Journal of Prime Research in Mathematics Vol. **12***(2016) 91-109*

A NOVEL APPROACH TO APPROXIMATE UNSTEADY SQUEEZING FLOW THROUGH POROUS MEDIUM

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Abstract. In this article, a new alteration of the Homotopy Perturbation Method (HPM) is proposed to approximate the solution of unsteady axisymmetric flow of Newtonian fluid. The flow is squeezed between two circular plates and passes through a porous medium channel. The alteration extends the Homotopy Perturbation with a Laplace transform, which is referred to as the Laplace Transform Homotopy Perturbation Method (LTHPM) in this manuscript. A single fourth order non-linear ordinary differential equation is obtained using similarity transformations. The resulting boundary value problem is then solved through LTHPM, HPM and fourth order Implicit Runge Kutta Method (IRK4). Convergence of the proposed scheme is checked by finding absolute residual errors of various order solutions. Also, the validity is confirmed by comparing numerical and analytical (LTHPM) solutions. The comparison of obtained residual errors shows that LTHPM is an effective scheme that can be applied to various initial and boundary value problems in science and engineering.

Key words : Squeezing Flow, Porous Media, Laplace Transform Homotopy Perturbation Method.
AMS SUBJECT : Primary 34K10, 34K28, 76S99.

1. INTRODUCTION

A porous medium; identified as a material that contains fluid-filled pores, is always characterized by properties such as porosity and permeability. Porosity defines the quantity of fluid that can be held by the material, whereas permeability is the amount of fluid that can pass through it. The analysis of porous medium has become an interesting topic of research since the introduction

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of modified Darcy's Law [1]. Various applications where this analysis can be applied can be found in ground water hydrology, chemical reactors, irrigation, drainage, seepage, and recovery of crude oil from pores of reservoir rocks [2–7]. These applications can specifically be classified to engineering fields such as petroleum, reservoir, and chemical engineering.

Squeezing flows can be induced by applying normal stresses or vertical velocities by means of a moving boundary. It has been the focus of the research community primarily due to its wide use in hydro-dynamical tools and machines. Applications of squeezing flows can be found in chemical engineering, mechanical engineering, industrial engineering, bio-mechanics, and the food industry. Some specific examples are that of polymer processing, or modeling of lubrication systems.

The initial work on squeezing flows has been carried out by Stefan [8] where he proposed an ad hoc asymptotic solution of Newtonian fluids. Thorp proposed an explicit solution of the squeezing flow while considering inertial terms [9]. Kuzma [10] studied the consequences of fluid inertia in squeeze film between circular plates. Gupta et al. [11] showed that the solution given in [9] is unable to satisfy boundary conditions. A squeeze film between two plane annuli considering fluid inertia effects was later studied by Elkouh [12]. Verma [13] and Singh et al. [14] worked towards the numerical solutions of the squeezing flows between parallel plates. Leider and Bird [15] conducted the theoretical analysis for squeezing flow of power-law fluid between parallel plates. Similarly, squeeze film lubrication of a short porous journal bearing with couple stress fluids has been done by Naduvinamani et al. [16]. For steady axisymmetric squeezing flows in a porous medium channel, analysis is presented by Islam et al. [17]. Hamza investigated squeeze films while considering magneto-hydro-dynamic (MHD) effects [18]. Domairry et al. [19] provided analytic solution for squeezing flow considering suction and injection effects, while porosity and squeezing effects have been investigated by Qayyum et al. [20] for unsteady squeezing flow of visco-elastic Jeffery fluids between parallel disks. Apart from this body of work, other articles concerning different theoretical and experimental studies of squeezing flows are presented in [21–27].

Real world phenomena are typically modeled using non-linear differential equations. From the body of literature, a variety of perturbation techniques that can analytically solve non-linear boundary value problems can be found. A limitation of these techniques is the assumption of small or large parameters. In this regard, He [28–31] proposed a new method based on the combination of traditional perturbation methods with homotopy. This technique then came about to be known as the Homotopy Perturbation Method (HPM), which eventually was applied to a number of papers discussing non-linear boundary value problems [28–31]. In addition, HPM has also been applied to various non-linear differential equations [32–37]. In fluid dynamics, Siddiqui et al. [32, 33] applied this technique for solving non-linear boundary value problems arising in Newtonian and non-Newtonian fluids. Khan et al. [38] compare HPM with various analytical and numerical techniques while solving higher order nonlinear ordinary differential equations. In addition, some researchers proposed different alterations in classical HPM to improve the solution process. For instance, Herisanu and Marinca [34] proposed optimal HPM for non conservative dynamical system of a rotating electrical machine. Moreover, an extended HPM for nonlinear differential equations was introduced and applied by Wang et al. [35]. Gulshan et al. [36] proposed a modified homotopy perturbation method coupled with the Fourier transform for nonlinear and singular Lane-Embed equation.

In this context, the aim of this article is to put forward a new alteration of HPM, wherein the combination of the traditional HPM with a Laplace transform is proposed for satisfying all boundary conditions. This proposed method is referred to as the Laplace Transform Homotopy Perturbation Method (LTHPM) in the rest of the manuscript. The proposed approach is applied to approximate the solution of an unsteady squeezing flow between two circular plates through a porous medium. The movement of the circular plates in the problem is considered to be symmetric about the axial line, while the fluid is considered to be Newtonian, incompressible and viscous. A comparison of LTHPM is then made with HPM and IRK4 by means of residual errors for verifying its efficiency and reliability. In the remaining part of the manuscript, section 2 includes mathematical formulation of the problem. Sections 3 and 4 present the basic idea of LTHPM and its application. Section 5 comprises the results and discussion. Finally, conclusion is presented in section 6.

2. MATHEMATICAL FORMULATION

The basic governing equations of motion [39] are;

$$\nabla \cdot \mathbf{U} = \mathbf{0} \tag{1}$$

$$\rho \left[\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} \right] = \rho \mathrm{f} + \nabla \cdot \mathbf{T} + \tilde{\mathbf{r}} \tag{2}$$

where

$$\mathbf{T} = -\mathbf{pI} + \mu \mathbf{A} \tag{3}$$

$$\mathbf{A} = \nabla \mathbf{U} + (\nabla \mathbf{U})^{\mathbf{t}} \tag{4}$$

and $\mathbf{U}$ is the velocity vector, p is the pressure, f is the body force, $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{A}$ is the Rivlin-Ericksen tensor and η is the coefficient of viscosity, and $\tilde{\mathbf{r}}$ is the Darcy's resistance. According to Breugem equation [40], $\tilde{\mathbf{r}}$ can be written as:

$$\tilde{\mathbf{r}} = -\frac{\eta}{k}\,\mathbf{U} \tag{5}$$

where k is the permeability constant.

In this manuscript, an unsteady axisymmetric squeezing flow of incompressible Newtonian fluid passing through porous medium channel is considered. The fluid with density ρ, viscosity η, and kinematic viscosity ν is squeezed between two circular plates having speed $\epsilon(t)$. It is assumed that at any time t, the distance between the two circular plates is $2H(t)$. Also, it is assumed that $r - axis$ is the central axis of the channel while $z - axis$ is normal to it. The plates move symmetrically with respect to the central axis $z = 0$ while the flow is axisymmetric about $r = 0$. The longitudinal and normal velocity components in radial and axial directions are $u_r(r, z, t)$ and $u_z(r, z, t)$ respectively. The geometrical representation of the problem is given in Fig. 1.

Figure 1. Geometry of the problem

An unsteady two-dimensional flow through porous medium can now be formulated. It can be assumed that:

$$\mathbf{U} = [u_r(r, z, t), 0, u_z(r, z, t)] \tag{6}$$

and the vorticity function $\Omega(r,z,t)$ and generalized pressure $\widehat{P}(r,z,t)$ can be introduced as:

$$\Omega(r,z,t) = \frac{\partial u_z}{\partial r} - \frac{\partial u_r}{\partial z} \tag{7}$$

$$\widehat{P}(r,z,t) = \frac{\rho}{2}\left[u_r^2 + u_z^2\right] + p \tag{8}$$

Equations (1) and (2) can be reduced to

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0 \tag{9}$$

$$\frac{\partial \widehat{P}}{\partial r} + \rho\left(\frac{\partial u_r}{\partial t} - u_z\Omega\right) = -\eta\left(\frac{\partial \Omega}{\partial z} + \frac{u_r}{k}\right) \tag{10}$$

$$\frac{\partial \widehat{P}}{\partial z} + \rho\left(\frac{\partial u_z}{\partial t} + u_r\Omega\right) = \eta\left(\frac{1}{r}\frac{\partial}{\partial r}(r\Omega) - \frac{u_z}{k}\right) \tag{11}$$

The boundary conditions on $u_r(r,z,t)$ and $u_z(r,z,t)$ are

$$\begin{aligned} &u_r(r,z,t) = 0, \quad u_z(r,z,t) = \epsilon(t) \qquad at\ z = H \\ &\frac{\partial}{\partial z}u_r(r,z,t) = 0, \quad u_z(r,z,t) = 0 \qquad at\ z = 0 \end{aligned} \tag{12}$$

where $\epsilon(t) = \frac{dH}{dt}$ is the velocity of the plates. The boundary conditions (12) are due to no-slip at the upper plate when $z = H$ and symmetry at $z = 0$. Considering the dimensionless parameter as:

$$\sigma = \frac{z}{H(t)} \tag{13}$$

Equations (7), (9), (10) and (11) are then converted to

$$\Omega(r,z,t) = \frac{\partial u_z}{\partial r} - \frac{1}{H}\frac{\partial u_r}{\partial \sigma} \tag{14}$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{H}\frac{\partial u_z}{\partial \sigma} = 0 \tag{15}$$

$$\frac{\partial \widehat{P}}{\partial r} + \rho\left(\frac{\partial u_r}{\partial t} - u_z\Omega\right) = -\eta\left(\frac{1}{H}\frac{\partial \Omega}{\partial \sigma} + \frac{u_r}{k}\right) \tag{16}$$

$$\frac{1}{H}\frac{\partial \widehat{P}}{\partial \sigma} + \rho\left(\frac{\partial u_z}{\partial t} + u_r\Omega\right) = \eta\left(\frac{1}{r}\frac{\partial}{\partial r}(r\Omega) - \frac{u_z}{k}\right) \tag{17}$$

The boundary conditions on $u_r and\ u_z$ are given as:

$$\begin{aligned} &u_r = 0, \quad u_z(r,z,t) = \epsilon(t) \qquad at\ \sigma = 1 \\ &\frac{\partial u_r}{\partial \sigma} = 0, \quad u_z(r,z,t) = 0 \qquad at\ \sigma = 0 \end{aligned} \tag{18}$$

Elimination of generalized pressure between (16) and (17) results in:

$$\rho\left[\frac{\partial\Omega}{\partial t}+u_r\frac{\partial\Omega}{\partial r}+\frac{u_z}{H}\frac{\partial\Omega}{\partial\sigma}-\frac{u_r}{r}\Omega\right]=\eta\left[\nabla^2\Omega-\left(\frac{1}{r^2}+\frac{1}{k}\right)\Omega\right] \tag{19}$$

where ∇^2 is the Laplacian operator.

Defining velocity components as [11]

$$u_r=-\frac{r\epsilon(t)}{2H(t)}F'(\sigma), \qquad u_z=\epsilon(t)F(\sigma) \tag{20}$$

It cas be seen that (15) is identically satisfied and (19) becomes:

$$\frac{d^4F}{d\sigma^4}+R\left[(\sigma-F)\frac{d^3F}{d\sigma^3}+2\frac{d^2F}{d\sigma^2}\right]-Q\frac{d^2F}{d\sigma^2}-M\frac{d^2F}{d\sigma^2}=0 \tag{21}$$

where

$$R=\frac{H\epsilon}{\nu}, \quad Q=\frac{H^2}{\nu\epsilon}\frac{d\epsilon}{dt} \quad and \quad M=\frac{H^2}{k} \tag{22}$$

Here R, Q and M are functions of t but we consider R, Q and M constants for similarity solution. Since $\epsilon=\frac{dH}{dt}$, Integrating the first equation of (22) results in:

$$H(t)=(at+b)^{\frac{1}{2}} \tag{23}$$

where a and b are constants. When $a>0$ and $b>0$, it is implied that the plates move away from each other symmetrically with respect to σ . When $a<0,\ b>0$ and $H(t)>0$, the plates approach each other. From (22) and (23) it follows that $Q=-R$. Then (21) becomes

$$\frac{d^4F}{d\sigma^4}+R\left[(\sigma-F)\frac{d^3F}{d\sigma^3}+3\frac{d^2F}{d\sigma^2}\right]-M\frac{d^2F}{d\sigma^2}=0 \tag{24}$$

After using (18) and (20), the following boundary conditions in case of no-slip at the upper plate can be established:

$$\begin{aligned} F(0)&=0, & F''(0)&=0\\ F(1)&=1, & F'(1)&=0 \end{aligned} \tag{25}$$

3. Basic Idea of LTHPM

The basic idea can be presented by applying LTHPM to the following differential equation:

$$L[v(x)]+N[v(x)]-f(x)=0 \tag{26}$$

where x represents an independent variable, $L,\ N$ are linear, nonlinear operators respectively, $v(x)$ is an unknown function and $f(x)$ is a known function.

According to LTHPM, homotopy can be constructed as $V(x,p):\mathbb{R}\times[0,1]\rightarrow\mathbb{R}$ such that it satisfies:

$$(1-p)\left[L[V(x,p)]-f(x)\right]+p\left[L[V(x,p)]+N[V(x,p)]-f(x)\right]=0 \tag{27}$$

where $x \in \mathbb{R}$, $p \in [0,1]$ is an embedding parameter, and $V(x,p)$ is an unknown function. Ckearly, when $p = 0$ and $p = 1$, it holds that $V(x,0) = V_0(x)$ and $V(x,1) = \tilde{V}(x)$ respectively.
Thus, as p varies from 0 to 1, the solution $V(x,p)$ approaches from $V_0(x)$ to $\tilde{V}(x)$.
To obtain an approximate solution, expanding $V(x,p)$ using Taylor series about p gives:

$$V(x,p) = V_0(x) + \sum_{k=1}^{\infty} V_k p^k \tag{28}$$

Substituting (28) in (27) and equating the coefficents of like powers of p, the following different order problems can be obtained:
The zero-th order problem is:

$$L[V_0(x)] - f(x) = 0 \tag{29}$$

Application of Laplace Transform on both side of (29) gives:

$$\mathcal{L}\left[LV_0(x)\right] - \mathcal{L}\left[f(x)\right] = 0$$

Using the differential property of Laplace Transform results in:

$$\mathcal{L}\left[V_0(x)\right] = \frac{1}{s^n}\left[s^{n-1}V_0(\alpha) + s^{n-2}V_0^{'}(\alpha) + ... + V_0^{n-1}(\alpha) + \mathcal{L}(f(x))\right]$$

Application of inverse Laplace Transform on both sides gives:

$$V_0(x) = \mathcal{L}^{-1}\left[\frac{1}{s^n}\left(s^{n-1}V_0(\alpha) + s^{n-2}V_0^{'}(\alpha) + ... + V_0^{n-1}(\alpha) + \mathcal{L}[f(x)]\right)\right] \tag{30}$$

The general, k^{th} order problem is:

$$L\left[V_k(x)\right] - N_{k-1}\left[V_0, V_1, ..., V_{k-1}\right] = 0, \quad k = 1, 2, 3, ... \tag{31}$$

Application of Laplace Transform on both side of (31) gives:

$$\mathcal{L}\left[L(V_k(x)) - N_{k-1}\left(V_0, V_1, ..., V_{k-1}\right)\right] = 0$$

Using the differential property of Laplace Transform results in:

$$\mathcal{L}\left[V_k(x)\right] = \frac{1}{s^n}\begin{cases} s^{n-1}V_k(\alpha) + s^{n-2}V_k^{'}(\alpha) + ... + V_k^{n-1}(\alpha) \\ +\mathcal{L}\left[N_{k-1}(V_0, V_1, ..., V_{k-1})\right] \end{cases}$$

Application of inverse Laplace Transform on both sides gives:

$$V_k(x) = \mathcal{L}^{-1}\begin{cases} \frac{1}{s^n}\{s^{(n-1)}V_k(\alpha) + s^{(n-2)}V_k^{'}(\alpha) + ... + V_k^{(n-1)}(\alpha) \\ +\mathcal{L}\left[N_{k-1}(V_0, V_1, ..., V_{k-1})\right]\} \end{cases} \tag{32}$$

Let the initial approximation be of the form $V_k(\alpha) = a_0, V_k^{'}(\alpha) = a_1, ..., V_k^{n-1}(\alpha) = a_{n-1}$. Then, the approximate solution may be obtained as:

$$\tilde{U} = V_0 + V_1 + V_2 + ... \tag{33}$$

Substituting (33) in (26), the expression for residual is:

$$R(x) = L[\tilde{U}(x)] + N[\tilde{U}(x)] - f(x) \tag{34}$$

If $R = 0$, then $\tilde{U}$ will be the exact solution but usually this does not happen in non linear problems.

4. Application of LTHPM

Using (24) and (25), various order problems are as follows:

Zeroth order problem:

$$\begin{gathered} v_0^{(iv)}(\sigma) = 0, \\ v_0(0) = 0, \qquad v_0'(0) = A, \qquad v_0''(0) = 0, \qquad v_0'''(0) = B \end{gathered} \tag{35}$$

For the solution of the zeroth order problem, the Laplace Transform is applied on both sides of (35), giving:

$$\mathcal{L}\left[v_0^{(iv)}(\sigma)\right] = 0,$$

$$\mathcal{L}\left[v_0(s)\right] = \frac{1}{s^4}\left[s^3 v_0(0) + s^2 v_0^{'}(0) + s v_0^{''}(0) + v_0^{'''}(0)\right]$$

Application of the inverse Laplace Transform on both sides gives:

$$\mathcal{L}^{-1}\left[v_0(s)\right] = \mathcal{L}^{-1}\left[\frac{1}{s^4}\left(s^3 v_0(0) + s^2 v_0^{'}(0) + s v_0^{''}(0) + v_0^{'''}(0)\right)\right]$$

$$v_0(\sigma) = A\sigma + \frac{B\sigma^3}{6}$$

First order problem:

$$\begin{aligned} &v_1^{(iv)}(\sigma) - Mv_0''(\sigma) + 3Rv_0''(\sigma) + R\sigma v_0'''(\sigma) - Rv_0(\sigma)\,v_0'''(\sigma) = 0, \\ &v_1(0) = 0, \qquad v_1'(0) = 0, \qquad v_1''(0) = 0, \qquad v_1'''(0) = 0 \end{aligned} \tag{36}$$

Solution of the first order problem is:

$$v_1(\sigma) = \frac{1}{20}\left(BM - 4BR + ABR\right)\sigma^5 + \frac{B^2R\sigma^7}{5040}$$

Second order problem:

$$\begin{aligned} &v_2^{(iv)}(\sigma) - Mv_1''(\sigma) + 3Rv_1''(\sigma) - Rv_1(\sigma)\,v_0'''(\sigma) + R\sigma v_1'''(\sigma) \\ &\qquad -Rv_0(\sigma)\,v_1'''(\sigma) = 0, \\ &v_2(0) = 0, \qquad v_2'(0) = 0, \qquad v_2''(0) = 0, \qquad v_2'''(0) = 0 \end{aligned} \tag{37}$$

Solution of the second order problem is:

$$\begin{aligned} v_2(\sigma) = \;& \frac{1}{5040}\left(BM^2 - 10BMR + 4ABMR + 24BR^2 - 18ABR^2\right)\sigma^7 \\ & + \frac{1}{90720}\left(3B^2MR - 13B^2R^2 + 4AB^2R^2\right)\sigma^9 + \frac{B^3R^2\sigma^{11}}{1108800} \end{aligned}$$

Third order problem:

$$\begin{aligned} &v_3^{(iv)}(\sigma) - Mv_2''(\sigma) + 3Rv_2''(\sigma) - Rv_2(\sigma)\,v_0'''(\sigma) - Rv_1(\sigma)\,v_1'''(\sigma) \\ &\qquad +R\sigma v_2'''(\sigma) - Rv_0(\sigma)\,v_2'''(\sigma) = 0, \\ &v_3(0) = 0, \qquad v_3'(0) = 0, \qquad v_3''(0) = 0, \qquad v_3'''(0) = 0 \end{aligned} \tag{38}$$

Solution of the third order problem is:

$$\begin{aligned} v_3(\sigma) = \;& \frac{1}{362880}\begin{pmatrix} BM^2 - 18BM^2R + 9ABM^2R + 104BR^2 \\ -100ABMR^2 + 23A^2BMR^2 - 192BR^3 \\ +264ABR^3 - 114A^2BR^3 + 15A^3BR^3 \end{pmatrix}\sigma^9 \\ & + \frac{1}{39916800}\begin{pmatrix} 69B^2M^2R - 700B^2MR^2 + 286AB^2MR^2 \\ +1720B^2R^3 - 1340AB^2R^3 + 241A^2B^2R^3 \end{pmatrix}\sigma^{11} \\ & + \frac{1}{3113510400}\left(\;609B^2MR^2 - 2750B^3R^3 + 923AB^3R^3\;\right)\sigma^{13} \\ & + \frac{1051B^4R^3\sigma^{15}}{217945728000} \end{aligned}$$

A similar approach can be used to find $v_4(\sigma)$ and $v_5(\sigma)$.

Considering the fifth order solution:

$$\tilde{v}(\sigma) = \sum_{j=0}^{5} v_j(\sigma) \tag{39}$$

Using the boundary conditions $v(1) = 1$ and $v'(1) = 0$, the unknowns A and B for fixed values of R and M in (38) can be found. In particular, when $R = 1$ and $M = 2$, then $A = 1.5168$ and $B = -3.19019$. The fifth order solution is therefore:

$$\begin{aligned}\tilde{v}(\sigma) = \; & 1.64238\sigma - 0.789068\sigma^3 + 0.146579\sigma^5 + 4.89615 \times 10^{-3}\sigma^7 \\ & -4.95186 \times 10^{-3}\sigma^9 - 6.49614 \times 10^{-5}\sigma^{11} + 2.52107 \times 10^{-}4\sigma^{13} \\ & -1.05994 \times 10^{-5}\sigma^{15} - 1.33999 \times 10^{-5}\sigma^{17} + 1.58831 \times 10^{-6}\sigma^{19} \\ & +8.15214 \times 10^{-7}\sigma^{21} + 4.88136 \times 10^{-8}\sigma^{23}\end{aligned} \tag{40}$$

The residual error of the problem is:

$$Res\ Error = \frac{d^4\tilde{v}}{d\sigma^4} + R(\sigma - \tilde{v})\frac{d^3\tilde{v}}{d\sigma^3} + (3R - M)\frac{d^2\tilde{v}}{d\sigma^2} \tag{41}$$

5. Results and Discussion

In this paper, an unsteady axisymmetric squeezing flow of incompressible Newtonian fluid passing through porous medium is considered. The resulting boundary value problem is solved by LTHPM and the results are compared with HPM and IRK4 solutions.

In the current problem, two parameters Reynolds number R and constant containing permeability M are involved. Firstly, the problem is solved for various values of R using LTHPM, HPM and IRK4, and presented in tables 1-3. Analyses of these tables indicate that results from LTHPM are consistent and more accurate for various values of R as compared to HPM and IRK4. Consistency is determined by comparison of average absolute residual errors at different values of R. It can also be noted that although IRK4 results are also consistent, the accuracy is very less when compared with LTHPM and HPM. The problem is then solved for various values of M using LTHPM, HPM and IRK4 and presented in tables 4, 5, and 6. These tables also indicate that results obtained from LTHPM are consistent and more accurate for various values of M, as compared to the HPM and IRK4. It is also observed that an increase in R does not affect the consistency and accuracy of LTHPM solution. In contrast, the accuracy in case of HPM and IRK4 is significantly reduced.

FIGURE 2. Convergence of LTHPM solution

FIGURE 3. Comparison of LTHPM and IRK4 solutions

Moreover, an increase in M does not affect the accuracy and consistency of LTHPM and IRK4 solutions but on the other hand it enhances the accuracy in case of HPM. It can also be observed that for various values of R and M, IRK4 results in the domain [0, 1] show more accuracy near the center.

Figure 4. LTHPM residual error of fifth order solution for $R = 1$ and $M = 2$

Figure 5. HPM residual error of fifth order solution for $R = 1$ and $M = 2$

The convergence of proposed scheme is confirmed by finding various order solutions along with absolute residual errors in Table 7. In this table, it can be clearly observed that the LTHPM solution considerably improves as the order of approximation is increased for fixed values of R and M. Finally, the validity of the proposed scheme is checked by comparing the analytical (LTHPM) and numerical (IRK4) solutions, and presented in Table 8. Analysis of this table shows that an increase in R, decreases the similarity between LTHPM and

FIGURE 6. IRK4 residual error for $R = 1$ and $M = 2$

FIGURE 7. Comparison of residual errors obtained using LTHPM, HPM and IRK4 for fixed R and M

IRK4 solutions when M is considered to be fixed. On the other hand, for a fixed value of R, an increase in M increases the similarity in the solutions of LTHPM and IRK4. All the tables signify the efficiency of the proposed scheme.

Convergence of LTHPM solution is given in Fig. 2. This plot represents the average absolute residuals against different order approximations and it is clearly seen that LTHPM solution is convergent.

	$R = 1$		$R = 1.5$		$R = 2$	
σ	**Solution**	**Error**	**Solution**	**Error**	**Solution**	**Error**
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.156127	0.	0.160847	2×10^{-16}	0.166566	1×10^{-15}
0.2	0.30846	9×10^{-15}	0.31732	3×10^{-13}	0.328044	5×10^{-12}
0.3	0.453286	2×10^{-12}	0.465187	5×10^{-11}	0.479568	6×10^{-10}
0.4	0.587056	2×10^{-10}	0.600508	3×10^{-9}	0.616723	3×10^{-8}
0.5	0.706467	7×10^{-9}	0.719778	1×10^{-7}	0.735767	1×10^{-6}
0.6	0.808545	1×10^{-7}	0.820069	2×10^{-6}	0.833844	2×10^{-5}
0.7	0.890728	2×10^{-6}	0.899163	3×10^{-5}	0.909182	3×10^{-4}
0.8	0.95094	2×10^{-5}	0.955671	3×10^{-4}	0.961244	3×10^{-3}
0.9	0.987665	1×10^{-4}	0.98912	3×10^{-3}	0.990816	2×10^{-2}
1.0	1.0	9×10^{-4}	1.0	1×10^{-2}	1.0	1×10^{-1}

Table 1. Fifth order LTHPM solutions along with absolute residual errors for various R when $M = 0.3$

	$R = 1$		$R = 1.5$		$R = 2$	
σ	**Solution**	**Error**	**Solution**	**Error**	**Solution**	**Error**
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.156127	6×10^{-7}	0.160842	4×10^{-5}	0.166529	6×10^{-4}
0.2	0.308459	2×10^{-5}	0.31731	2×10^{-4}	0.327972	9×10^{-4}
0.3	0.453285	7×10^{-5}	0.465173	1×10^{-3}	0.479471	6×10^{-3}
0.4	0.587055	1×10^{-4}	0.600492	2×10^{-3}	0.616612	1×10^{-2}
0.5	0.706466	2×10^{-4}	0.719763	3×10^{-3}	0.735656	2×10^{-2}
0.6	0.808544	$3. \times 10^{-4}$	0.820055	4×10^{-3}	0.833748	2×10^{-2}
0.7	0.890727	2×10^{-4}	0.899153	3×10^{-3}	0.909113	2×10^{-2}
0.8	0.95094	8×10^{-5}	0.955665	1×10^{-3}	0.961206	8×10^{-3}
0.9	0.987665	1×10^{-4}	0.989118	2×10^{-3}	0.990805	1×10^{-2}
1.0	1.	3×10^{-4}	1.	4×10^{-3}	1.	2×10^{-2}

Table 2. Fifth order HPM solutions along with absolute residual errors for various R when $M = 0.3$

Validity of LTHPM solution is demonstrated in Fig. 3, where LTHPM and IRK4 solutions are compared for fixed values of R and M. Here, it can be observed that LTHPM solution is in good agreement with IRK4 solution.

Fig. 4 and Fig. 5 represent the residual errors of the fifth order solutions obtained by LTHPM and HPM respectively for $R = 1$ and $M = 2$. Fig. 6 demonstrates the residual error in case of IRK4. In addition, comparison of residual errors of LTHPM, HPM and IRK4 is presented in Fig. 7.

	$R = 1$		$R = 1.5$		$R = 2$	
σ	**Solution**	**Error**	**Solution**	**Error**	**Solution**	**Error**
0.0	0.	4×10^{-3}	0.	7×10^{-3}	0.	9×10^{-3}
0.1	0.156127	2×10^{-4}	0.160847	7×10^{-4}	0.166565	1×10^{-3}
0.2	0.30846	5×10^{-5}	0.317319	1×10^{-4}	0.328041	4×10^{-4}
0.3	0.453286	1×10^{-5}	0.465186	5×10^{-5}	0.479564	1×10^{-4}
0.4	0.587056	4×10^{-6}	0.600508	1×10^{-5}	0.616719	3×10^{-5}
0.5	0.706466	7×10^{-9}	0.719778	1×10^{-6}	0.735762	9×10^{-6}
0.6	0.808545	4×10^{-6}	0.820068	1×10^{-5}	0.833839	8×10^{-6}
0.7	0.890728	1×10^{-5}	0.899162	4×10^{-5}	0.909178	3×10^{-5}
0.8	0.95094	5×10^{-5}	0.95567	1×10^{-4}	0.961242	1×10^{-4}
0.9	0.987665	2×10^{-4}	0.98912	5×10^{-4}	0.9908015	3×10^{-4}
1.0	1.	4×10^{-3}	1.	7×10^{-3}	1.	9×10^{-3}

Table 3. IRK4 solutions along with absolute residual errors for various R when $M = 0.3$

	$M = 0.5$		$M = 1$		$M = 2$	
σ	**Solution**	**Error**	**Solution**	**Error**	**Solution**	**Error**
0.0	0.	0.	0.	0.	0.	0.
0.1	0.155489	1×10^{-16}	0.153958	0.	0.151148	5×10^{-17}
0.2	0.307263	1×10^{-14}	0.30439	1×10^{-14}	0.29911	2×10^{-14}
0.3	0.451679	2×10^{-12}	0.447819	1×10^{-12}	0.440715	2×10^{-12}
0.4	0.585241	1×10^{-10}	0.580877	4×10^{-11}	0.572825	4×10^{-11}
0.5	0.704672	4×10^{-9}	0.700351	8×10^{-10}	0.692354	3×10^{-10}
0.6	0.806992	8×10^{-8}	0.803248	1×10^{-8}	0.79629	2×10^{-9}
0.7	0.889592	1×10^{-6}	0.886845	2×10^{-7}	0.88172	8×10^{-9}
0.8	0.950303	1×10^{-5}	0.948758	2×10^{-6}	0.94586	2×10^{-8}
0.9	0.987469	9×10^{-5}	0.986992	1×10^{-5}	0.986092	5×10^{-9}
1.0	1.	6×10^{-4}	1.	1×10^{-4}	1.	7×10^{-7}

Table 4. Fifth order LTHPM solutions along with absolute residual errors for various M when $R = 1$

6. Conclusion

In this article, a similarity solution for an unsteady axisymmetric squeezing flow of incompressible Newtonian fluid through porous medium is presented using a novel alteration of the Homotopy Perturbation Method. The alteration is introduced in this paper as the Homotopy Perturbation Laplace Method (LTHPM). The analysis of the residual errors of the solution confirms the effectiveness of the proposed scheme. The convergence of the proposed scheme

$M = 0.5$			$M = 1$		$M = 2$	
σ	**Solution**	**Error**	**Solution**	**Error**	**Solution**	**Error**
0.0	0.	0.	0.	0.	0.	0.
0.1	0.155489	3×10^{-6}	0.153958	3×10^{-6}	0.151148	9×10^{-8}
0.2	0.307262	6×10^{-6}	0.304389	2×10^{-6}	0.29911	1×10^{-7}
0.3	0.451678	4×10^{-5}	0.447819	4×10^{-6}	0.440715	2×10^{-7}
0.4	0.58524	9×10^{-5}	0.580877	1×10^{-5}	0.572825	1×10^{-7}
0.5	0.704672	1×10^{-4}	0.700351	3×10^{-5}	0.692354	2×10^{-8}
0.6	0.806992	1×10^{-4}	0.803248	5×10^{-5}	0.79629	4×10^{-7}
0.7	0.889591	1×10^{-4}	0.886845	4×10^{-5}	0.88172	7×10^{-7}
0.8	0.950303	6×10^{-5}	0.948758	2×10^{-5}	0.94586	6×10^{-7}
0.9	0.987469	8×10^{-5}	0.986992	1×10^{-5}	0.986092	1×10^{-7}
1.0	1.	2×10^{-4}	1.	5×10^{-5}	1.	5×10^{-7}

Table 5. Fifth order HPM solutions along with absolute residual errors for various M when $R = 1$

$M = 0.5$			$M = 1$		$M = 2$	
σ	**Solution**	**Error**	**Solution**	**Error**	**Solution**	**Error**
0.0	0.	3×10^{-3}	0.	2×10^{-3}	0.	4×10^{-4}
0.1	0.155489	2×10^{-4}	0.153958	1×10^{-4}	0.151148	2×10^{-5}
0.2	0.307263	5×10^{-5}	0.30439	3×10^{-5}	0.29911	7×10^{-6}
0.3	0.451679	1×10^{-5}	0.447819	1×10^{-5}	0.440715	2×10^{-6}
0.4	0.585241	4×10^{-6}	0.580877	3×10^{-6}	0.572825	5×10^{-7}
0.5	0.704672	6×10^{-8}	0.700351	1×10^{-7}	0.692354	1×10^{-7}
0.6	0.806992	4×10^{-6}	0.803248	3×10^{-6}	0.79629	1×10^{-6}
0.7	0.889592	1×10^{-5}	0.886845	1×10^{-5}	0.88172	3×10^{-6}
0.8	0.950303	5×10^{-5}	0.948758	4×10^{-5}	0.94586	1×10^{-5}
0.9	0.987469	2×10^{-4}	0.986992	1×10^{-4}	0.986092	5×10^{-5}
1.0	1.	4×10^{-3}	1.	3×10^{-3}	1.	1×10^{-3}

Table 6. IRK4 solutions along with absolute residual errors for various M when $R = 1$

is also verified using the residual errors of various order approximate solutions. Validity of the proposed method is confirmed by solving the problem using HPM and IRK4 and comparing the residual errors. This comparison confirms that obtained analytic results using LTHPM are in good agreement with the IRK4 numerical scheme. From the above facts it is concluded that LTHPM can be efficiently used in various areas of science and engineering as it promises a higher degree of accuracy.

	First Order		Third Order		Fifth Order	
σ	**Solution**	**Error**	**Solution**	**Error**	**Solution**	**Error**
0.0	0.	0.	0.	0.	0.	0.
0.1	0.151157	1×10^{-4}	0.153959	4×10^{-11}	0.151148	5×10^{-17}
0.2	0.299127	1×10^{-3}	0.30439	6×10^{-9}	0.29911	2×10^{-14}
0.3	0.44074	3×10^{-3}	0.44782	1×10^{-7}	0.440715	2×10^{-12}
0.4	0.572857	5×10^{-3}	0.580878	1×10^{-6}	0.572825	4×10^{-11}
0.5	0.692391	6×10^{-3}	0.700353	1×10^{-5}	0.692354	3×10^{-10}
0.6	0.796328	1×10^{-4}	0.803249	7×10^{-5}	0.79629	2×10^{-9}
0.7	0.881754	2×10^{-2}	0.886847	4×10^{-4}	0.88172	8×10^{-9}
0.8	0.945885	7×10^{-2}	0.948759	1×10^{-3}	0.94586	2×10^{-8}
0.9	0.986102	1×10^{-1}	0.986993	7×10^{-3}	0.986092	5×10^{-9}
1.0	1.	3×10^{-1}	1.	2×10^{-2}	1.	7×10^{-7}

Table 7. Different order solutions along with absolute residual errors of LTHPM for fixed R and M

	$M = 0.3$(Fixed)			$R = 1$(Fixed)		
σ	$R = 1$	$R = 1.5$	$R = 2$	$M = 0.5$	$M = 1$	2
0.0	0.	0.	0.	0.	0.	0.
0.1	6×10^{-9}	1×10^{-7}	1×10^{-6}	3×10^{-9}	7×10^{-10}	3×10^{-12}
0.2	1×10^{-8}	2×10^{-7}	2×10^{-6}	7×10^{-9}	1×10^{-9}	6×10^{-12}
0.3	1×10^{-8}	3×10^{-7}	3×10^{-6}	1×10^{-8}	2×10^{-9}	9×10^{-12}
0.4	2×10^{-8}	4×10^{-7}	4×10^{-6}	1×10^{-8}	2×10^{-9}	1×10^{-11}
0.5	2×10^{-8}	4×10^{-7}	4×10^{-6}	1×10^{-8}	2×10^{-9}	1×10^{-11}
0.6	2×10^{-8}	4×10^{-7}	4×10^{-6}	1×10^{-8}	2×10^{-9}	1×10^{-11}
0.7	1×10^{-8}	4×10^{-7}	3×10^{-6}	1×10^{-8}	2×10^{-9}	9×10^{-12}
0.8	1×10^{-8}	2×10^{-7}	2×10^{-6}	8×10^{-9}	1×10^{-9}	5×10^{-12}
0.9	6×10^{-9}	1×10^{-7}	1×10^{-6}	3×10^{-9}	7×10^{-10}	1×10^{-12}
1.0	8×10^{-17}	5×10^{-15}	2×10^{-16}	7×10^{-16}	1×10^{-16}	2×10^{-17}

Table 8. Comparison of LTHPM and IRK4 solution for various R and M

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