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RATIONAL CUBIC SPIRALS

AYESHA SHAKEEL[1], MARIA HUSSAIN[2] AND MALIK ZAWWAR HUSSAIN[3]

Abstract. A parametric rational cubic approximation scheme is presented to preserve the monotone curvature profile of the given curve. The rational cubic curve has four control points and two free parameters. Values of control points are attained by C^1-approximation. Simple sufficient data dependent constraints are obtained on the free parameters to preserve the monotonicity of curvature of given curve. Devised curvature-preserving approximation scheme is simple and robust.

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1. Introduction

Designing of visually pleasing fair curves is a fundamental issue in computer aided design (CAD), especially for automobiles and other consumer goods such as household appliances where sales largely depend on the appearance of the products. A curve is fair if it has monotone curvature profile [3, 9] and spirals are best for fitting fair curves. Well known spirals like Cornu spirals, logarithmic spirals, generalized Cornu spirals (GCS) are defined in terms of arc length parameter. CAD system is based on parametric curves. Thus these spirals cannot be directly used in CAD. Therefore approximation of spirals by parametric curves has been in focus of the researchers [2, 4, 5, 6, 10].

Generalized Cornu spirals (GCSs) [1] are a family of spirals which can be characterized as straight lines, circles, Cornu spirals and logarithmic spirals. GCS [1] has rational linear curvature given by

$$k(s) = \frac{(a+bs)}{(S+rs)},\ 0 \leq s \leq S. \tag{1}$$

Here s and S are arc length parameters, a and b are real constants and $r \geq -1$ is the shape factor. For different values of a, b and r in (1), GCS reduces to

[1,3] Department of Mathematics, University of the Punjab, Lahore, Pakistan. E-mail: ayesha031@hotmail.com and malikzawwar.math@pu.edu.pk
[2] Lahore College for Women University, Lahore, Pakistan. E-mail: maria.hussain@lcwu.edu.pk.

Cornu spiral and logarithmic spiral. Cornu spiral is a well-known elegant curve with monotone curvature profile. This property made Cornu spiral a perfect adequate transition curve for connecting the circles and straight lines in railway and highway design [7].

Efforts have been put by many authors and researchers for fair curve designing. Ali et al. [1] used generalized Cornu spiral (1) for span generation with given values of end points and end unit tangents of curves. End curvatures (k_0, k_1), arc length (S) and shape factor (r) were treated as free parameters. Firstly, arbitrary values were assigned to these free parameters. Then these values were readjusted to obtain a GCS that matched the given end points and end unit tangents of span to obtain desired curve. Baumgarten and Farin [2] described a method for approximating logarithmic spiral segments by rational cubic spline curves. Burchard et al. [3] had discussed about the relation between aesthetic and monotonicity of curvature. Ling and Ali [7] discussed the application of generalized Cornu spiral in aesthetic design. Aim of using GCS in aesthetic design is because it has excellent curvature properties. Cripps et al. [4] proposed a method of constructing an approximation to a generalized Cornu spiral. Zhu [10] developed the non-rational (quartic and quintic) and rational (cubic, quartic and quintic) Bézier G^2-approximation schemes for GCS. The G^2-constraints introduced length of end tangents as free parameters. Optimized values of length of end tangents were obtained by minimizing the relative curvature error of these approximation schemes. Numerical experiments proved G^2 non-rational quintic Bézier curve approximation scheme better than the other approximation schemes discussed in [10].

In this research paper, a parametric rational cubic curve is used to preserve the monotonicity of curvature of given curves. The rational cubic curve has four control points and two free parameters. These free parameters are used to change the shape and curvature profile of curve without changing the data. Although the computation of non-rational curves is easy but rational curves are preferred over non-rational curves due to presence of free parameters and their ability to model functions with poles. In this research paper, the control points are evaluated by C^1-constraints i.e. by matching end points and end tangents of the parametric rational cubic curve and given curve with monotone curvature. Substituting the values of these control points in (1), a C^1-parametric rational cubic curve with two free parameters is obtained. Curvature $k_P(t)$ of C^1-parametric rational cubic curve is differentiated w.r.t. arc length parameter s. Simple sufficient data dependent constraints are derived on free parameter to guarantee $\frac{dk_P}{ds} > 0$ over the whole domain. Constraints are also derived on free parameters to guarantee $\frac{dk_P}{ds} < 0$ over the whole domain for curves with monotonically decreasing curvature profile. Substituting these values of free parameter in (2) and (5), the C^1-parametric rational cubic curve preserving the monotonicity of curvature profile of the given curve is obtained.

Parametric curvature-preserving scheme developed in this research paper has many advantages over the existing schemes. The details are as follows:

- In [1], GCS was used for span generation. End curvatures of the GCS, arc length and shape factor r were used as free parameters. Values of these free parameters were repeatedly adjusted by a subroutine to obtain the desired curve. Parametric curvature-preserving scheme developed in this research paper works for the given values of end curvatures and arc length. Therefore, unlike [1] it does not require the modification of end curvatures and arc length. Moreover, for a given data set a fixed value of free parameters (α, β) is obtained which guarantee monotone curvature profile of the parametric rational cubic curve (2). On the other hand, in [1] the value of shape factor is adjusted to obtain the perceived result.
- In [10], optimized values of the free parameters were obtained by minimizing the relative curvature error of the concerned G^2-approximation scheme through well- known numerical search algorithm known as Fast Evolutionary Programming technique(FEP). Use of optimization technique increased the complexity and decreased efficiency of the G^2-approximation scheme [10]. The curvature-preserving scheme introduced in this research paper does not involve any optimization technique for the finding the values of free parameters. Thus it is robust and simpler than [10].

The rest of the paper is schemed as follows. In the Section 2, the parametric rational cubic curve with two free parameters is introduced. In Section 3, constraints are developed on free parameters to preserve monotone curvature profile of the given curve. Section 4 concludes the paper.

2. C^1 PARAMETRIC RATIONAL CUBIC CURVE

The parametric form of rational cubic curve [8] is given by

$$P(t) = \frac{\sum_{i=0}^{3}(1-t)^{3-i}t^{i}p_i}{\alpha + \beta(1-t)(t)}, \ t \in [0,1], \ p_i \in \Re^2. \tag{2}$$

Here, p_i, $i = 0,1,2,3$, are the control points and α, β, are the free parameters. Let $C(s)$ be a given curve with arc length parameter s. Let two end points of $C(s)$ be $c_i, i = 0,1$ and two end tangents of $C(s)$ be $T_i, i = 0,1$. If S is the total arc length of the given curve then

$$C(0) = c_0, \ C(S) = c_1, \ C^{'}(0) = T_0, \ C^{'}(S) = T_1. \tag{3}$$

The C^1-approximation of the given curve $C(s)$, by the parametric rational cubic curve (2) is attained by the following end conditions:

$$P(0) = C(0),\ P(1) = C(S),\ P^{'}(0) = C^{'}(0),\ P^{'}(1) = C^{'}(S). \tag{4}$$

Applying the end conditions (4) to (2), the following values of the control points of the parametric rational cubic curve (2) are obtained:

$$\begin{aligned} p_0 &= \alpha(x_0, y_0),\ p_1 = ((3\alpha+\beta)x_0 + \alpha t_0, (3\alpha+\beta)y_0 + \alpha t_1), \\ p_2 &= ((3\alpha+\beta)x_3 - \alpha t_2, (3\alpha+\beta)y_3 - \alpha t_3),\ p_3 = \alpha(x_3, y_3). \end{aligned} \tag{5}$$

with

$$c_0 = (x_0, y_0), c_1 = (x_3, y_3), T_0 = (t_0, t_1), T_1 = (t_2, t_3).$$

3. Preserving monotone curvature profile using C^1 parametric rational cubic curve

In this section, constraints are developed on the free parameters α and β to preserve the monotonicity of curvature of the given curve.

Suppose that the planar curve considered in (3) is a fair curve i.e. it has monotone curvature profile w.r.t arc length. Let $k_C(s)$ be the curvature of the considered curve $C(s)$. By our supposition, the following end condition holds:

$$\begin{aligned} k_C^{'}(0) > 0 \ and \ k_C^{'}(S) > 0, \\ k_C^{'}(0) < 0 \ and \ k_C^{'}(S) < 0. \end{aligned} \tag{6}$$

For the control points given in (5), the parametric equations of the parametric rational cubic curve (2), $P(t) = (x(t), y(t))$, are defined as:

$$x(t) = \frac{f_1(t)}{g(t)} \ and \ y(t) = \frac{f_2(t)}{g(t)}. \tag{7}$$

Here,

$$\begin{aligned} f_1(t) &= \alpha x_0(1-t)^3 + ((3\alpha+\beta)x_0 + \alpha t_0)(1-t)^2 t + ((3\alpha+\beta)x_3 - \alpha t_2) \times \\ &\quad (1-t)t^2 + \alpha x_3 t^3, \\ f_2(t) &= \alpha y_0(1-t)^3 + ((3\alpha+\beta)y_0 + \alpha t_1)(1-t)^2 t + ((3\alpha+\beta)y_3 - \alpha t_3) \times \\ &\quad (1-t)t^2 + \alpha y_3 t^3, \\ g(t) &= \alpha + \beta(1-t)t. \end{aligned}$$

The curvature of the parametric curve, $P(t) = (x(t), y(t))$, is given by

$$k_P(t) = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2\}^{\frac{3}{2}}} \tag{8}$$

The C^1-parametric rational cubic curve (2), preserves the monotone curvature profile of the given curve $C(s)$, if the curvature $k_P(t)$ is also monotone function of arc length s. The desired condition is expressed mathematically as:

$$\frac{dk_P}{ds} > 0 \tag{9}$$

Here, $\frac{dk_P}{ds}$, is obtained by the chain rule as follows:

$$\frac{dk_P}{ds} = \frac{dk_P(t)}{dt}.\frac{dt}{ds} \tag{10}$$

The parameter t and the arc length s of the C^1-parametric rational cubic curve (2), are related as:

$$s = \int_0^t \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt \tag{11}$$

A simple computation can reproduce the following:

$$\frac{dk_P(t)}{dt} = \frac{(\frac{dx}{dt}\frac{d^3y}{dt^3} - \frac{dy}{dt}\frac{d^3x}{dt^3})((\frac{dx}{dt})^2 + (\frac{dy}{dt})^2) - 3(\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2})(\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2})}{\{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2\}^{\frac{5}{2}}}, \tag{12}$$

and

$$\frac{ds}{dt} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}. \tag{13}$$

Substituting the values from (12) and (13) in (10), we have

$$\frac{dk_P}{ds} = \frac{(\frac{dx}{dt}\frac{d^3y}{dt^3} - \frac{dy}{dt}\frac{d^3x}{dt^3})}{((\frac{dx}{dt})^2 + (\frac{dy}{dt})^2)^2} - \frac{3(\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2})}{((\frac{dx}{dt})^2 + (\frac{dy}{dt})^2)^3}.(\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}). \tag{14}$$

It follows from above expression that $\frac{dk_P}{ds} > 0$, if

$$\frac{dx}{dt} > 0,\ \frac{dy}{dt} > 0,\ \frac{d^3x}{dt^3} < 0,\ \frac{d^3y}{dt^3} > 0,\ \frac{d^2x}{dt^2} < 0,\ \frac{d^2y}{dt^2} < 0,\ (\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}) > 0, \tag{15}$$

with

$$t_0 > \Delta_1 > t_2, t_1 > \Delta_2 > t_3, t_0 > 0, t_1 > 0, t_2 > 0, t_3 > 0, \Delta_1 > 0, \Delta_2 > 0,$$

$$\Delta_1 = x_3 - x_0, \Delta_2 = y_3 - y_0.$$

Now, to preserve the monotonicity of curvature of the given curve by the parametric rational cubic curve (2), we shall determine the constraint on free parameters for which the inequalities in (15) hold.

The first, second and third order derivatives of the parametric rational cubic curve (2), are given by

$$\begin{aligned} \frac{dx}{dt} &= \frac{A(t)}{(g(t))^2}, \frac{dy}{dt} = \frac{B(t)}{(g(t))^2}, \frac{d^2x}{dt^2} = \frac{C(t)}{(g(t))^3}, \frac{d^2y}{dt^2} = \frac{D(t)}{(g(t))^3}, \frac{d^3x}{dt^3} = \frac{E(t)}{(g(t))^4}, \\ \frac{d^3y}{dt^3} &= \frac{F(t)}{(g(t))^4}. \end{aligned} \tag{16}$$

Here,

$$A(t) = \sum_{i=0}^{4}(1-t)^{4-i}t^i A_i, \tag{17}$$

$$\begin{aligned}
A_0 &= \alpha^2 t_0, \\
A_1 &= 2\alpha\{(3\alpha+\beta)\Delta_1 - \alpha t_2\}, \\
A_2 &= 3\alpha^2\Delta_1 + (3\alpha+\beta)\{(3\alpha+\beta)\Delta_1 - \alpha t_2 - \alpha t_0\}, \\
A_3 &= 2\alpha\{(3\alpha+\beta)\Delta_1 - \alpha t_0\}, \\
A_4 &= \alpha^2 t_2.
\end{aligned}$$

$$B(t) = \sum_{i=0}^{4}(1-t)^{4-i}t^i B_i, \tag{18}$$

$$\begin{aligned}
B_0 &= \alpha^2 t_1, \\
B_1 &= 2\alpha\{(3\alpha+\beta)\Delta_2 - \alpha t_3\}, \\
B_2 &= 3\alpha^2\Delta_2 + (3\alpha+\beta)\{(3\alpha+\beta)\Delta_2 - \alpha t_3 - \alpha t_1\}, \\
B_3 &= 2\alpha\{(3\alpha+\beta)\Delta_2 - \alpha t_1\}, \\
B_4 &= \alpha^2 t_3.
\end{aligned}$$

$$C(t) = \sum_{i=0}^{5}(1-t)^{5-i}t^i C_i, \tag{19}$$

$$\begin{aligned}
C_0 &= 2\alpha^2\{(3\alpha+\beta)\Delta_1 - \alpha t_2 - (2\alpha+\beta)t_0\}, \\
C_1 &= 2\alpha^2\{7\alpha(\Delta_1 - t_0) + 2\beta(\Delta_1 - t_0) + 2\alpha(\Delta_1 - t_2)\}, \\
C_2 &= 2\alpha\{(6\alpha^2+\alpha\beta)\Delta_1 - (8\alpha^2+\alpha\beta)t_0 + 2\alpha^2 t_2\}, \\
C_3 &= 2\alpha\{(-6\alpha^2-\alpha\beta)\Delta_1 + (8\alpha^2+\alpha\beta)t_2 - 2\alpha^2 t_0\}, \\
C_4 &= 2\alpha^2\{7\alpha(t_2 - \Delta_1) + 2\beta(t_2 - \Delta_1) - 2\alpha(\Delta_1 - t_0)\}, \\
C_5 &= 2\alpha^2\{-(3\alpha+\beta)\Delta_1 + \alpha t_0 + (2\alpha+\beta)t_2\}.
\end{aligned}$$

$$D(t) = \sum_{i=0}^{5}(1-t)^{5-i}t^i D_i, \tag{20}$$

$$\begin{aligned}
D_0 &= 2\alpha^2\{(3\alpha+\beta)\Delta_2 - \alpha t_3 - (2\alpha+\beta)t_1\}, \\
D_1 &= 2\alpha^2\{7\alpha(\Delta_2 - t_1) + 2\beta(\Delta_2 - t_1) + 2\alpha(\Delta_2 - t_3)\}, \\
D_2 &= 2\alpha\{(6\alpha^2+\alpha\beta)\Delta_2 - (8\alpha^2+\alpha\beta)t_1 + 2\alpha^2 t_3\}, \\
D_3 &= 2\alpha\{(-6\alpha^2-\alpha\beta)\Delta_2 + (8\alpha^2+\alpha\beta)t_3 - 2\alpha^2 t_1\}, \\
D_4 &= 2\alpha^2\{7\alpha(t_3 - \Delta_2) + 2\beta(t_3 - \Delta_2) - 2\alpha(\Delta_2 - t_1)\}, \\
D_5 &= 2\alpha^2\{-(3\alpha+\beta)\Delta_2 + \alpha t_1 + (2\alpha+\beta)t_3\}.
\end{aligned}$$

$$E(t) = \sum_{i=0}^{6} (1-t)^{6-i} t^i E_i, \tag{21}$$

$$\begin{aligned}
E_0 &= 6\alpha^2\{-(2\alpha^2 + 4\alpha\beta + \beta^2)\Delta_1 + (\alpha^2 + 3\alpha\beta + \beta^2)t_0 + (\alpha^2 + \alpha\beta)t_2\},\\
E_1 &= 12\alpha^2\{(3\alpha^2 + 2\alpha\beta)(t_2 - \Delta_1) - (3\alpha^2 + 7\alpha\beta)(\Delta_1 - t_0) - 2\beta^2(\Delta_1 - t_0)\},\\
E_2 &= 2\alpha^2\{(45\alpha^2 + 13\alpha\beta)(t_2 - \Delta_1) - (45\alpha^2 + 83\alpha\beta + 16\beta^2)(\Delta_1 - t_0)\},\\
E_3 &= 12\alpha^2\{-(20\alpha^2 + 10\alpha\beta + \beta^2)\Delta_1 + (10\alpha^2 + 10\alpha\beta + \beta^2)t_0 + 10\alpha^2 t_2\},\\
E_4 &= 6\alpha^2\{-(30\alpha^2 - \beta^2)\Delta_1 + 5\alpha(3\alpha + \beta)t_0 + 5\alpha(3\alpha - \beta)t_2\},\\
E_5 &= 12\alpha^2\{(3\alpha^2 - 2\alpha\beta)(t_2 - \Delta_1) - (3\alpha^2 - \alpha\beta)(\Delta_1 - t_0) - \beta^2(t_2 - \Delta_1)\},\\
E_6 &= 6\alpha^2\{-(2\alpha^2 - 4\alpha\beta - \beta^2)\Delta_1 + (\alpha^2 - \alpha\beta)t_0 + (\alpha^2 - \alpha\beta - \beta^2)t_2\}.
\end{aligned}$$

$$F(t) = \sum_{i=0}^{6} (1-t)^{6-i} t^i F_i, \tag{22}$$

$$\begin{aligned}
F_0 &= 6\alpha^2\{-(2\alpha^2 + 4\alpha\beta + \beta^2)\Delta_2 + (\alpha^2 + 3\alpha\beta + \beta^2)t_1 + (\alpha^2 + \alpha\beta)t_3\},\\
F_1 &= 12\alpha^2\{(3\alpha^2 + 2\alpha\beta)(t_3 - \Delta_2) - (3\alpha^2 + 7\alpha\beta)(\Delta_2 - t_1) - 2\beta^2(\Delta_2 - t_1)\},\\
F_2 &= 2\alpha^2\{(45\alpha^2 + 13\alpha\beta)(t_3 - \Delta_2) - (45\alpha^2 + 83\alpha\beta + 16\beta^2)(\Delta_2 - t_1)\},\\
F_3 &= 12\alpha^2\{-(20\alpha^2 + 10\alpha\beta + \beta^2)\Delta_2 + (10\alpha^2 + 10\alpha\beta + \beta^2)t_1 + 10\alpha^2 t_3\},\\
F_4 &= 6\alpha^2\{-(30\alpha^2 - \beta^2)\Delta_2 + 5\alpha(3\alpha + \beta)t_1 + 5\alpha(3\alpha - \beta)t_3\},\\
F_5 &= 12\alpha^2\{(3\alpha^2 - 2\alpha\beta)(t_3 - \Delta_2) - (3\alpha^2 - \alpha\beta)(\Delta_2 - t_1) - \beta^2(t_3 - \Delta_2)\},\\
F_6 &= 6\alpha^2\{-(2\alpha^2 - 4\alpha\beta - \beta^2)\Delta_2 + (\alpha^2 - \alpha\beta)t_1 + (\alpha^2 - \alpha\beta - \beta^2)t_3\}.
\end{aligned}$$

From (17), $\frac{dx}{dt} > 0$ if

$$A_i > 0,\ \ i = 0, 1, 2, 3, 4 \ \textit{and}\ (g(t))^2 > 0.$$

$(g(t))^2$ is always positive so $\frac{dx}{dt} > 0$ if

$$\alpha > 0, \beta > 0, A_i > 0,\ \ i = 0, 1, 2, 3, 4.$$

Obviously, $A_0 >\ 0$ and $A_4 > 0$, from the necessary conditions of monotonicity of curvature.
Moreover, $A_1 > 0$ if

$$\beta > \frac{\alpha t_2}{\Delta_1}. \tag{23}$$

$A_2 > 0$ if

$$\beta > \frac{\alpha(t_0 + t_2)}{\Delta_1}. \tag{24}$$

$A_3 > 0$ if

$$\beta > \frac{\alpha t_0}{\Delta_1}. \tag{25}$$

From (23),(24) and (25), $A_i > 0, i = 0, 1, 2, 3, 4$ if

$$\alpha > 0,$$

and

$$\beta > max\{0, \frac{\alpha t_0}{\Delta_1}, \frac{\alpha(t_0 + t_2)}{\Delta_1}, \frac{\alpha t_2}{\Delta_1}\}.$$

$\frac{dy}{dt} > 0$ if

$$B_i > 0, i = 0, 1, 2, 3, 4 \ and \ (g(t))^2 > 0.$$

Obviously, $B_0 > 0$ and $B_4 > 0$, from the necessary conditions of monotonicity of curvature.
Moreover, $B_1 > 0$ if

$$\beta > \frac{\alpha t_3}{\Delta_2}. \tag{26}$$

$B_2 > 0$ if

$$\beta > \frac{\alpha(t_1 + t_3)}{\Delta_2}. \tag{27}$$

$B_3 > 0$ if

$$\beta > \frac{\alpha t_1}{\Delta_2}. \tag{28}$$

From (26),(27) and (28), $B_i > 0, i = 0, 1, 2, 3, 4$ if

$$\alpha > 0,$$

and

$$\beta > max\{0, \frac{\alpha t_1}{\Delta_2}, \frac{\alpha(t_1 + t_3)}{\Delta_2}, \frac{\alpha t_3}{\Delta_2}\}.$$

$\frac{d^2x}{dt^2} < 0$ if

$$\widetilde{C}_i > 0, i = 0, 1, 2, 3, 4, 5 \ and \ (g(t))^3 > 0.$$

Here, $\widetilde{C}_i = -C_i, i = 0, 1, 2, 3, 4, 5$. $C_i's$ are already defined in (19).
$(g(t))^3 > 0$ if

$$\alpha > 0, \beta > 0.$$

$\widetilde{C}_0, \widetilde{C}_1 > 0$ if

$$\beta > \frac{(t_2 - \Delta_1)\alpha}{t_0 - \Delta_1}. \tag{29}$$

$\widetilde{C}_2 > 0$ if

$$\beta > \frac{2(t_2 - t_0)\alpha}{t_0 - \Delta_1}. \tag{30}$$

$\widetilde{C}_3 > 0$ if

$$\beta > \frac{2(t_2 - t_0)\alpha}{\Delta_1 - t_2}. \tag{31}$$

$\widetilde{C}_4, \widetilde{C}_5 > 0$ if
$$\beta > \frac{(t_0 - \Delta_1)\alpha}{\Delta_1 - t_2}. \tag{32}$$
From (29),(30),(31) and (32), $\widetilde{C}_i > 0, i = 0, 1, 2, 3, 4, 5,$ if
$$\alpha > 0,$$
and
$$\beta > max\{0, \frac{(t_2 - \Delta_1)\alpha}{t_0 - \Delta_1}, \frac{2(t_2 - t_0)\alpha}{t_0 - \Delta_1}, \frac{2(t_2 - t_0)\alpha}{\Delta_1 - t_2}, \frac{(t_0 - \Delta_1)\alpha}{\Delta_1 - t_2}\}.$$
$\frac{d^2y}{dt^2} < 0$ if
$$\widetilde{D}_i > 0, i = 0, 1, 2, 3, 4, 5 \; and \; (g(t))^3 > 0.$$
Here, $\widetilde{D}_i = -D_i, i = 0, 1, 2, 3, 4, 5$. $D_i's$ are already defined in (20).
$(g(t))^3 > 0$ if
$$\alpha > 0, \beta > 0.$$
$\widetilde{D}_0, \widetilde{D}_1 > 0$ if
$$\beta > \frac{(\Delta_2 - t_3)\alpha}{t_1 - \Delta_2}. \tag{33}$$
$\widetilde{D}_2 > 0$ if
$$\beta > \frac{2(t_3 - t_1)\alpha}{t_1 - \Delta_2}. \tag{34}$$
$\widetilde{D}_3 > 0$ if
$$\beta > \frac{2(t_3 - t_1)\alpha}{\Delta_2 - t_3}. \tag{35}$$
$\widetilde{D}_4, \widetilde{D}_5 > 0$ if
$$\beta > \frac{(t_1 - \Delta_2)\alpha}{\Delta_2 - t_3}. \tag{36}$$
From (33),(34),(35) and (36), $\widetilde{D}_i > 0, i = 0, 1, 2, 3, 4, 5,$ if
$$\alpha > 0,$$
and
$$\beta > max\{0, \frac{(\Delta_2 - t_3)\alpha}{t_1 - \Delta_2}, \frac{2(t_3 - t_1)\alpha}{t_1 - \Delta_2}, \frac{2(t_3 - t_1)\alpha}{\Delta_2 - t_3}, \frac{(t_1 - \Delta_2)\alpha}{\Delta_2 - t_3}\}.$$
$\frac{d^3x}{dt^3} < 0$ if
$$\widetilde{E}_i > 0, i = 0, 1, 2, 3, 4, 5, 6 \; and \; (g(t))^4 > 0.$$
Here, $\widetilde{E}_i = -E_i, i = 0, 1, 2, 3, 4, 5, 6$. $E_i's$ are already defined in (21).
$(g(t))^4 > 0$ is always positive.
$\widetilde{E}_0, \widetilde{E}_1 > 0$ if
$$\beta > \frac{(t_2 - \Delta_1)\alpha}{\Delta_1 - t_0}. \tag{37}$$

$\widetilde{E}_2 > 0$ if

$$\beta > \frac{13(t_2 - \Delta_1)\alpha}{16(\Delta_1 - t_0}. \tag{38}$$

$\widetilde{E}_3 > 0$ if

$$\beta > \frac{(t_2 - \Delta_1)\alpha}{\Delta_1 - t_0}. \tag{39}$$

$\widetilde{E}_4$ if

$$\beta > \frac{3(t_0 + t_2)\alpha}{t_2 - t_0}. \tag{40}$$

$\widetilde{E}_5, \widetilde{E}_6 > 0$ if

$$\beta > \frac{(\Delta_1 - t_0)\alpha}{t_2 - \Delta_1}. \tag{41}$$

From (37)-(41), $\widetilde{E}_i > 0, i = 0, 1, 2, 3, 4, 5, 6,$ if

$$\alpha > 0,$$

and

$$\beta > max\{0, \frac{(t_2 - \Delta_1)\alpha}{\Delta_1 - t_0}, \frac{13(t_2 - \Delta_1)\alpha}{16(\Delta_1 - t_0}, \frac{(t_2 - \Delta_1)\alpha}{\Delta_1 - t_0}, \frac{3(t_0 + t_2)\alpha}{t_2 - t_0}, \frac{(\Delta_1 - t_0)\alpha}{t_2 - \Delta_1}\}.$$

$\frac{d^3y}{dt^3} > 0$ if

$$F_i > 0, i = 0, 1, 2, 3, 4, 5, 6 \; and \; (g(t))^4 > 0$$

$F_0, F_1 > 0$ if

$$\beta > \frac{(t_3 - \Delta_2)\alpha}{\Delta_2 - t_1}. \tag{42}$$

$F_2 > 0$ if

$$\beta > \frac{13(t_3 - \Delta_2)\alpha}{16(\Delta_2 - t_1}. \tag{43}$$

$F_3 > 0$ if

$$\beta > \frac{(t_1 - 2(\Delta_2) + t_3)\alpha}{\Delta_2 - t_1}. \tag{44}$$

$F_4 > 0$ if

$$\beta > \frac{5(t_3 - t_1)\alpha}{t_3 - \Delta_2}. \tag{45}$$

$F_5, F_6 > 0$ if

$$\beta > \frac{(\Delta_2 - t_1)\alpha}{t_3 - \Delta_2}. \tag{46}$$

From (42)-(46), $F_i > 0, i = 0, 1, 2, 3, 4, 5, 6,$ if

$$\alpha > 0,$$

and

$$\beta > max\{0, \frac{(t_3 - \Delta_2)\alpha}{\Delta_2 - t_1}, \frac{13(t_3 - \Delta_2)\alpha}{16(\Delta_2 - t_1}, \frac{(t_1 - 2\Delta_2 + t_3)\alpha}{\Delta_2 - t_1}, \frac{5(t_3 - t_1)\alpha}{t_3 - \Delta_2}, \frac{(\Delta_2 - t_1)\alpha}{t_3 - \Delta_2}\}.$$

$\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2} > 0$ if

$$\beta > max\{0, \frac{(t_0t_3 - t_1t_2)\alpha}{\Delta_2 t_0 - \Delta_1 t_1}, \frac{(t_0t_3 - t_1t_2)\alpha}{\Delta_1 t_3 - \Delta_2 t_2}\}.$$

The above discussion can be summarized in the following theorem.

Theorem 1. The parametric rational cubic curve defined in (2), preserves monotonicity of curvature of given curve if the free parameters α and β satisfy the following constraints:

$$\alpha \ > 0 \ and \ \beta \ > U. \tag{47}$$

Here,

$$U = max\{0, H_k, k = 1, 2, 3, ..., 22\} \tag{48}$$

$$\begin{aligned}
H_1 &= \frac{\alpha t_0}{\Delta_1}, H_2 = \frac{\alpha(t_0 + t_2)}{\Delta_1}, H_3 = \frac{\alpha t_2}{\Delta_1}, H_4 = \frac{\alpha t_3}{\Delta_2}, H_5 = \frac{\alpha(t_1 + t_3)}{\Delta_2}, \\
H_6 &= \frac{\alpha t_1}{\Delta_2}, H_7 = \frac{(t_2 - \Delta_1)\alpha}{(t_0 - \Delta_1)}, H_8 = \frac{2(t_2 - t_0)\alpha}{(t_0 - \Delta_1)}, H_9 = \frac{2(t_2 - t_0)\alpha}{(\Delta_1 - t_2)}, \\
H_{10} &= \frac{(t_0 - \Delta_1)\alpha}{(\Delta_1 - t_2)}, H_{11} = \frac{(\Delta_2 - t_3)\alpha}{(t_1 - \Delta_2)}, H_{12} = \frac{2(t_3 - t_1)\alpha}{(t_1 - \Delta_2)}, \\
H_{13} &= \frac{2(t_3 - t_1)\alpha}{\Delta_2 - t_3}, H_{14} = \frac{(t_1 - \Delta_2)\alpha}{\Delta_2 - t_3}, H_{15} = \frac{(t_2 - \Delta_1)\alpha}{\Delta_1 - t_0}, \\
H_{16} &= \frac{13(t_2 - \Delta_1)\alpha}{16(\Delta_1 - t_0)}, H_{17} = \frac{3(t_0 + t_2)\alpha}{(t_2 - t_0)}, H_{18} = \frac{13(t_3 - \Delta_2)\alpha}{16(\Delta_2 - t_1)}, \\
H_{19} &= \frac{(t_1 - 2\Delta_2 + (t_3)\alpha}{(\Delta_2 - t_1)}, H_{20} = \frac{5(t_3 - t_1)\alpha}{(t_3 - \Delta_2)}, H_{21} = \frac{(t_0t_3 - t_1t_2)\alpha}{\Delta_2 t_0 - \Delta_1 t_1}, \\
H_{22} &= \frac{(t_0t_3 - t_1t_2)\alpha}{\Delta_1 t_3 - \Delta_2 t_2}.
\end{aligned}$$

Remark 1. The Theorem 1 can only be used to preserve the monotonically increasing curvature profile of spirals. However there are spirals with monotonically decreasing curvature profile. To deal with these spirals it is necessary to develop parametric rational cubic function (2) with monotonically decreasing curvature. It can be easily concluded from (14) that parametric rational cubic function (2) will have monotonically decreasing curvature profile i.e. $\frac{dk_P}{ds} < 0$ if

$$\frac{dx}{dt} > 0, \ \frac{dy}{dt} > 0, \ \frac{d^3x}{dt^3} > 0, \ \frac{d^3y}{dt^3} < 0, \ \frac{d^2x}{dt^2} > 0, \ \frac{d^2y}{dt^2} > 0, \ (\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}) > 0. \tag{49}$$

Values of these derivatives have been already defined. Substituting these values in Eqn. (49), it is observed that the conditions on derivatives given in (49) are true if the free parameters α and β of the parametric rational cubic curve (2) satisfy the following constraints:

$$\alpha > 0,\ \beta > W \tag{50}$$

where

$$W = max\{0, G_k, k = 1, 2, 3, ..., 17\},$$

$$\begin{aligned}
G_1 &= \frac{\alpha t_0}{\Delta_1}, G_2 = \frac{\alpha t_1}{\Delta_2}, G_3 = \frac{\alpha t_2}{\Delta_1}, G_4 = \frac{\alpha t_3}{\Delta_2}, G_5 = \frac{\alpha(t_0 + t_2)}{\Delta_1},\\
G_6 &= \frac{\alpha(t_1 + t_3)}{\Delta_2}, G_7 = \frac{(t_2 - \Delta_1)\alpha}{(\Delta_1 - t_0)}, G_8 = \frac{2(t_0 - t_2)\alpha}{(\Delta_1 - t_0)}, G_9 = \frac{2(t_0 - t_2)\alpha}{(t_2 - \Delta_1)},\\
G_{10} &= \frac{(\Delta_1 - t_0)\alpha}{(t_2 - \Delta_1)}, G_{11} = \frac{(t_3 - \Delta_2)\alpha}{(\Delta_2 - t_1)}, G_{12} = \frac{2(t_1 - t_3)\alpha}{(\Delta_2 - t_1)}, G_{13} = \frac{2(t_1 - t_3)\alpha}{(t_3 - \Delta_2)},\\
G_{14} &= \frac{(\Delta_2 - t_1)\alpha}{(t_3 - \Delta_2)}, G_{15} = \frac{2(t_1 - t_3)\alpha}{(t_3 - \Delta_2)}, G_{16} = \frac{(t_0 t_3 - t_1 t_2)\alpha}{\Delta_2 t_0 - \Delta_1 t_1},\\
G_{17} &= \frac{(t_0 t_3 - t_1 t_2)\alpha}{\Delta_1 t_3 - \Delta_2 t_2}.
\end{aligned}$$

Algorithm 1.

Step 1: Input end points (x_0, y_0, x_3, y_3) and end tangents (t_0, t_1, t_2, t_3) of the given curve.
Step 2: Assign any positive value to free parameter α.
Step 3: If $k_C^{'}(0) > 0$ & $k_C^{'}(S) > 0$, then calculate the value U from relation(47). If $k_C^{'}(0) < 0$ & $k_C^{'}(S) < 0$, then calculate the value W from relation (50).
Step 4: If $k_C^{'}(0) > 0$ & $k_C^{'}(S) > 0$, then $\beta = U + v_1$, $v_1 > 0$. If $k_C^{'}(0) < 0$ & $k_C^{'}(S) < 0$, then $\beta = W + v_1$, $v_1 > 0$.
Step 5: Substitute the value of end points, end tangents and free parameters α and β from Steps 1, 2 and 4 in (2) and (5) to obtain a C^1-parametric rational cubic curve with monotone curvature profile.

Remark 2. In Algorithm 1, v_1 is a positive real free parameter, so any positive real value can be assigned to it. However, to avoid interval tension, it is advised to assign small positive value to free parameter v_1.

4. Conclusion

In this research paper, C^1-parametric rational cubic curve with two parameters α and β is used to preserve the monotone curvature of the given curve. Unlike [1], the developed approximation scheme of this research paper does not require modification of arc length, end curvatures and free parameters. It does not involve any optimization technique therefore it is simpler than [10].

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