the vertex-to-clique detour diameter, respectively, of some connected graph. Also it is shown that for any three positive integers a, b, c with $2 \leq a \leq b < c$, there exists a connected graph G such that $r_1 = a$, $R_1 = b$, $R = c$ and for any three positive integers a, b, c with $2 \leq a \leq b < c$ and $a + c \leq 2b$, there exists a connected graph G such that $d_1 = a$, $D_1 = b$, $D = c$.

Key words : vertex-to-clique distance, vertex-to-clique detour distance, vertex-to-clique detour center, vertex-to-clique detour periphery.
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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected simple graph. For basic graph theoretic terminologies, we refer to Chartrand and Zhang [2]. If $X \subseteq V$, then $\langle X \rangle$ is the subgraph induced by X. A clique C of a graph G is a maximal complete subgraph and we denote it by its vertices. A $u - v$ path P beginning with u and ending with v in G is a sequence of distinct vertices such that consecutive vertices in the sequence are adjacent in G. For a graph G, the length of a path is the number of edges on the path.

For example if one is locating an emergency facility like police station, fire station, hospital, school, college, library, ambulance depot, emergency care center, etc., then the primary aim is to minimize the distance between the facility and the location of a possible emergency. In 1964, Hakimi [3] considered the facility location problems as vertex-to-vertex distance in graphs. For any two vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u-v$ path in G. A $u-v$ path of length $d(u, v)$ is called a $u - v$ geodesic in G. For a vertex v in G, the eccentricity of v is defined by $e(v) = \max\{d(v, u) : u \in V\}$. A vertex u of G such that $d(u, v) = e(v)$ is called an eccentric vertex of v. The radius r and diameter d of G are defined by $r = rad(G) = \min\{e(v) : v \in V\}$ and $d = diam(G) = \max\{e(v) : v \in V\}$ respectively. A vertex v in G is called a central vertex if $e(v) = r$ and the center of G is defined by $C(G) = Cen(G) = \langle\{v \in V : e(v) = r\}\rangle$. A vertex v in G is called a peripheral vertex if $e(v) = d$ and the periphery of G is defined by $P(G) = Per(G) = \langle\{v \in V : e(v) = d\}\rangle$. If every vertex of a graph is central vertex then G is called self-centered.

For example if one is making an election canvass or circular bus service the distance from the location is to be maximized. In 2005, Chartrand et. al. [1] introduced and studied the concepts of detour distance in graphs. For any two vertices u and v in a connected graph G, the detour distance $D(u, v)$ is the length of a longest $u - v$ path in G. A $u - v$ path of length $D(u, v)$ is called a $u - v$ detour in G. For a vertex v in G, the detour eccentricity of v is defined by $e_D(v) = \max\{D(v, u) : u \in V\}$. A vertex u of G such that $D(u, v) = e_D(v)$ is called a detour eccentric vertex of v. The detour radius R and detour diameter D of G are defined by $R = rad_D G = \min\{e_D(v) : v \in V\}$ and $D = diam_D(G) = \max\{e_D(v) : v \in V\}$ respectively. A vertex v in G is called a detour central vertex if $e_D(v) = R$ and the detour center of G is defined by $C_D(G) = Cen_D(G) = \langle\{v \in V : e_D(v) = R\}\rangle$. A vertex v in G is called a detour peripheral vertex if $e_D(v) = D$ and the detour periphery of G is defined by $P_D(G) = Per_D(G) = \langle\{v \in V : e_D(v) = D\}\rangle$. If every vertex of a graph is detour central vertex then G is called detour self-centered.

For example when a railway line, pipe line or highway is constructed, the distance between the respective structure and each of the communities to be served is to be minimized. In a social network a clique represents a group of individuals having a common interest. Thus the centrality with respect to cliques have interesting applications in social networks. In 2002, Santhakumaran and Arumugam [6] introduced the facility locational problem as vertex-to-clique distance in graphs as follows: For a vertex u and a clique C in a connected graph G, the vertex-to-clique distance is defined by $d(u, C) = \min\{d(u, v) : v \in C\}$. The vertex-to-clique eccentricity of u is defined by $e_1(u) = \max\{d(u, C) : C \in \zeta\}$, where ζ is the set of all cliques in G. A clique C of G such that $e_1(u) = d(u, C)$ is called a vertex-to-clique eccentric vertex of u. The vertex-to-clique radius r_1 is defined by $r_1 = \min\{e_1(v) : v \in V\}$ and the vertex-to-clique diameter d_1 of G is defined by $d_1 = \max\{e_1(v) : v \in V\}$. A vertex v in G is called a vertex-to-clique central vertex if $e_1(v) = r_1$ and the vertex-to-clique center of G is defined by $Z_1(G) = \langle\{v \in V : e_1(v) = r_1\}\rangle$. For our convenience $Z_1(G)$ is denoted by $C_1(G)$. A vertex v in G is called a vertex-to-clique peripheral vertex if $e_1(v) = d_1$ and the vertex-to-clique periphery of G is defined by $P_1(G) = \langle\{v \in V : e_1(v) = d_1\}\rangle$. If every vertex of a graph is vertex-to-clique central vertex then G is called vertex-to-clique self-centered.

For example when a dam, lake, pond, river or channel is constructed, the maximum number of places should be covered between the respective structure and each of the communities to be served. These motivated us to introduce a distance called the vertex-to-clique detour distance in graphs and investigate certain results related to vertex-to-clique detour distance and other distances in graphs. Further these ideas have interesting applications in channel assignment problem in radio technologies. Also there are useful applications of these concepts to security based communication network design. Throughout this paper, G denotes a connected graph with at least two vertices.

2. Vertex-To-Clique Detour Distance

Definition 1. *Let u be a vertex and C a clique in a connected graph G. A vertex-to-clique $u - C$ path P is a $u - v$ path, where v is a vertex in C such that P contains no vertices of C other than v. The vertex-to-clique detour distance $D(u, C)$ is the length of a longest $u - C$ path. A $u - C$ path of length $D(u, C)$ is called a vertex-to-clique $u - C$ detour or simply $u - C$ detour. For our convenience a $u - C$ path of length $d(u, C)$ is called a vertex-to-clique $u - C$ geodesic or simply $u - C$ geodesic.*

Example 1. *Consider the graph G given in Fig 2.1. For the vertex u and the clique $C = \{x, y, z\}$, the paths $P_1 : u, t, s, x$, $P_2 : u, w, v, r, z$ and $P_3 : u, z$ are $u - C$ paths, while the paths $Q_1 : u, t, s, x, y, z$ and $Q_2 : u, w, v, r, z, y, x$ are not $u - C$ paths. Now the vertex-to-clique distance $d(u, C) = 1$ and the vertex-to-clique detour distance $D(u, C) = 4$. Thus the vertex-to-clique distance is* *different from the vertex-to-clique detour distance. Also P_2 is a $u-C$ detour and P_3 is a $u-C$ geodesic. Note that the $x-C$, $y-C$ and $z-C$ paths are trivial paths of length 0 and any non-trivial $u-C$ path does not contain a simplicial vertex of C.*

y

x z

s r

t v

u w

Fig. 2.1: G

Since the length of a $u-C$ path between a vertex u and a clique C in a graph G of order n is at most $n-2$, we have the following observation.

Observation 1. *For any vertex u and a clique C in a non-trivial connected graph G of order n, $0 \leq d(u,C) \leq D(u,C) \leq n-2$. The bounds are sharp. If G is a complete graph of order n, then $d(u,C) = D(u,C) = 0$ for every vertex u in G and if G is a path $P : u = u_1, u_2, \ldots, u_{n-1}, u_n$ of order n, then $d(u,C) = D(u,C) = n-2$, where $C = \{u_{n-1}, u_n\}$. Also we note that if G is a tree, then $d(u,C) = D(u,C)$ and if G is an even cycle with $u \notin C$, then $d(u,C) < D(u,C)$ for every vertex u in G.*

Since a vertex of degree $n-1$ in a graph G of order n, belongs to every clique C in G, we have the following observation.

Observation 2. *Let G be a connected graph of order n and C a clique in G. If u is a vertex of degree $n-1$, then $D(u,C) = 0$. The converse is not true. Consider the graph G given in Fig. 2.1, $D(u,C) = 0$, where $C = \{u, z\}$, but $deg(u) \neq n-1$.*

Theorem 1. *Let $K_{n,m} (n < m)$ be a complete bipartite graph with the partition V_1, V_2 of $V(K_{n,m})$ such that $|V_1| = n$ and $|V_2| = m$. Let u be a vertex and C a clique such that $u \notin C$ in $K_{n,m}$, then*

$$D(u,C) = \begin{cases} 2n-2, & \textit{if } u \in V_1 \\ 2n-1 & \textit{if } u \in V_2 \end{cases}$$

Proof. Let $V_1 = \{x_1, x_2, \ldots, x_n\}$ and $V_2 = \{y_1, y_2, \ldots, y_n, y_{n+1}, \ldots, y_m\}$. Since any edge is a clique, without loss of generality assume that $C = \{x_n, y_n\}$ is a clique and $u = x_1$ or $u = y_1$.

Case 1. $u = x_1$. Let $P_1 : u = x_1, y_1, x_2, y_2, \ldots, x_{n-1}, y_{n-1}, x_n$ be a longest $u - x_n$ path, which has n vertices of V_1 and $n-1$ vertices of V_2. It must contain $2n-1$ vertices of $K_{n,m}$. So that its length is $2n-2$. Also $P_2 : u = x_1, y_1, x_2, y_2, \ldots, x_{n-1}, y_n$ be a longest $u - y_n$, which has $n-1$ vertices of V_1 and $n-1$ vertices of V_2. It must contain $2n-2$ vertices of $K_{n,m}$. So that its length is $2n-3$. Thus $D(u, C) = 2n-2$ if $u \in V_1$.
Case 2. $u = y_1$. Let $Q_1 : u = y_1, x_1, y_2, x_2, \ldots, y_{n-1}, x_{n-1}, y_{n+1}, x_n$ be a longest $u - x_n$ path, which has n vertices of V_1 and n vertices of V_2. It must contain $2n$ vertices of $K_{n,m}$. So that its length is $2n-1$. Also Q_2 : $u = y_1, x_1, y_2, x_2, \ldots, y_{n-1}, x_{n-1}, y_n$ be a longest $u - y_n$ path, which has $n-1$ vertices of V_1 and n vertices of V_2. It must contain $2n-1$ vertices of $K_{n,m}$. So that its length is $2n-2$. Thus $D(u, C) = 2n-1$ if $u \in V_2$. □

Corollary 2. *Let u be a vertex and C a clique in a complete bipartite graph $K_{n,n}$ such that $u \notin C$, then $D(u, C) = 2n-2$.*

Since every tree has unique $u - C$ path between a vertex u and a clique C, we have the following observation.

Observation 3. *If G is a tree, then $d(u, C) = D(u, C)$ for every vertex u and a clique C in G. The converse is not true. For the graph G obtained from a complete bipartite graph $K_{2,n} (n \geq 2)$ by joining the vertices of degree n by an edge. In such a graph every clique C is isomorphic to K_3 and for every vertex u in G with $u \notin C$, $d(u, C) = D(u, C) = 1$ and for every vertex u in G with $u \in C$, $d(u, C) = D(u, C) = 0$, but G is not tree.*

3. Vertex-To-Clique Detour Central Concepts

Definition 2. *The vertex-to-clique detour eccentricity $e_{D1}(u)$ of a vertex u in a connected graph G is defined as $e_{D1}(u) = \max\{D(u, C) : C \in \zeta\}$, where ζ is the set of all cliques in G. A clique C for which $e_{D1}(u) = D(u, C)$ is called a vertex-to-clique detour eccentric clique of u. The vertex-to-clique detour radius of G is defined as, $R_1 = rad_{D1}(G) = \min\{e_{D1}(v) : v \in V\}$ and the vertex-to-clique detour diameter of G is defined as, $D_1 = diam_{D1}(G) = \max\{e_{D1}(v) : v \in V\}$. A vertex v in G is called a vertex-to-clique detour central vertex if $e_{D1}(v) = R_1$ and the vertex-to-clique detour center of G is defined as, $C_{D1}(G) = Cen_{D1}(G) = \langle\{v \in V : e_{D1}(v) = R_1\}\rangle$. A vertex v in G is called a vertex-to-clique detour peripheral vertex if $e_{D1}(v) = D_1$ and the vertex-to-clique detour periphery of G is defined as, $P_{D1}(G) = Per_{D1}(G) = \langle\{v \in V : e_{D1}(v) = D_1\}\rangle$. If every vertex of G is a vertex-to-clique detour central vertex, then G is called a vertex-to-clique detour self centered graph. If G is a vertex-to-clique detour self-centered graph, then G is its own vertex-to-clique detour periphery.*

Example 2. *For the connected graph G given in Fig. 3.1, the set of all cliques in G are given by,* $\zeta = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}\}$ *where* $C_1 = \{v_1, v_2, v_3\}$, $C_2 = \{v_3, v_4\}$, $C_3 = \{v_4, v_5\}$, $C_4 = \{v_5, v_6\}$, $C_5 = \{v_6, v_7\}$, $C_6 = \{v_7, v_8\}$, $C_7 = \{v_8, v_{10}\}$, $C_8 = \{v_9, v_{10}\}$, $C_9 = \{v_4, v_9\}$, *and* $C_{10} = \{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$.

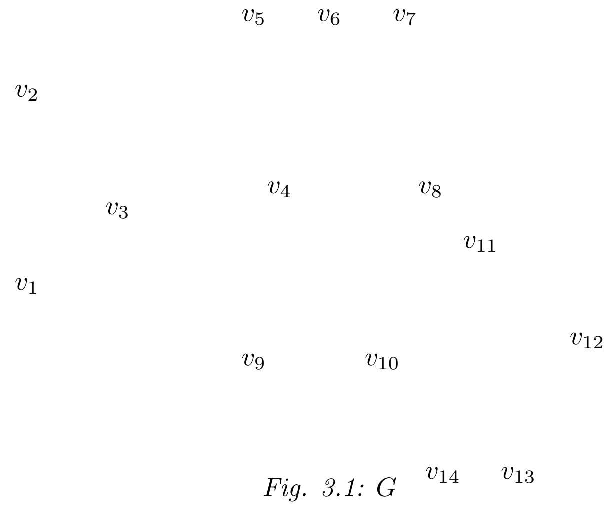

Fig. 3.1: G

The eccentricity $e(v)$, vertex-to-clique eccentricity $e_1(v)$, detour eccentricity $e_D(v)$, vertex-to-clique detour eccentricity $e_{D1}(v)$ of all the vertices of G are given in Table 1.

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}
$e(v)$	5	5	4	3	4	4	5	5	3	4	5	5	5	5
$e_1(v)$	5	5	4	3	3	3	4	4	3	3	4	4	4	4
$e_D(v)$	12	12	10	9	9	8	9	10	10	8	12	12	12	12
$e_{D1}(v)$	8	8	6	5	7	6	5	6	7	6	10	10	10	10

Table 1

The vertex-to-clique detour eccentric clique of all the vertices of G are given in Table 2. The radius $r = 3$, the diameter $d = 5$, the vertex-to-clique radius $r_1 = 3$, the vertex-to-clique diameter $d_1 = 5$, the detour radius $R = 8$, the detour diameter $D = 12$, the vertex-to-clique detour radius $R_1 = 5$ and the vertex-to-clique detour diameter $D_1 = 10$. Also the center $C(G) = \langle\{v_4, v_9\}\rangle$, the periphery $P(G) = \langle\{v_1, v_2, v_7, v_8, v_{11}, v_{12}, v_{13}, v_{14}\}\rangle$, the vertex-to-clique center $C_1(G) = \langle\{v_4, v_5, v_6, v_9, v_{10}\}\rangle$, the vertex-to-clique periphery $P_1(G) = \langle\{v_1, v_2\}\rangle$, the detour center $C_D(G) = \langle\{v_6, v_{10}\}\rangle$, the detour periphery $P_D(G) = \langle\{v_1, v_2, v_{11}, v_{12}, v_{13}, v_{14}\}\rangle$, the vertex-to-clique detour center $C_{D1}(G) = \langle\{v_4, v_7\}\rangle$ and the vertex-to-clique detour periphery $P_{D1}(G) =$

$\langle\{v_{11}, v_{12}, v_{13}, v_{14}\}\rangle$. In a connected graph G, $C(G)$, $C_1(G)$, $C_D(G)$, $C_{D1}(G)$ and $P(G)$, $P_1(G)$, $P_D(G)$, $P_{D1}(G)$ need not be same. For the the graph G given in Fig 3.1, $C(G)$, $C_1(G)$, $C_D(G)$, $C_{D1}(G)$ and $P(G)$, $P_1(G)$, $P_D(G)$, $P_{D1}(G)$ are distinct. Also no cut vertex in a connected graph G is a vertex-to-clique detour peripheral vertex of G.

Vertex v	*Vertex-to-Clique Detour Eccentric Clique C*
v_1, v_2, v_3, v_4	C_4, C_8, C_{10}
v_7	C_1, C_4, C_7, C_{10}
v_8	C_{10}
$v_5, v_6, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}$	C_1

Table 2

The vertex-to-clique detour radius R_1, the vertex-to-clique detour diameter D_1 of some standard graphs are given in Table 3.

Graph G	K_n	P_n	$C_n (n \geq 4)$	$W_n (n \geq 5)$	$K_{n,m} (m \geq n)$
R_1	*0*	$\left\lfloor \frac{n-2}{2} \right\rfloor$	$n-2$	*0*	$2(n-1)$
D_1	*0*	$n-2$	$n-2$	$n-3$	$\begin{cases} 2(n-1), & \text{if } n = m \\ 2n-1 & \text{if } n > m \end{cases}$

Table 3

Remark 1. *In a connected graph G, $C_{D1}(G)$ and $P_{D1}(G)$ need not be connected. For the graph G given in Fig 3.2, $C_{D1}(G) = \langle\{v_2, v_4\}\rangle$ and $P_{D1}(G) = \langle\{v_1, v_3, v_5\}\rangle$ are disconnected.*

v_1

v_5 v_2

v_4 v_3

Fig. 3.2: G

Example 3. *The complete graph K_n, the cycle C_n and the complete bipartite graph $K_{n,n}$ are vertex-to-clique detour self centered graphs.*

Remark 2. *A self-centered graph need not be a vertex-to-clique detour self centered graph. For the graph G given in Fig 3.3, $C(G) = \langle V(G)\rangle$ and $C_{D1}(G) = \langle\{v_{11}\}\rangle$.*

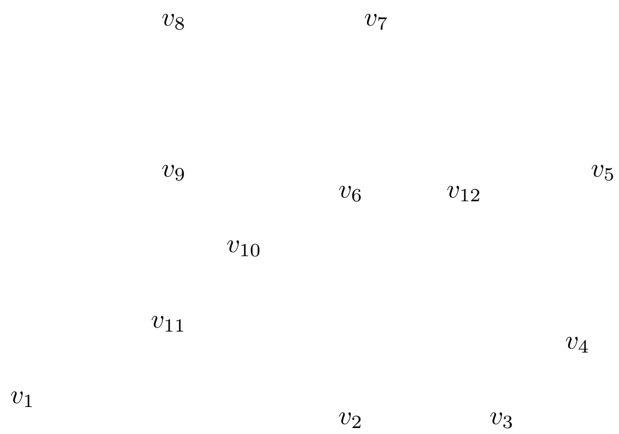

Fig. 3.3: G

Remark 3. *A detour self-centered graph need not be a vertex-to-clique detour self centered graph. For the wheel* $W_n = K_1 + C_{n-1} (n \geq 4)$, $C_D(W_n) = \langle V(W_n) \rangle$ *and* $C_{D1}(W_n) = \langle V(K_1) \rangle$.

Remark 4. *A vertex-to-clique self-centered graph need not be a vertex-to-clique detour self centered graph. For the graph G given in Fig 3.4,* $C_1(G) = \langle V(G) \rangle$ *and* $C_{D1}(G) = \langle \{v_2, v_3, v_7\} \rangle$.

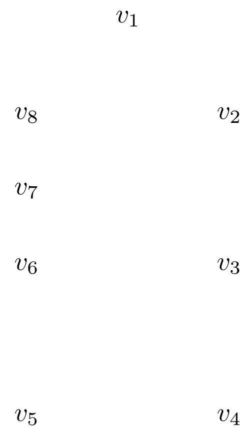

Fig. 3.4: G

Santhakumaran et. al. [6] showed that for any graph G, either $C(G) \subseteq C_1(G)$ or $C_1(G) \subseteq C(G)$. The following example shows that the similar result is not true for the detour center and the vertex-to-clique detour center.

Example 4. *For the graph G given in Fig 3.5,* $C_{D1}(G) = \langle \{v_3\} \rangle$ *and* $C_D(G) = \langle \{v_4\} \rangle$.

v_5

v_1 v_2 v_3 v_4 v_6

v_7

Fig. 3.5: G

Santhakumaran et. al. [6] showed that for any graph G, either $P(G) \subseteq P_1(G)$ or $P_1(G) \subseteq P(G)$. The similar result is true for the detour periphery and the vertex-to-clique detour periphery in G. Now we have the following observation.

Observation 4. *For any graph G, either $P_D(G) \subseteq P_{D1}(G)$ or $P_{D1}(G) \subseteq P_D(G)$.*

Since the vertex-to-clique eccentricity is the maximum vertex-to-clique distance and the vertex-to-clique detour eccentricity is the maximum vertex-to-clique detour distance, the following observation is a consequence of Observation 1.

Observation 5. *For any vertex v in a non-trivial connected graph G of order n, $0 \leq e_1(v) \leq e_{D1}(v) \leq n-2$. The bounds are sharp. If G is a complete graph of order n, then $e_1(v) = e_{D1}(v) = 0$ for every vertex v in G and if G is a path $P : u = u_1, u_2, ..., u_{n-1}, u_n$ of order n, then $e_1(u) = e_{D1}(u) = n-2$. Also we note that if G is a tree, then $e_1(v) = e_{D1}(v)$ for every vertex v in G and if G is an even cycle, then $e_1(v) < e_{D1}(v)$ for every vertex v in G.*

Since the vertex-to-clique radius (diameter) is the minimum (maximum) vertex-to-clique eccentricity and the vertex-to-clique detour radius (diameter) is the minimum (maximum) vertex-to-clique detour eccentricity, the following observation is a consequence of Observation 5.

Observation 6. *Let G be a connected graph. Then*
(i) $0 \leq r_1 \leq R_1 \leq n-2$.
(ii) $0 \leq d_1 \leq D_1 \leq n-2$.

We observe that for every vertex v in a graph G, either $e(v) \leq e_{D1}(v)$ or $e(v) > e_{D1}(v)$ and $e_{D1}(v) \neq e_D(v)$, but $e_{D1}(v) \leq e_D(v) - 1$. Now we have the following observation.

Observation 7. *Let G be a connected graph. Then*
(i) $e_1(v) \leq e_{D1}(v) \leq e_D(v) - 1$ *for every vertex v in G.*
(ii) $r_1 \leq R_1 \leq R - 1$.
(iii) $d_1 \leq D_1 \leq D - 1$.

Chartrand et. al. [2] showed that in a connected graph, the radius and diameter are related by $r \leq d \leq 2r$ and the detour radius and detour diameter are related by $R \leq D \leq 2R$. Also Santhakumaran et. al. [6] showed that the vertex-to-clique radius and vertex-to-clique diameter are related by $r_1 \leq d_1 \leq 2r_1 + 1$. The following example shows that the upper inequality does not hold for the vertex-to-clique detour distance.

Example 5. *For the cycle* $C_n (n \geq 4), D_1 < 2R_1$ *and* $D_1 < 2R_1 + 1$*, the path* P_{2n}*,* $D_1 = 2R_1$ *and the wheel* $W_n (n \geq 5), D_1 > 2R_1$ *and* $D_1 > 2R_1 + 1$*.*

Ostrand [5] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter respectively of some connected graph and Chartrand et. al. [1] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the detour radius and detour diameter respectively of some connected graph. Also Santhakumaran et. al. [6] showed that every two positive integers a and b with $a \leq b \leq 2a + 1$ are realizable as the vertex-to-clique radius and vertex-to-clique diameter respectively of some connected graph. Now we have a realization theorem for the vertex-to-clique detour radius and the vertex-to-clique detour diameter of some connected graph.

Theorem 3. *For each pair* a, b *of positive integers with* $2 \leq a \leq b$*, there exists a connected graph* G *with* $R_1 = a$ *and* $D_1 = b$*.*

Proof. **Case 1.** $a = b$. Let $G = C_{a+2} : u_1, u_2, ..., u_{a+2}, u_1$ be a cycle of order $a + 2$. Then $e_{D1}(u_i) = a$ for $1 \leq i \leq a + 2$. Thus $R_1 = a$ and $D_1 = b$ as $a = b$.
Case 2. $2 \leq a < b \leq 2a$. Let $C_{a+2} : u_1, u_2, \ldots, u_{a+2}, u_1$ be a cycle of order $a+2$ and $P_{b-a+1} : v_1, v_2, \ldots, v_{b-a+1}$ be a path of order $b-a+1$. We construct the graph G of order $b + 2$ as shown in the Fig 3.6 by identifying the vertex u_1 of C_{a+2} with v_1 of P_{b-a+1}.

u_{a+2}
u_{a+1}
v_1 v_2 v_3 v_{b-a} v_{b-a+1}
u_1
u_2
u_3

Fig. 3.6: G

It is easy to verify that

$e_{D1}(u_i) = a$ for $i = 1$

$$e_{D1}(u_i) = \begin{cases} b - i + 2, & \text{if } 2 \leq i \leq \lceil \frac{a+2}{2} \rceil \\ b - a + i - 2, & \text{if } \lceil \frac{a+2}{2} \rceil < i \leq a + 2 \end{cases}$$

$e_{D1}(v_i) = a + i - 1$ for $1 \leq i \leq b - a + 1$
In particular $e_{D1}(u_i) = e_{D1}(v_i) = a$ for $i = 1$
$e_{D1}(u_i) = b$ for $i = 2, a + 2$
$e_{D1}(v_i) = b$ for $i = b - a + 1$
It is easy to verify that there is no vertex x in G with $e_{D1}(x) < a$ and there is no vertex y in G with $e_{D1}(y) > b$. Thus $R_1 = a$ and $D_1 = b$ as $a < b$.
Case 3. $2 \leq a < b > 2a$. Let $P : v_1, v_2, \ldots, v_{2a+1}$ be a path P_{2a+1} of order $2a + 1$. Let $Q : u_1, u_2, \ldots, u_{b-2a+1}$ be a path P_{b-2a+1} of order $b - 2a + 1$. We construct the graph G of order $b + 2$ as shown in the Fig 3.7 by joining each vertex in Q with v_1 in P.

Fig. 3.7: G

u_{b-2a+1} u_{b-2a} u_3 u_2 u_1 v_1 v_2 v_3 v_a v_{a+1} v_{2a} v_{2a+1}

It is easy to verify that

$$e_{D1}(u_i) = \begin{cases} b - i + 1, & \text{if } 1 \leq i \leq \left\lceil \frac{b-2a+1}{2} \right\rceil \\ b - a + i, & \text{if } \left\lceil \frac{b-2a+1}{2} \right\rceil < i \leq b - 2a + 1 \end{cases}$$

$$e_{D1}(v_i) = \begin{cases} 2a - i, & \text{if } 1 \leq i \leq a \\ i - 1, & \text{if } a + 1 \leq i < 2a + 1 \end{cases}$$

$e_{D1}(u_i) = b$ for $i = 1, b - 2a + 1$
$e_{D1}(v_i) = a$ for $i = a, a + 1$
It is easy to verify that there is no vertex x in G with $e_{D1}(x) < a$ and there is no vertex y in G with $e_{D1}(y) > b$. Thus $R_1 = a$ and $D_1 = b$ as $b > 2a$. □

Now we have a realization theorem for the vertex-to-clique radius, vertex-to-clique detour radius and the detour radius of some connected graph.

Theorem 4. *For any three positive integers a, b, c with $2 \leq a \leq b < c$, there exists a connected graph G such that $r_1 = a$, $R_1 = b$ and $R = c$.*

Proof. Let $K = K_{b-a+2}$ be the complete graph of order $b - a + 2$ with $V(K) = \{z_1, z_2, \ldots, z_{b-a+2}\}$. Let $P_1 : x_1, x_2, \ldots, x_{a+1}$ and $P_2 : y_1, y_2, \ldots, y_{a+1}$ be two copies of the path P_{a+1} of order $a+1$. Let $P_3 : s_1, s_2, \ldots, s_{c-b}$ and $P_4 : t_1, t_2, \ldots, t_{c-b}$ be two copies of the path P_{c-b} of order $c-b$. We construct the graph G of order $2c-b+a$ as follows: (i) identify the vertices x_1 in P_1 with z_1 in K, also identify the vertices y_1 in P_2 with z_{b-a+2} in K, (ii) identify the vertices s_1 in P_3 with x_{a+1} in P_1 and t_1 in P_4 with y_{a+1} in P_2, (iii) join every vertex in P_3 other than s_1 with x_a in P_1 and join every vertex in P_4 other than t_1 with y_a in P_2. The resulting graph G is shown in Fig. 3.8.

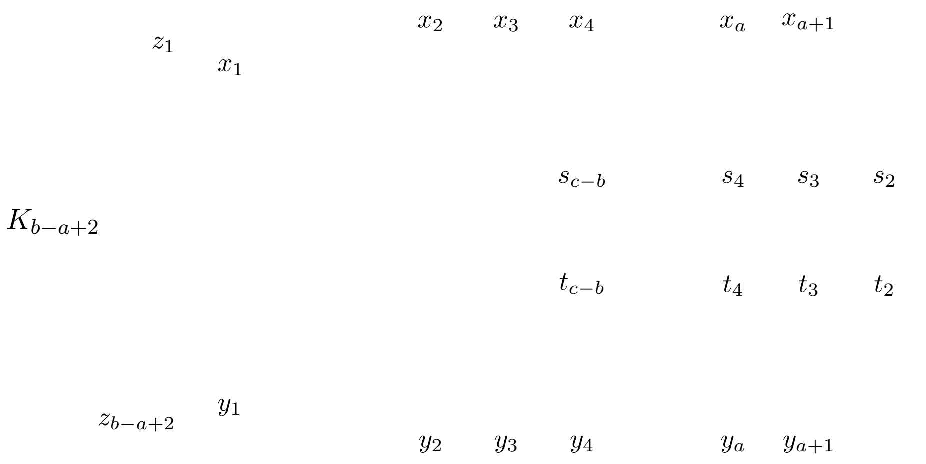

Fig. 3.8: G

It is easy to verify that

$e_1(z_i) = a$ for $1 \le i \le b-a+2$
$e_{D1}(z_i) = b$ for $1 \le i \le b-a+2$
$e_D(z_i) = c$ for $1 \le i \le b-a+2$
$e_1(x_i) = a+i-1$ for $1 \le i \le a$
$e_{D1}(x_i) = b+i-1$ for $1 \le i \le a$
$e_D(x_i) = c+i-1$ for $1 \le i \le a$
$e_1(s_i) = 2a$ for $1 \le i \le c-b$
$e_1(t_i) = 2a$ for $1 \le i \le c-b$

$$e_{D1}(s_i) = \begin{cases} c+a-i, & \text{if } 1 \le i \le \lceil \frac{c-b}{2} \rceil \\ b+a+i-1, & \text{if } \lceil \frac{c-b}{2} \rceil < i \le c-b \end{cases}$$

$$e_{D1}(t_i) = \begin{cases} c+a-i, & \text{if } 1 \le i \le \lceil \frac{c-b}{2} \rceil \\ b+a+i-1, & \text{if } \lceil \frac{c-b}{2} \rceil < i \le c-b \end{cases}$$

$$e_D(s_i) = \begin{cases} 2c + a - b - i, & \text{if } 1 \leq i \leq \left\lceil \frac{c-b}{2} \right\rceil \\ c + a + i - 1, & \text{if } \left\lceil \frac{c-b}{2} \right\rceil < i \leq c - b \end{cases}$$

$$e_D(t_i) = \begin{cases} 2c + a - b - i, & \text{if } 1 \leq i \leq \left\lceil \frac{c-b}{2} \right\rceil \\ c + a + i - 1, & \text{if } \left\lceil \frac{c-b}{2} \right\rceil < i \leq c - b \end{cases}$$

It is easy to verify that there is no vertex x in G with $e_1(x) < a$, $e_{D1}(x) < b$ and $e_D(x) < c$. Thus $r_1 = a$, $R_1 = b$ and $R = c$ as $a \leq b < c$. □

Now we have a realization theorem for the vertex-to-clique diameter, vertex-to-clique detour diameter and the detour diameter of some connected graph.

Theorem 5. *For any three positive integers a, b, c with $2 \leq a \leq b < c$ and $a + c \leq 2b$, there exists a connected graph G such that $d_1 = a$, $D_1 = b$ and $D = c$.*

Proof. Let $K = K_{b-a+2}$ be the complete graph of order $b-a+2$ with $V(K) = \{w_1, w_2, \ldots, w_{b-a+2}\}$. Let $P : u_1, u_2, \ldots, u_{a+1}$ be a path P_{a+1} of order $a+1$. Also let $Q : v_1, v_2, \ldots, v_{c-b}$ be a path of order $c-b$. We construct the graph G of order $c+1$ as follows: (i) identify the vertices u_1 in P with w_1 in K and also identify the vertices u_{a+1} in P with v_1 in Q. (ii) join each vertex in Q other than v_1 with u_a in P. The resulting graph G is shown in Fig. 3.9.

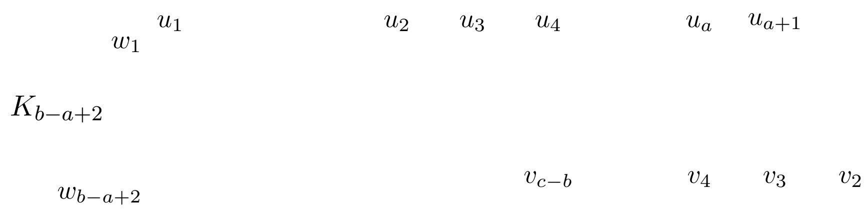

Fig. 3.9: G

It is easy to verify that

$$e_1(w_i) = \begin{cases} a - 1, & \text{if } i = 1 \\ a, & \text{if } 2 \leq i \leq b - a + 2 \end{cases}$$

$$e_{D1}(w_i) = \begin{cases} b - 1, & \text{if } i = 1 \\ b, & \text{if } 2 \leq i \leq b - a + 2 \end{cases}$$

$$e_D(w_i) = \begin{cases} c - 1, & \text{if } i = 1 \\ c, & \text{if } 2 \leq i \leq b - a + 2 \end{cases}$$

$$e_1(u_i) = \begin{cases} a-i, & \text{if } 1 \le i \le \left\lfloor \frac{a+1}{2} \right\rfloor \\ i-1, & \text{if } \left\lfloor \frac{a+1}{2} \right\rfloor < i \le a+1 \end{cases}$$

$$e_{D1}(u_i) = \begin{cases} a-i, & \text{if } 1 \le i \le \left\lfloor \frac{a+1}{2} \right\rfloor \\ i-1, & \text{if } \left\lfloor \frac{a+1}{2} \right\rfloor < i \le a+1 \end{cases}$$

$$e_D(u_i) = \begin{cases} b-a+i, & \text{if } 1 \le i \le a,\ b-a+i \ge c-b+a-i \\ a+c-b-i, & \text{if } 1 \le i \le a,\ b-a+i \le c-b+a-i \\ c, & \text{if } i = a+1 \end{cases}$$

$e_1(v_i) = a$ for $1 \le i \le c-b$

$$e_{D1}(v_i) = \begin{cases} c-b+a-i, & \text{if } 1 \le i \le \left\lfloor \frac{c-b}{2} \right\rfloor \\ a+i-1, & \text{if } \left\lfloor \frac{c-b}{2} \right\rfloor < i \le c-b \end{cases}$$

$$e_D(v_i) = \begin{cases} c+1-i, & \text{if } 1 \le i \le \left\lfloor \frac{c-b}{2} \right\rfloor \\ b+i, & \text{if } \left\lfloor \frac{c-b}{2} \right\rfloor < i \le c-b \end{cases}$$

It is easy to verify that there is no vertex x in G with $e_1(x) > a$, $e_{D1}(x) > b$ and $e_D(x) > c$. Thus $d_1 = a$, $D_1 = b$ and $D = c$ as $a \le b < c$. □

Problem 1. *For any three positive integers a, b, c with $2 \le a \le b < c$ and $a + c > 2b$, does there exist a connected graph G for which $d_1 = a$, $D_1 = b$ and $D = c$?.*

Harary and Norman [4] showed that the center of every connected graph G lies in a single block of G and Chartrand et. al. [1] showed that the detour center of every connected graph G lies in a single block of G. Also Santhakumaran et.al. [6] showed that the vertex-to-clique center of every connected graph G lies in a single block of G. Now we have the following theorem for the vertex-to-clique detour center of a graph.

Theorem 6. *The vertex-to-clique detour center of every connected graph G lies in a single block of G.*

Proof. Suppose that the vertex-to-clique detour center of a connected graph G lies in more than one block. Then G contains a cut vertex v such that $G-v$ has two components G_1 and G_2, each of which contains a vertex-to-clique detour central vertices of G. Let C be a vertex-to-clique detour eccentric clique of v and let P be a vertex-to-clique longest path in G. At least one of G_1 and G_2 contains no vertices of P, say G_2 contains no vertex of P. Let w be a vertex-to-clique detour central vertex in G that belongs to G_2 and let Q be a $w-v$ longest path in G. Since v is a cut vertex, P followed by Q produces a $w-C$ longest path, whose length is greater than that of P. Hence $e_{D1}(w) \ge D(w,v) + D(v,C) = D(w,v) + e_{D1}(v) > e_{D1}(v)$. Thus $e_{D1}(w) > e_{D1}(v)$. So w is not a vertex-to-clique detour central vertex in G, which is contradiction. Hence $C_{D1}(G)$ lies within a block of G. □

Corollary 7. *The vertex-to-clique detour center of every tree is isomorphic to either K_1 or K_2.*

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