Journal of Prime Research in Mathematics Vol. **13***(2017), 01-07*

VERTEX EQUITABLE LABELING FOR LADDER AND SNAKE RELATED GRAPHS

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ABSTRACT. Let G be a graph with p vertices and q edges and $A = \{0, 1, 2, \cdots, \lceil \frac{q}{2} \rceil\}$. A vertex labeling $f : V(G) \to A$ induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv. For $a \in A$, let $v_f(a)$ be the number of vertices v with $f(v) = a$. A graph G is said to be vertex equitable if there exists a vertex labeling f such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \cdots, q$. In this paper, we prove that triangular ladder TL_n, $L_n \odot mK_1$, $Q_n \odot K_1$, $TL_n \odot K_1$ and alternate triangular snake $A(T_n)$ are vertex equitable graphs.

Key words : Vertex equitable labeling, ladder, snake.
AMS SUBJECT : 05C78.

1. INTRODUCTION

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminologies of graph theory as in [2]. A labeling of a graph is a map that carries the graph elements to the set of numbers, usually to the set of non-negative or positive integers. If the domain is the set of vertices, then the labeling is called vertex labeling. If the domain is the set of edges, then the labeling is called edge labeling. If the labels are assigned to both vertices and edges, then the labeling is called total labeling. For a dynamic survey of various graph labeling, we refer to Gallian [1]. A harmonious labeling f is an injection from the vertex set of a graph G to the set $\{0, 1, 2, \cdots, q-1\}$ that induces for each edge uv the label $f^*(uv) = f(u) + f(v)$ $(mod\ q)$ such that the resulting edge labels are distinct. The notion of vertex equitable labeling is a natural generalization

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of harmonious labeling. The concept of vertex equitable labeling was due to Lourdusamy and Seenivasan [5] and further studied in [3]. Here, we prove that triangular ladder TL_n, $L_n \odot mK_1$, $Q_n \odot K_1$, $TL_n \odot K_1$ and alternate triangular snake $A(T_n)$ admit vertex equitable labeling.

Definition 1.1. *Let G be a graph with p vertices and q edges and $A = \{0, 1, 2, \cdots, \lceil \frac{q}{2} \rceil\}$. A vertex labeling $f : V(G) \to A$ induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv. For $a \in A$, let $v_f(a)$ be the number of vertices v with $f(v) = a$. A graph G is said to be vertex equitable if there exists a vertex labeling f such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \cdots, q$.*

Definition 1.2. *A triangular ladder is a graph obtained from L_n by adding the edges $u_i v_{i+1}, 1 \leq i \leq n-1$, where u_i and $v_i, 1 \leq i \leq n$ are the vertices of L_n such that $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ are two paths of length n in the graph L_n.*

Definition 1.3. *A quadrilateral snake Q_n is obtained from a path $u_1, u_2, \cdots, u_n$ by joining u_i and u_{i+1} to two new vertices v_i and w_i, $1 \leq i \leq n-1$ respectively and then joining v_i and w_i.*

Definition 1.4. *An alternate triangular snake $A(T_n)$ is obtained from a path $v_1, v_2, \cdots, v_n$ where $n \equiv 2 \pmod 4$ by adding vertices $u_1, u_2, \cdots, u_{\frac{n}{2}}$ and edges $u_i v_{2i-1}$, $u_i v_{2i}$ for $i = 1, 2, \cdots, \frac{n}{2}$.*

Definition 1.5. *The graph $P_n \times P_2$ is called a ladder graph L_n.*

Definition 1.6. *The corona $G_1 \odot G_2$ of two graphs $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.*

2. Main Results

Theorem 2.1. *The triangular ladder TL_n is vertex equitable.*

Proof. Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be two paths of length n. Join u_i and v_i for $1 \leq i \leq n$. Join u_i and v_{i+1} for $1 \leq i \leq n-1$ in order to obtain $G = TL_n$. Let $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{u_i u_{i+1}; v_i v_{i+1}; u_i v_{i+1} : 1 \leq i \leq n-1; u_i v_i : 1 \leq i \leq n\}$. Then, G is of order $2n$ and size $4n-3$.
Define $f : V(G) \to A = \{0, 1, \cdots, \lceil \frac{4n-3}{2} \rceil\}$ as follows:

$$f(u_i) = 2i - 1, \ 1 \leq i \leq n;$$
$$f(v_i) = 2i - 2, \ 1 \leq i \leq n.$$

Then, the induced edge labels are

$$f^*(u_i u_{i+1}) = 4i, \ 1 \leq i \leq n-1;$$
$$f^*(u_i v_i) = 4i - 3, \ 1 \leq i \leq n;$$
$$f^*(v_i v_{i+1}) = 4i - 2, \ 1 \leq i \leq n-1;$$

$$f^*(u_iv_{i+1}) = 4i - 1,\ 1 \le i \le n-1.$$

We observe that $|v_f(a) - v_f(b)| \le 1$ for all $a, b \in A$.
Thus, TL_n is vertex equitable. □

Example 2.2. *A vertex equitable labeling of* TL_5 *is shown in Figure* 2.1.

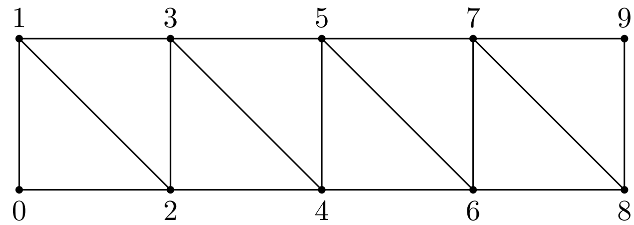

Figure 2.1

Theorem 2.3. *The graph* $L_n \odot mK_1$ *is vertex equitable.*

Proof. Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of L_n. Let $x_{i,j}$ and $y_{i,j}$ be two vertices joined to u_i and v_i respectively for $1 \le i \le n$, $1 \le j \le m$ in order to obtain $G = L_n \odot mK_1$. Let $V(G) = \{u_i, v_i : 1 \le i \le n\} \bigcup \{x_{i,j}, y_{i,j} : 1 \le i \le n, 1 \le j \le m\}$ and $E(G) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \bigcup \{u_iv_i : 1 \le i \le n\} \bigcup \{u_ix_{i,j}, v_iy_{i,j} : 1 \le i \le n, 1 \le j \le m\}$. Then, G is of order $2n + 2mn$ and size $2mn + 3n - 2$.
Define $f : V(G) \to A = \{0, 1, \cdots, \lceil \frac{2mn+3n-2}{2} \rceil\}$ as follows:

$$f(u_i) = \begin{cases} \frac{(2m+3)(i-1)}{2} & \text{if } i \text{ is odd and } 1 \le i \le n \\ \frac{(2m+3)i-2}{2} & \text{if } i \text{ is even and } 1 \le i \le n; \end{cases}$$

$$f(v_i) = \begin{cases} \frac{(2m+3)i-1}{2} & \text{if } i \text{ is odd and } 1 \le i \le n \\ \frac{(2m+3)(i-1)+1}{2} & \text{if } i \text{ is even and } 1 \le i \le n; \end{cases}$$

For $1 \le j \le m$,

$$f(x_{i,j}) = \begin{cases} \frac{(2m+3)(i-1)}{2} + j & \text{if } i \text{ is odd and } 1 \le i \le n \\ \frac{(2m+3)(i-1)+1}{2} + j & \text{if } i \text{ is even and } 1 \le i \le n; \end{cases}$$

$$f(y_{i,j}) = \begin{cases} \frac{(2m+3)(i-1)}{2} + j & \text{if } i \text{ is odd and } 1 \le i \le n \\ \frac{(2m+3)(i-1)+1}{2} + (j-1) & \text{if } i \text{ is even and } 1 \le i \le n. \end{cases}$$

Then, the induced edge labels are

$$f^*(u_iu_{i+1}) = (2m+3)i - 1,\ 1 \le i \le n-1;$$
$$f^*(u_iv_i) = (2m+3)i - m - 2,\ 1 \le i \le n;$$
$$f^*(v_iv_{i+1}) = (2m+3)i,\ 1 \le i \le n-1;$$

For $1 \le j \le m$,

$$f^*(u_ix_{i,j}) = \begin{cases} (2m+3)(i-1) + j & \text{if } i \text{ is odd and } 1 \le i \le n \\ (2m+3)i - m - 2 + j & \text{if } i \text{ is even and } 1 \le i \le n; \end{cases}$$

$$f^*(v_iy_{i,j}) = \begin{cases} (2m+3)i - m - 2 + j & \text{if } i \text{ is odd and } 1 \le i \le n \\ (2m+3)(i-1) + j & \text{if } i \text{ is even and } 1 \le i \le n. \end{cases}$$

We observe that $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.
Thus, $L_n \odot mK_1$ is vertex equitable. □

Example 2.4. *A vertex equitable labeling of $L_5 \odot 2K_1$ is shown in Figure 2.2.*

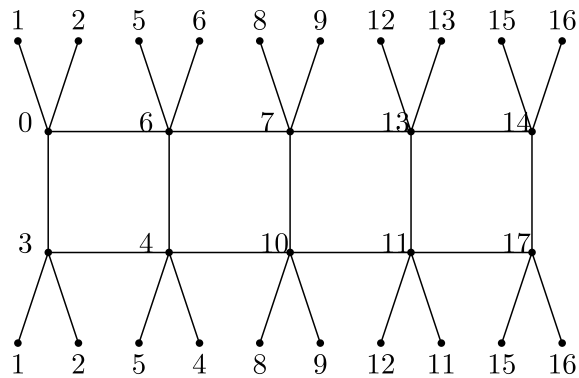

Figure 2.2

Theorem 2.5. *The graph $Q_n \odot K_1$ is vertex equitable.*

Proof. Let $u_1, u_2, \cdots, u_n$ be a path. Let v_i and w_i be two vertices joined to u_i and u_{i+1} respectively and then join v_i and w_i, $1 \leq i \leq n-1$. Let x_i be the new vertex joined to u_i,$1 \leq i \leq n$. Let y_i be the new vertex joined to v_i,$1 \leq i \leq n-1$. Let z_i be the new vertex joined to w_i,$1 \leq i \leq n-1$ in order to obtain $G = Q_n \odot K_1$. Let $V(G) = \{u_i, x_i : 1 \leq i \leq n \, ; v_i, w_i, y_i, z_i : 1 \leq i \leq n-1\}$ and $E(G) = \{u_iu_{i+1}, u_iv_i, u_{i+1}w_i, v_iw_i, v_iy_i, w_iz_i : 1 \leq i \leq n-1; u_ix_i : 1 \leq i \leq n\}$.
Then, G is of order $6n-4$ and size $7n-6$.
Define $f : V(G) \to A = \{0, 1, \cdots, \lceil \frac{7n-6}{2} \rceil\}$ as follows:

$f(u_{2i-1}) = 7i - 7, \ 1 \leq i \leq \lceil \frac{n}{2} \rceil;$
$f(u_{2i}) = 7i - 3, \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor;$
$f(x_{2i-1}) = 7i - 6, \ 1 \leq i \leq \lceil \frac{n}{2} \rceil;$
$f(x_{2i}) = 7i - 3, \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor;$
$f(v_{2i-1}) = 7i - 5, \ 1 \leq i \leq \lceil \frac{n-1}{2} \rceil;$
$f(v_{2i}) = 7i - 2, \ 1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor;$
$f(y_{2i-1}) = 7i - 6, \ 1 \leq i \leq \lceil \frac{n-1}{2} \rceil;$
$f(y_{2i}) = 7i - 2, \ 1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor;$
$f(w_{2i-1}) = 7i - 4, \ 1 \leq i \leq \lceil \frac{n-1}{2} \rceil;$
$f(w_{2i}) = 7i, \ 1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor;$
$f(z_{2i-1}) = 7i - 4, \ 1 \leq i \leq \lceil \frac{n-1}{2} \rceil;$
$f(z_{2i}) = 7i - 1, \ 1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor.$

Then, the induced edge labels are

$f^*(u_iu_{i+1}) = 7i - 3, \ 1 \leq i \leq n-1;$
$f^*(u_ix_i) = 7i - 6, \ 1 \leq i \leq n;$

$f^*(u_iv_i) = 7i - 5,\ 1 \leq i \leq n-1;$
$f^*(u_{i+1}w_i) = 7i,\ 1 \leq i \leq n-1;$
$f^*(v_iw_i) = 7i - 2,\ 1 \leq i \leq n-1;$
$f^*(v_iy_i) = 7i - 4,\ 1 \leq i \leq n-1;$
$f^*(w_iz_i) = 7i - 1,\ 1 \leq i \leq n-1.$

We observe that $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.
Thus, $Q_n \odot K_1$ is vertex equitable. □

Example 2.6. *A vertex equitable labeling of $Q_5 \odot K_1$ is shown in Figure* 2.3.

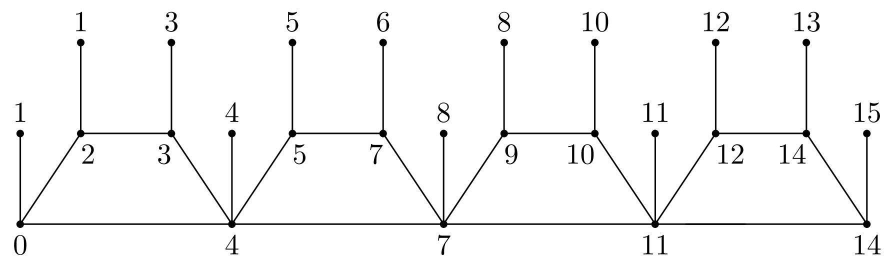

Figure 2.3

Theorem 2.7. *The graph $TL_n \odot K_1$ is vertex equitable.*

Proof. Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of TL_n. Let x_i and y_i be two vertices joined to u_i and v_i respectively for $1 \leq i \leq n$. Let $G = TL_n \odot K_1$ with $V(G) = \{u_i, v_i, x_i, y_i : 1 \leq i \leq n\}$ and $E(G) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_{i+1} : 1 \leq i \leq n-1;\ u_ix_i, v_iy_i, u_iv_i : 1 \leq i \leq n\}$. Then, G is of order $4n$ and size $6n - 3$.
Define $f : V(G) \to A = \{0, 1, \cdots, \lceil \frac{6n-3}{2} \rceil\}$ as follows:
For $1 \leq i \leq n$,

$$f(u_i) = \begin{cases} 0 & \text{if } i = 1 \\ 3i - 2 & \text{if } i \text{ is odd} \\ 3i - 1 & \text{if } i \text{ is even ;} \end{cases}$$

$$f(v_i) = \begin{cases} 2 & \text{if } i = 1 \\ 3i - 3 & \text{if } i \text{ is odd} \\ 3i - 2 & \text{if } i \text{ is even ;} \end{cases}$$

$$f(x_i) = \begin{cases} 1 & \text{if } i = 1 \\ 3i - 1 & \text{if } i \text{ is odd} \\ 3i - 3 & \text{if } i \text{ is even ;} \end{cases}$$

$$f(y_i) = \begin{cases} 1 & \text{if } i = 1 \\ 3i - 1 & \text{if } i \text{ is odd} \\ 3i - 3 & \text{if } i \text{ is even.} \end{cases}$$

Then, the induced edge labels are

For $1 \le i \le n-1$,

$$f^*(u_iu_{i+1}) = \begin{cases} 5 & \text{if } i = 1 \\ 6i & \text{if } 2 \le i \le n-1; \end{cases}$$

$$f^*(v_iv_{i+1}) = \begin{cases} 6 & \text{if } i = 1 \\ 6i-2 & \text{if } 2 \le i \le n-1; \end{cases}$$

$$f^*(u_iv_{i+1}) = \begin{cases} 4 & \text{if } i = 1 \\ 6i-1 & \text{if } 2 \le i \le n-1; \end{cases}$$

For $1 \le i \le n$,

$$f^*(u_iv_i) = \begin{cases} 2 & \text{if } i = 1 \\ 6i-5 & \text{if } i \text{ is odd} \\ 6i-3 & \text{if } i \text{ is even }; \end{cases}$$

$$f^*(u_ix_i) = \begin{cases} 1 & \text{if } i = 1 \\ 6i-3 & \text{if } i \text{ is odd} \\ 6i-4 & \text{if } i \text{ is even }; \end{cases}$$

$$f^*(v_iy_i) = \begin{cases} 3 & \text{if } i = 1 \\ 6i-4 & \text{if } i \text{ is odd} \\ 6i-5 & \text{if } i \text{ is even }. \end{cases}$$

We observe that $|v_f(a) - v_f(b)| \le 1$ for all $a, b \in A$.
Thus, $TL_n \odot K_1$ is vertex equitable. □

Example 2.8. *A vertex equitable labeling of* $TL_5 \odot K_1$ *is shown in Figure* 2.4.

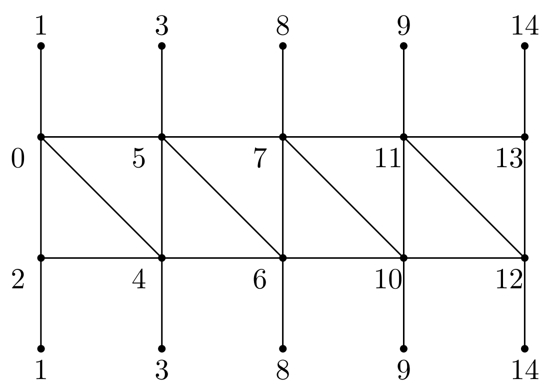

Figure 2.4

Theorem 2.9. *The alternate triangular snake* $A(T_n)$ *is vertex equitable.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of path P_n. The graph $A(T_n)$ is obtained by joining the vertices v_iv_{i+1} (alternately) to new vertex $u_i, 1 \le i \le n-1$ for n is even and $1 \le i \le n-2$ for n is odd. Let $V(G) = \{v_i : 1 \le i \le n; u_i : 1 \le i \le \frac{n}{2}\}$ and $E(G) = \{v_iv_{i+1} : 1 \le i \le n-1; v_{2i-1}u_i : 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor; u_iv_{2i} :$

$1 \leq i \leq \lceil \frac{n}{2} \rceil\}$. Then

$$|V(G)| = \begin{cases} \frac{3n-1}{2} \text{ for } n \text{ is odd} \\ \frac{3n}{2} \text{ for } n \text{ is even} \end{cases}$$

$$|E(G)| = \begin{cases} 2n-2 \text{ for } n \text{ is odd} \\ 2n-1 \text{ for } n \text{ is even} \end{cases}$$

Define $f : V(G) \to A = \{0, 1, 2, \cdots, n-1\}$ as follows:

$f(u_{2i-1}) = 2i - 2,\ 1 \leq i \leq \lceil \frac{n}{2} \rceil$;
$f(u_{2i}) = 2i,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$;
$f(v_i) = 2i - 1,\ 1 \leq i \leq \frac{n}{2}$.

Then, the induced edge labels are

$f^*(u_i u_{i+1}) = 2i,\ 1 \leq i \leq n-1$;
$f^*(u_{2i-1} v_i) = 4i - 3,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$;
$f^*(u_{2i} v_i) = 4i - 1,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

We observe that $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.
Thus, $A(T_n)$ is vertex equitable. □

Example 2.10. *A vertex equitable labeling of $A(T_6)$ is shown in Figure 2.5.*

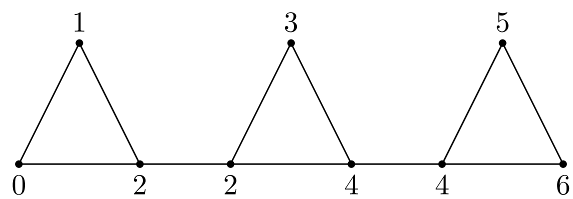

Figure 2.5

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