Journal of Prime Research in Mathematics Vol. **13***(2017), 16-29*

PROPERTIES OF CO-INTERSECTION GRAPH OF SUBMODULES OF A MODULE

LOTF ALI MAHDAVI[1], YAHYA TALEBI[2]

Abstract. Let R be a ring with identity and M be a unitary left R-module. The co-intersection graph of proper submodules of M, $\Omega(M)$ is an undirected simple graph whose vertices are non-trivial submodule of M in which two vertices N and K are joined by an edge, if and only if $N + K \neq M$. In this paper, we study several properties of $\Omega(M)$. We prove that, if $\Omega(M)$ is a path, then $\Omega(M) \cong P_2$ or $\Omega(M) \cong P_3$. We show that, if $\Omega(M)$ is a forest, then each component of $\Omega(M)$ is complete or star graph. We determine the conditions under which $\Omega(M)$ is weakly perfect. Moreover, we introduce the universal vertices and the dominating sets of $\Omega(M)$ and their relationship with the non-trivial small submodules of M.

Key words : co-intersection graph, forest, weakly perfect graph, universal vertex, dominating set.
AMS SUBJECT : 05C15,05C25,05C69,16D10.

1. Introduction

The graphs associated with the algebraic structures are attractive subjects and research about these topics is extended by many mathematicians. One of the most important graphs is the intersection graph of the algebraic structures. The idea of studying the intersection graph of algebraic structures, first appeared in [9] by J. Bosak where defined the intersection graph of proper subsemigroups of a semigroup in 1964. The graphs related to the group and ring structures has been studied extensively by several authors, for example see [2], [3], [4], [7], [9], [10], [13], [14], [16] and [19]. Recently various constructions of graphs related to the module structure are found in [1], [5], [6], [15] and [18]. Motivated by previous studies on the intersection graph of algebraic

[1]Department of Mathematics, Faculty of Mathematical Sciences,
University of Mazandaran, Babolsar, Iran. Email: l.a.mahdavi154@gmail.com
[2]Department of Mathematics, Faculty of Mathematical Sciences,
University of Mazandaran, Babolsar, Iran. Email: talebi@umz.ac.ir.

structures, we define the *co-intersection graph* of submodules of a module in [15]. Now, our main goal is to study more about the connection between the algebraic properties of a module M and the graph theoretic properties of the graph $\Omega(M)$ which is associated with it.

Throughout this paper R is a ring with identity and M is a unitary left R-module. By a non-trivial submodule of an R-module M, we mean a non-zero proper left submodule of M. The co-intersection graph of an R-module M, denoted by $\Omega(M)$, is defined to be the undirected simple graph with the vertex set $V(\Omega)$ whose vertices are in one to one correspondence with all non-trivial submodules of M and two distinct vertices are adjacent if and only if the sum of the corresponding submodules of M is not equal to M. A submodule N of an R-module M is called *small* in M (we write $N \ll M$), if $N + L \neq M$ for every proper submodule L of M. A submodule K of non-zero R-module M is said to be *essential* in M (we write $K \trianglelefteq M$), if $K \cap L \neq (0)$ for every non-zero proper submodule L of M. If every non-zero submodule of M is essential, then M is called a *uniform* module. A non-zero R-module M is called *local*, if it has a unique maximal submodule that contains all other proper submodules. A submodule X of an R-module M is called the *complement* of the submodule Y of M, if $M = X \oplus Y$. We say that an R-module M is *complemented*, if every submodule of M has a complement in M. A non-zero R-module M is called *indecomposable*, if it is not a direct sum of two non-zero submodules. The ring of all endomorphisms of an R-module M is denoted by $End_R(M)$. The *radical* of an R-module M, denoted by $Rad(M)$, is the intersection of all maximal submodules of M. The *socle* of an R-module M, denoted by $Soc(M)$, is the sum of all simple submodules of M. For an R-module M, the length of M, is denoted by $l_R(M)$.
Let Ω=($V(\Omega)$, $E(\Omega)$) be a graph with the vertex set $V(\Omega)$ and the edge set $E(\Omega)$, where an edge is an unordered pair of distinct vertices of Ω. Graph Ω is finite, if $card(V(\Omega)) < \infty$, otherwise Ω is infinite. A graph Ω is *empty*, if $card(V(\Omega)) = 0$. A subgraph of a graph Ω is a graph Γ such that $V(\Gamma) \subseteq V(\Omega)$ and $E(\Gamma) \subseteq E(\Omega)$. By order of Ω, we mean the number of vertices of Ω and we denoted it by $|\Omega|$. If x and y are two adjacent vertices of Ω, then we write $x - y$. The *degree* of a vertex v in a graph Ω, denoted by $deg(v)$, is the number of edges incident with v. The maximum and minimum degree of Ω are $\Delta(\Omega)$ and $\delta(\Omega)$, respectively. A vertex u is called *universal*, if it is adjacent to all other vertices. A vertex v is called *isolated*, if $deg(v) = 0$. A vertex w is called *end vertex*, if $deg(w) = 1$. Let x and y be two distinct vertices of Ω. An *x, y-path* is a path with starting vertex x and end vertex y. A path with n vertices is denoted by P_n. A *cycle* in a graph is a path of length at least 3 through distinct vertices which begins and ends at the same vertex. We mean of (x, y, z) is a 3-cycle. A graph is said to be *null*, if it has no edge. A graph

is said to be *connected*, if there is a path between every pair of vertices of the graph. A *star graph* is a tree consisting of one universal vertex. A complete graph of order n is denoted by K_n. The *complement graph* of Ω, denoted by $\overline{\Omega}$. By a *clique* in a graph Ω, we mean a complete subgraph of Ω and the number of vertices in a largest clique of Ω, is called the clique number of Ω and is denoted by $\omega(\Omega)$. For a graph Ω, let $\chi(\Omega)$, denote the *chromatic number* of Ω, i.e., the minimum number of colors which can be assigned to the vertices of Ω such that every two adjacent vertices have different colors. A *weakly perfect graph*, is a graph Ω in which $\omega(\Omega) = \chi(\Omega)$.

2. Forest and weakly perfect graphs $\Omega(M)$

In this section, we determine some conditions on the module M for which $\Omega(M)$ is empty, null, path, tree, forest and weakly perfect graph. We consider the relationship between the maximum degree $\Delta = \Delta(\Omega(M))$ and the minimum degree $\delta = \delta(\Omega(M))$ with the length of the module M and study the clique number, the chromatic number of the weakly perfect graph $\Omega(M)$.

In the following proposition we determine the conditions under which the graph $\Omega(M)$ is empty or it has at least an empty subgraph.

Proposition 1. *Let R be a commutative ring and M be a non-zero R-module. Then the following statements hold:*
(1) If N is a maximal submodule of M, then $\Omega(M/N)$ is an empty graph.
(2) If M is semisimple, then M has a submodule N such that $\Omega(N)$ is an empty graph.
(3) If M is complemented, then M has a submodule K such that $\Omega(K)$ is an empty graph.
(4) If M is Artinian, then M has a submodule L such that $\Omega(L)$ is an empty graph.

Proof. (1) It is clear.
(2) Suppose that M is a semisimple R-module. Then M has a non-zero element x which Rx is a non-zero cyclic submodule of M. Since Rx is non-zero and finitely generated, it contains a maximal submodule $N^\star$ and since M is semisimple, Rx is also so. Thus by Theorem 9.6(e) of [8, p. 117], there exists a submodule N of Rx such that $Rx = N \bigoplus N^\star$. However, since $N \cong Rx/N^\star$ and $N^\star$ is maximal, N is simple and $\Omega(N)$ is an empty graph.
(3) In order to establish this part, assume that M is a complemented R-module and $0 \neq x \in M$. We consider Rx as a submodule of M. Since M is complemented, Rx is also so and since Rx is cyclic, there exists a left ideal I of R such that $Rx \cong R/I$ as a R-module. As R has identity, then by Zorn's Lemma, there exists a left maximal ideal $I^\star$ such that $I \subseteq I^\star$ and consequently R/I has the maximal submodule $I^\star/I$. Since $Rx \cong R/I$, Rx has a maximal

submodule $K^\star$ and since Rx is complemented, there exists a submodule K such that $Rx = K \bigoplus K^\star$. However, as $K \cong Rx/K^\star$ and $K^\star$ is maximal, then K is simple and $\Omega(K)$ is an empty graph.
(4) Let M be an Artinian R-module. If M is simple, we set $L = M$ and $\Omega(L)$ is an empty graph. Otherwise, there exists a submodule M_1 such that $M_1 \subset M$. If M_1 is a simple, we set $L = M_1$, otherwise, there exists a submodule M_2 of M_1 such that $M_2 \subset M_1 \subset M$. Consequently there exists an infinite strictly decreasing sequence $M \supset M_1 \supset M_2 \supset \ldots$ and since M is an Artinian R-module, this chain should be stationary. Hence there exists a positive integer n such that M_n is simple. We set $L = M_n$ and $\Omega(L)$ is an empty graph. □

The following theorem is fundamental for the connectivity of the graph $\Omega(M)$.

Theorem 2. ([**15**, **Theorem 2.1**]) *Let M be an R-module. Then the graph $\Omega(M)$ is not connected if and only if M is a direct sum of two simple R-modules.*

Now, we have some corollaries which are immediate consequences of Theorem 2.

Corollary 3. *Let M be an R-module which is not simple. Then $\Omega(M)$ is connected if and only if either M is not semisimple or $M = \oplus_{i=1}^{i=n} M_i$, where $n \geq 3$ and M_i is a simple R-module.*

Corollary 4. *Let M be an R-module and $|\Omega(M)| \geq 2$. If $\Omega(M)$ has at least an edge, then $\Omega(M)$ is a connected graph.*

Corollary 5. *Let M be an R-module and $|\Omega(M)| \geq 2$. Then $\Omega(M)$ is a null graph if and only if $l_R(M) \leq 2$.*

Now, we consider the conditions of the module M such that $\Omega(M)$ is a tree or path and obtain some submodules of M with at most one degrees.

Proposition 6. *Let M be an R-module and N a maximal submodule of M. If the graph $\Omega(M)$ has no 3-cycle, then N is either an isolated vertex or an end vertex of $\Omega(M)$.*

Proof. Let N be a maximal submodule of M and $deg(N) \geq 2$. Then there exist at least two distinct non-trivial submodules K and L such that both are adjacent to the vertex N of the graph $\Omega(M)$. Hence $N + K \neq M$ and $N + L \neq M$. Since $N \subseteq N + K \neq M$ and $N \subseteq N + L \neq M$, the maximality of N implies that $N + K = N = N + L$. Then $K \subseteq N$ and $L \subseteq N$, thus $K + L \subseteq N \neq M$ and this implies that K and L are two adjacent vertices of $\Omega(M)$. Thus there is a 3-cycle of the form (N, K, L) in $\Omega(M)$, a contradiction. Therefore, $deg(N) = 0$ or $deg(N) = 1$ and consequently N is either an isolated vertex or an end vertex of $\Omega(M)$. □

Theorem 7. *Let M be an R-module. Then $\Omega(M) \cong P_2$ if and only if M have only two distinct non-trivial submodules which are one minimal and the other one maximal.*

Proof. Suppose that $\Omega(M) \cong P_2$. Let X and Y be two vertices of $\Omega(M)$. Clearly $(0) \neq X+Y \neq M$, thus $X+Y$ is a vertex of $\Omega(M)$ and $X+Y = X$ or $X+Y = Y$. If $X+Y = X$, then $Y \subseteq X$ and this implies that Y is a minimal submodule and X is a maximal submodule of M. Similarly, if $X+Y = Y$, we conclude that X is a minimal submodule and Y is a maximal submodule of M. The converse is straightforward. □

Example 1. *Let p be a prime number. Consider $\mathbb{Z}_{p^3}$ as a $\mathbb{Z}$-module. There are only two non-trivial submodules $p\mathbb{Z}_{p^3}$ and $p^2\mathbb{Z}_{p^3}$ such that $p\mathbb{Z}_{p^3} + p^2\mathbb{Z}_{p^3} \neq \mathbb{Z}_{p^3}$. Thus $\Omega(\mathbb{Z}_{p^3}) \cong P_2$.*

Lemma 8. *Let M be an R-module and V be an end vertex of the graph $\Omega(M)$. Then V is a maximal submodule or a minimal submodule of M.*

Proof. Suppose that V is an end vertex of $\Omega(M)$. Then there exist only a vertex W of $\Omega(M)$ such that W is adjacent to V. Thus $(0) \neq V+W \neq M$. Since $V+W+V = V+W \neq M$ and $W+V+W = V+W \neq M$ and since $deg(V) = 1$, $V+W = V$ or $V+W = W$. Thus $V \subseteq W$ or $W \subseteq V$. Now, if $V \subseteq W$, then we show that V is a minimal submodule of M. Suppose that there exists a non-trivial submodule X of M such that $(0) \subset X \subset V$, then $X+V = V \neq M$ and X is adjacent to V and thus $deg(V) \geq 2$, a contradiction. However, if $V \subseteq W$, we show that V is a maximal submodule of M. Assume that there exists a non-trivial submodule U of M such that $V \subset U \subset M$, then $V+U = U \neq M$ and U is adjacent to V and again, a contradiction. □

Lemma 9. *Let M be an R-module and the graph $\Omega(M)$ be a path as sequence $M_1, M_2, \ldots, M_n$. If M_1 is a minimal submodule of M, then $\Omega(M) \cong P_2$.*

Proof. Suppose that the graph $\Omega(M)$ is a path as sequence $M_1, M_2, \ldots, M_n$ and M_1 is a minimal submodule of M. Since $(0) \neq M_1+M_2 \neq M$, $M_1+M_2 = M_1$ or two vertices M_1+M_2 and M_1 are adjacent. If $M_1+M_2 = M_1$, then $M_2 \subseteq M_1$ and the minimality of M_1 implies that $M_1 = M_2$, a contradiction. Hence, M_1+M_2 and M_1 are adjacent. Consequently $M_1+M_2 = M_2$, thus $M_1 \subseteq M_2$. Now let us get $n \geq 3$. If $M_2+M_3 = M_2$, then $M_3 \subseteq M_2$ and since $M_1 \subseteq M_2$, $M_1+M_3 \subseteq M_2 \neq M$ and thus M_1 and M_3 are adjacent, a contradiction. Otherwise $M_2+M_3 = M_1$ or $M_2+M_3 = M_3$. If $M_2+M_3 = M_1$, then $M_3 \subseteq M_1$ and $M_1+M_3 = M_1 \neq M$, a contradiction. If $M_2+M_3 = M_3$, then $M_2 \subseteq M_3$ and since $M_1 \subseteq M_2$, $M_1 \subseteq M_3$, then $M_1+M_3 = M_3 \neq M$, a contradiction. Therefore, $\Omega(M) \cong P_2$. □

Theorem 10. *Let M be an R-module and $\Omega(M)$ be a path. Then $\Omega(M) \cong P_2$ or $\Omega(M) \cong P_3$. Moreover, if $\Omega(M) \cong P_3$ and $l_R(M) < \infty$, then M have only* *three submodules* M_1, M_2 *and* M_3 *such that* $l_R(M) = l_R(M_1)+l_R(M_3)-l_R(M_2)$ *and* $\Omega(M/M_1) \cong \Omega(M_3/M_2)$.

Proof. Suppose that the graph $\Omega(M)$ is a path as sequence $M_1, M_2, \ldots, M_n$. By Lemma 8, the submodule M_1 is minimal or maximal. Let M_1 be a maximal submodule of M. Since M_1 and M_2 are two adjacent vertices and $M_1 \subseteq M_1 + M_2 \neq M$, the maximality of M_1 implies that $M_1 + M_2 = M_1$ and thus $M_2 \subseteq M_1$. Let $n > 2$, then $M_2 + M_3 \neq M_2$. As $M_2 + M_3 = M_2$ implies $M_3 \subseteq M_2$ and since $M_2 \subseteq M_1$, then $M_3 \subseteq M_1$ and $M_3 + M_1 = M_1 \neq M$. Hence, M_1 and M_3 are two adjacent vertices of $\Omega(M)$ which is a contradiction. However, two vertices $M_2 + M_3$ and M_2 are adjacent, so $M_2 + M_3 = M_1$ or $M_2 + M_3 = M_3$. If $M_2 + M_3 = M_1$, then $M_1 = M_2 + M_3 = M_2 + M_3 + M_3 = M_1 + M_3$. Hence, M_1 and M_3 are adjacent and again a contradiction. So $M_2 + M_3 = M_3$ and thus $M_2 \subseteq M_3$. Thus, this is true for n=3. Let $n > 3$, then $M_4 + M_3 \neq M_3$. As $M_4 + M_3 = M_3$, implies that $M_4 \subseteq M_3$ and since $M_2 \subseteq M_3$, $M_4 + M_2 \subseteq M_3$. Hence, M_2 and M_4 are two adjacent vertices of $\Omega(M)$ which is a contradiction. Since two vertices $M_4 + M_3$ and M_3 are adjacent, $M_4+M_3 = M_2$ or $M_4+M_3 = M_4$. If $M_4+M_3 = M_2$, then $M_4 \subseteq M_2$ and $M_4 + M_2 = M_2 \neq M$. Hence, M_2 and M_4 are adjacent, a contradiction. However, if $M_4 + M_3 = M_4$, then $M_3 \subseteq M_4$ and since $M_2 \subseteq M_3$, $M_2 \subseteq M_4$ and $M_2 + M_4 = M_4 \neq M$. Hence, M_2 and M_4 are adjacent, a contradiction. Finally, we have $M_2 \subseteq M_1$ and $M_2 \subseteq M_3$. Therefore, $M_2 \subseteq M_1 \cap M_3 \subseteq M_1$ and this fact that M_1 and M_3 are not adjacent, we claim that $M_2 = M_1 \cap M_3$. To see this, let $M_2 \neq M_1 \cap M_3$. Since $M_1 \cap M_3 + M_2 = M_1 \cap M_3 \neq M$ and $M_1 \cap M_3 + M_1 = M_1 \neq M$, the vertex $M_1 \cap M_3$ is adjacent to vertices M_1 and M_2. Then $M_1 = M_1 \cap M_3$ or $M_2 = M_1 \cap M_3$. If $M_1 = M_1 \cap M_3$, then $M_1 \subseteq M_3$ and so M_1 and M_3 are adjacent, a contradiction. Hence, $M_2 = M_1 \cap M_3$ and thus $l_R(M) = l_R(M_1 + M_3) = l_R(M_1) + l_R(M_3) - l_R(M_2)$ and since $M/M_1 \cong M_3/M_2$, $\Omega(M/M_1) \cong \Omega(M_3/M_2)$. Also, if M_1 is a minimal submodule of M, then the result is an immediate consequence of Lemma 9. □

In Theorem 3.3 of [15], it was proved that for an R-module M with $|\Omega(M)| \geq 2$, $\Omega(M)$ is a star graph if and only if it is a tree. Now we have the following corollary.

Corollary 11. *Let* M *be an* R*-module and* $\Omega(M)$ *be a forest. Then each component of* $\Omega(M)$ *is either* K_1 *or a star graph.*

Proof. Suppose that $\Omega(M)$ is a forest. We know that each component of $\Omega(M)$ is a tree. If $|\Omega(M)| = 1$, then $\Omega(M) \cong K_1$. But, if $|\Omega(M)| \geq 2$, then by above argument $\Omega(M)$ is a star graph, and the proof is complete. □

Now, we consider the conditions of the module M such that $\Omega(M)$ is a forest or weakly perfect graph. First, we give some examples of the co-intersection graph of submodules of the non-semisimple and semisimple modules.

Example 2. *Let p and q be two distinct prime numbers. We know that $\mathbb{Z}_{pq^2}$ as a $\mathbb{Z}$-module is not semisimple and $\Omega(\mathbb{Z}_{pq^2})$ is connected and weakly perfect graph but it is not a forest. (See Figure 1)*

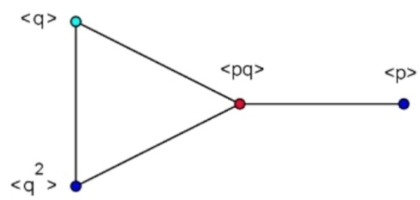

FIGURE 1. $\Omega(\mathbb{Z}_{pq^2})$

Example 3. *Let p, q and r be three distinct prime numbers. The $\mathbb{Z}$-modules $\mathbb{Z}_{pq} =< p > \oplus < q >$ and $\mathbb{Z}_{pqr} =< pq > \oplus < pr > \oplus < qr >$ are semisimple. Clearly, $\Omega(\mathbb{Z}_{pq}) \cong \overline{K_2}$ and $\omega(\Omega(\mathbb{Z}_{pq})) = \chi(\Omega(\mathbb{Z}_{pq})) = 1$. Hence, $\Omega(\mathbb{Z}_{pq})$ is both forest and weakly perfect graph. Also, $\omega(\Omega(\mathbb{Z}_{pqr})) = \chi(\Omega(\mathbb{Z}_{pqr})) = 3$. Therefore, $\Omega(\mathbb{Z}_{pqr})$ is a weakly perfect graph but not a forest. (See Figure 2)*

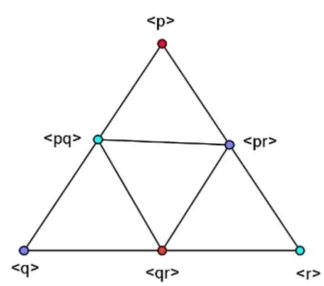

FIGURE 2. $\Omega(\mathbb{Z}_{pqr})$

Proposition 12. *Let M be an R-module. If $|\Omega(M)| \geq 2$ and $\Omega(M)$ is not connected, then $\Omega(M)$ is both forest and weakly perfect graph.*

Proof. Suppose that $|\Omega(M)| \geq 2$ and $\Omega(M)$ is not connected, then by [15, Corollary 2.4], $\Omega(M)$ is a null graph. Thus it has no cycle and we have $\omega(\Omega(M)) = \chi(\Omega(M) = 1$. Hence, $\Omega(M)$ is both forest and weakly perfect graph. □

Example 4. *We consider the co-intersection graph of submodules of the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_p$ such that p is a prime number. The order of all elements of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is 1 or p and since the only element of the order 1 is identity,* *then the number of elements of order p is $|\mathbb{Z}_p \oplus \mathbb{Z}_p| - 1 = p^2 - 1$. Also each non-identity element in $\mathbb{Z}_p \oplus \mathbb{Z}_p$ generates a submodule of order p, with $p-1$ non-identity elements. So p^2-1 counts each of the non-identity elements $p-1$ times. Therefore, $\mathbb{Z}_p \oplus \mathbb{Z}_p$ as a $\mathbb{Z}$-module has exactly $(p^2-1)/(p-1) = p+1$ non-trivial maximal and minimal submodules of order p which all are isolated vertices of this graph. Consequently, $\Omega(\mathbb{Z}_p \oplus \mathbb{Z}_p) \cong \overline{K_{p+1}}$ is both forest and weakly perfect graph.*

Theorem 13. *Let M be an R-module with the graph $\Omega = \Omega(M)$. If $\Delta = \Delta(\Omega) < \infty$ and $\delta = \delta(\Omega) \geq 1$, then the following statements hold:*
(1) $l_R(M) \leq \Delta + 1$.
(2) If M is except direct sum of two simple R-modules, then $l_R(M) < \infty$, $\omega(\Omega) < \infty$ and $\chi(\Omega) < \infty$.
(3) Every non-trivial submodule of M has finitely many submodules.
(4) Every non-trivial submodule of M is containing a minimal submodule and contained in a maximal submodule.

Proof. (1) In order to establish this part, suppose that $M_1 \subset M_2 \subset M_3 \subset \dots,$ is an infinite strictly increasing sequence of non-trivial submodules of M. Since $\delta \geq 1$, there exists a submodule X of M such that $M_{\Delta+1} + X \neq M$. Hence $M_i + X \neq M$, for $1 \leq i \leq \Delta + 1$ and thus $deg(X) \geq \Delta + 1$, which is a contradiction. Now, we assume that $N_1 \supset N_2 \supset N_3 \supset \dots,$ is an infinite strictly decreasing sequence of non-trivial submodules of M. Since $\delta \geq 1$, there exists a submodule Y of M such that $Y + N_1 \neq M$, then $Y + N_i \neq M$, for each $i \geq 1$, and thus $deg(Y) = \infty$, a contradiction. Therefore, $l_R(M) \leq \Delta + 1$.
(2) Clearly, $l_R(M) \leq \omega(\Omega) + 1$. Since M is except direct sum of two simple R-modules, by Theorem 2, Ω is a connected graph. Hence, by Theorem 10.3 Part 1 of [11, p. 289], $\omega(\Omega) \leq \chi(\Omega) \leq \Delta + 1$ and since $\Delta < \infty$, we have $l_R(M) < \infty$, $\omega(\Omega) < \infty$ and $\chi(\Omega) < \infty$.
(3) Let N be a non-trivial submodule of M. Since $\delta \geq 1$, there exists a submodule K of M such that $N + K \neq M$. Then for every submodule X of N, $X + K \neq M$ and since $\Delta < \infty$, the number of submodules of N is finite.
(4) Since $\Delta = \Delta(\Omega) < \infty$, by Part 1, $l_R(M) < \infty$. Then M is Noetherian and Artinian R-module. Let N be a non-trivial submodule of M. As M is Noetherian, it possesses a maximal submodule N^* such that $N \subseteq N^*$ and as M is Artinian, it possesses a minimal submodule N_* such that $N_* \subseteq N$. Hence, $N_* \subseteq N \subseteq N^*$. Thus every non-trivial submodule of M is containing a minimal submodule and contained in a maximal submodule. □

Theorem 14. *Let M be an R-module and $|\Omega(M)| > 2$. If M is Noetherian which contains a unique maximal submodule or it is hollow, then $\Omega(M)$ is a weakly perfect graph but not a forest.*

Proof. Suppose that M is a Noetherian R-module such that it contains a unique maximal submodule or M is a hollow R-module. Then by [15, Theorem

2.9] and [15, Theorem 2.11], $\Omega(M)$ is a complete graph. Since $|\Omega(M)| > 2$, $\Omega(M)$ is not a forest. However, we know that all complete graphs are their own maximal cliques and $\omega(\Omega(M)) \leq \chi(\Omega(M))$. Hence, if $\omega(\Omega(M)) = \infty$, there is nothing to prove. However, if assume that $\Omega(M) \cong K_n$, where $n \in \mathbb{N}$, then $\omega(\Omega(M)) = \chi(\Omega(M)) = n$, and this completes the proof. □

We now recall that an R-module M is said to be A-*projective* if for every epimorphism $g : A \to B$ and homomorphism $f : M \to B$, there exists a homomorphism $h : M \to A$ such that $gh = f$. A module P is projective if P is A-projective for every R-module A. If P is P-projective, then P is also called self-projective.

Corollary 15. *Let M be an R-module and $|\Omega(M)| > 2$. Then $\Omega(M)$ is weakly perfect graph but not forest, if one of the following statements holds:*
(1) If M is an indecomposable R-module such that every pair of non-trivial submodules of M, have zero intersection.
(2) If M is a local R-module.
(3) If M is a self-projective R-module and $End_R(M)$ is a local ring.

Proof. It is an immediate consequences of [15, Corollary 2.16] and Theorem 14. □

Example 5. *For every prime number p and $n \in \mathbb{N}$ with $n \geq 4$, since $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-modules are hollow, by Theorem 14, two graphs $\Omega(\mathbb{Z}_{p^n})$ and $\Omega(\mathbb{Z}_{p^\infty})$ are weakly perfect but they are not forests.*

Proposition 16. *Let M be an Artinian R-module such that it contains a unique minimal submodule and $\Delta = \Delta(\Omega(M)) < \infty$. Then M is Noetherian.*

Proof. Assume that M is an Artinian R-module such that it contains a unique minimal submodule. Then by [15, Theorem 2.10], $\Omega(M)$ is a connected graph. So $\delta = \delta(\Omega(M)) \geq 1$ and since $\Delta = \Delta(\Omega(M)) < \infty$, by Theorem 13 Part 1, $l_R(M) < \infty$. Hence, M is Noetherian. □

3. Universal vertices and dominating sets of $\Omega(M)$

Let M be an R-module. In this section, we study the universal vertices and the dominating sets of $\Omega(M)$ and also obtain their relationship with the non-trivial small submodules of M.

Lemma 17. *Let M be an R-module and N a non-trivial submodule of M. Then $N \ll M$ if and only if N is a universal vertex of the graph $\Omega(M)$.*

Proof. Obvious. □

Example 6. *Let p and q be two distinct prime numbers. We know that* $< pq >$ *is the only non-trivial small submodule of the* $\mathbb{Z}$*-module* $\mathbb{Z}_{pq^2}$ *and by Lemma 17, it is the only universal vertex of* $\Omega(\mathbb{Z}_{pq^2})$*. (See Example 2 and Fig.* 1*).*

Proposition 18. *Let* M*,* N*,* K *and* L *are* R*-modules. Then the following conditions hold:*
(1) If $f : M \to N$ *and* $g : N \to L$ *are two epimorphisms, then* $gof(X)$ *is a universal vertex of the graph* $\Omega(M)$ *if and only if for each two universal vertices* X *and* Y *of* $\Omega(M)$ *and* $\Omega(N)$ *respectively, we have* $f(X)$ *and* $g(Y)$ *are two universal vertices of the graphs* $\Omega(M)$ *and* $\Omega(N)$*, respectively.*
(2) If $K \subset L \subset M$*, then* L *is a universal vertex of the graph* $\Omega(M)$ *if and only if* L/K *is a universal vertex of the graph* $\Omega(M/K)$*.*
(3) If $V_1, V_2, \ldots, V_n$ *are universal vertices of the graph* $\Omega(M)$*, then* $V = \sum_{i=1}^{i=n} V_i$ *is a universal vertex of* $\Omega(M)$*.*
(4) For every homomorphism $\varphi : M \to N$*, if* K *is a universal vertex of the graph* $\Omega(M)$*, then* $\varphi(K)$ *is a universal vertex of the graph* $\Omega(N)$*.*
(5) If $K \subset L \subset M$ *and* L *is a direct summand of* M*, then* K *is a universal vertex of the graph* $\Omega(M)$ *if and only if* K *is a universal vertex of the subgraph* $\Omega(L)$*.*

Proof. By Lemma 17 and 19.3 of [17, p. 159]. □

Proposition 19. *Let* M *be an* R*-module. If* $\Delta = \Delta(\Omega) < \infty$ *and* $\delta = \delta(\Omega) \geq 1$*, then* M *is semisimple or the graph* $\Omega(M)$ *has at least a universal vertex.*

Proof. Since by Theorem 13 Part 1, $l_R(M) < \infty$, then M is Artinian and it possesses a simple submodule. Moreover, every non-zero submodule of M contains a simple submodule. Now, consider $Soc(M)$. If $Soc(M) = M$, then M is semisimple, otherwise $Soc(M)$ is a vertex of $\Omega(M)$. Now, for every non-trivial submodule N of M, if $Soc(M) + N = M$, then $deg(Soc(M)) = 0$, a contradiction. Therefore, $Soc(M) + N \neq M$ and so $Soc(M) \ll M$. Hence, by Lemma 17, $Soc(M)$ is a universal vertex of $\Omega(M)$. □

Let Λ be a non-empty set (class) of R-modules. An R-module M is said to be *finitely cogenerated by* Λ, or *finitely* Λ*-cogenerated*, if there is a monomorphism $M \to \Pi_{i \leq k} \lambda_i = \bigoplus_{i \leq k} \lambda_i$ with finitely many $\lambda_i \in \Lambda$ and $k \in \mathbb{N}$.

Proposition 20. *Let* M *be an* R*-module and* $Rad(M) \neq (0)$*. Then the graph* $\Omega(M)$ *has a universal vertex, if one of the following conditions holds:*
(1) The module M *is finitely generated.*
(2) The module M *is finitely cogenerated or uniform and* $Soc(M) \neq (0)$*.*
(3) The module M *is finitely cogenerated such that* $Rad(M/K) = (0)$ *for any non-zero submodule* K *of* M*.*
(4) Every non-trivial submodule of M *is contained in a maximal submodule.*

Proof. (1) Since M is finitely generated, $Rad(M) \ll M$ and since $Rad(M) \neq (0)$, the result yields by Lemma 17.
(2) Suppose that M is a finitely cogenerated R-module, then by 21.3 Part 1 of [17, p. 175], $Soc(M)$ is finitely generated and essential in M. Hence, $Soc(M) \cap Rad(M) \neq (0)$, thus by [8, Corollary 9.9], $Soc(Rad(M)) \neq (0)$. But by 2.8 Part 9 of [12, p. 13], $Soc(Rad(M)) \ll M$. Consequently, by Lemma 17, $Soc(Rad(M))$ is a universal vertex of $\Omega(M)$. Also, if M is a uniform R-module, then $Rad(M) \trianglelefteq M$ and $Soc(M) \trianglelefteq M$. Hence, by above argument the result yields again.
(3) Assume that M is a finitely cogenerated R-module. Then by Corollary 10.5 Part 2 of [8, p. 125], M has a minimal submodule. Moreover, for any non-zero submodule K of M, we have $(Rad(M)+K)/K \subseteq Rad(M/K)$ and as $Rad(M/K) = (0)$, then $(Rad(M) + K)/K = (0)$. Hence, $Rad(M)) \subseteq K$ and $Soc(Rad(M)) \subseteq Soc(K) \subseteq K$. Thus, for any non-zero submodule K of M, $Soc(Rad(M)) \subseteq K$. Therefore, $Soc(Rad(M))$ is a unique minimal submodule of M. However, by Part 2, $Soc(Rad(M))$ is a universal vertex of $\Omega(M)$. Consequently, M has a unique minimal submodule which is a universal vertex of the graph $\Omega(M)$.
(4) In order to establish this part, we consider $Rad(M)$. Clearly, $Rad(M) \neq M$, then it is a vertex of the graph $\Omega(M)$. Since every non-trivial submodule of M is contained in a maximal submodule, by [8, Proposition 9.18], $Rad(M)$ is the unique largest small submodule of M. Hence, by Lemma 17, $Rad(M)$ is a universal vertex of $\Omega(M)$. □

We recall that the *dominating set* (DS) of the graph Ω, is a subset D of vertex set $V(\Omega)$ such that every vertex in $\Omega \backslash D$ is adjacent to at least one vertex in D. A DS is called *minimal dominating set*, denoted by mDS, if for any subset S of DS with $S \neq DS$, S is not a DS. The *domination number* of Ω, written $|DS(\Omega)|$, is the smallest of the cardinalities of the minimal dominating sets of Ω. In this paper, a subset S of the vertex set of the graph $\Omega(M)$ is a dominating set (DS) if and only if for any non-trivial submodule X of M there is a Y in S such that $X + Y \neq M$.

Lemma 21. *Let M be an R-module and $|\Omega(M)| \geq 2$, then the following statements hold:*
(1) If S is a subset of vertex set of the graph $\Omega(M)$ which contains at least a universal vertex , then S is a DS in $\Omega(M)$.
(2) If $\Omega(M)$ has at least a universal vertex, then for each universal vertex U of $\Omega(M)$, the set $\{U\}$ is a mDS and $|DS(\Omega(M)| = 1$.

Proof. Obvious. □

Example 7. *Consider $\mathbb{Q}$ as $\mathbb{Z}$-module and $\mathbb{Z}_{pq^2}$ as $\mathbb{Z}_{pq^2}$- module such that p and q are two distinct prime numbers. As $\mathbb{Z} \ll \mathbb{Q}$ and $< pq > \ll \mathbb{Z}_{pq^2}$, then by*

Lemma 21, we have:
(1) The set $\{< pq >\}$ *is a mDS and* $|DS\Omega(\mathbb{Z}_{pq^2})| = 1$.
(2) The set $\{\mathbb{Z}\}$ *is a mDS and* $|DS\Omega(\mathbb{Q})| = 1$.

Corollary 22. *Let* M *be an* R*-module and* $Rad(M) \neq (0)$*. Then the following conditions hold:*
(1) If M *is finitely generated, then the set* $\{Rad(M)\}$ *is a mDS.*
(2) If M *is finitely cogenerated or uniform and* $Soc(M) \neq (0)$*, then the set* $\{Soc(Rad(M))\}$ *is a mDS.*
(3) If M *is finitely cogenerated such that* $Rad(M/K) = (0)$ *for any non-zero submodule* K *of* M*, then the mDS of the graph* $\Omega(M)$ *contains a unique minimal submodule of* M*.*
(4) If every non-trivial submodule of M *is contained in a maximal submodule, then the graph* $\Omega(M)$ *has a mDS.*

Proof. By Proposition 20.

□

Theorem 23. *Let* M *be an* R*-module with non-trivial submodule* $Rad(M)$ *and* $|\Omega(M)| \geq 2$*. If* M *is hollow, then the following statements hold:*
(1) Every subset of the vertex set of the graph $\Omega(M)$ *is a DS in* $\Omega(M)$*.*
(2) $|DS(\Omega(M)| = 1$*.*
(3) If $\Omega(M)$ *is a finite graph, then the number of the DS is equal to* $2^{|\Omega(M)|} - 2$*.*
(4) If $\Omega(M)$ *is an infinite graph, then the number of the DS is infinite.*

Proof. (1) Suppose that M is a hollow R-module. Then by [15, Proposition 2.11], $\Omega(M)$ is a complete graph. Hence, every subset of the vertex set of the graph $\Omega(M)$ is a DS in $\Omega(M)$.
(2) Since M is a hollow R-module, then every non-trivial submodules of M is small. Hence, by Lemmas 17 and 21, $|DS(\Omega(M)| = 1$.
(3) Let $|\Omega(M)| = n$, where $2 \leq n < +\infty$. As $\Omega(M)$ is a complete graph with n vertices, then the number of non-empty proper subsets of the vertex set $V(\Omega(M))$, which are DS, is equal to $\sum_{r=1}^{n-1} C(n,r) = 2^n - 2$, where $C(n,r)$ is an r-combination of $V(\Omega(M))$ with n elements, for a non-negative integer $r \leq n$.
(4) It is obvious by Part 3. □

Example 8. *For every prime number* p *and* $n \in \mathbb{N}$ *with* $n \geq 2$*, we have:*
(1) $|DS(\Omega(\mathbb{Z}_{p^\infty}))| = |DS(\Omega(\mathbb{Z}_{p^n})| = 1$*.*
(2) The number of the DS of the graph $\Omega(\mathbb{Z}_{p^n})$ *is* $2^{n-1} - 2$*.*
(3) The number of the DS of the graph $\Omega(\mathbb{Z}_{p^\infty})$ *is infinite.*

Corollary 24. *Let* M *be an* R*-module. Then every subset of the vertex set of the graph* $\Omega(M)$ *is a DS in* $\Omega(M)$ *and* $|DS(\Omega(M))| = 1$*, if one of the following statements holds:*

(1) The module M is Noetherian and contains a unique maximal submodule.
(2) The module M is an indecomposable R-module such that every pair of non-trivial submodules of M have zero intersection.
(3) The module M is local.
(4) The module M is self-projective and $End_R(M)$ is a local ring.

Proof. The Part 1 follows from [15, Theorem 2.9] and Lemmas 17 and 21. Also, the Parts 2, 3 and 4 are immediate consequences of [15, Corollary 2.16] and Theorem 23. □

References

[1] A. Abbasi, H. Roshan-Shekalgourabi and D. Hassanzadeh-Lelekaami: *Associated Graphs of Modules Over Commutative Rings*, Iranian Journal of Mathematical Sciences and Informatics, 10(1) (2015), 45-58.

[2] S. Akbari, M. Habibi, A. Majidinya and R. Manaviyat: *A note on comaximal graph of non-commutative rings*, Algebr. Represent. Theory, 16(2) (2013), 303-307.

[3] S. Akbari, M. Habibi, A. Majidinya and R. Manaviyat: *On the inclusion ideal graph of a ring*, Comm. Algebra, 43(7) (2015) 1-9.

[4] S. Akbari, R. Nikandish and M.J. Nikmehr: *Some results on the intersection graphs of ideals of rings*, J. Algebra Appl. 12(4) (2013), (13 pages).

[5] S. Akbari, A. Tavallaee and S. Khalashi Ghezelahmad: *On the complement of the intersection graph of submodules of a module module*, J. Algebra Appl. 14 (2015), 1550116 (11 pages).

[6] S. Akbari, A. Tavallaee and S. Khalashi Ghezelahmad: *Intersection graph of submodule of a module*, J. Algebra Appl. 11(1) (2012), 1250019 (8 pages).

[7] A. Amini, B. Amini and M.H. Shirdareh Haghighi: *On a graph of ideals*, Acta Math. Hungar. 134(3) (2012) 369-384.

[8] F.W. Anderson and K.R. Fuller: Rings and Categories of Modules, Springer-Verlag, New York, 1992.

[9] J. Bosak: *The graphs of semigroups*, in Theory of Graphs and Application, Academic Press, New York, 1964, pp. 119-125.

[10] I. Chakrabarty, S. Gosh, T.K. Mukherjee and M.K. Sen: *Intersection graphs of ideals of rings*, Discrete Math. 309 (2009), 5381-5392.

[11] G. Chartrand and O.R. Oellermann: Applied and Algorithmic Graph Theory, McGraw-Hill, Inc., New York, 1993.

[12] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer: Lifting Modules Supplements and Projectivity in Module Theory, Frontiers in Mathematics, Birkauser Verlag, 2006.

[13] B. Csakany and G. Pollak: *The graph of subgroups of a finite group*, Czech Math. J. 19 (1969) 241-247.

[14] S.H. Jafari and N. Jafari Rad: *Domination in the intersection graphs of ring and modules*, Ital. J. pure Appl. Math. 28 (2011), 17-20.

[15] L.A. Mahdavi and Y. Talebi: *Co-intersection graph of submodules of a module*, J. Algebra Discrete Math. 21(1) (2016), 128-143.

[16] A.A. Talebi: *A kind of intersection graphs on ideals of rings*, Journal of Mathematics and statistics, 8(1) (2012), 82-84.
[17] R. Wisbauer: Foundations of Module and Ring Theory, Gordon and Breach, Reading, 1991.
[18] E. Yaraneri: *Intersection graph of a module*, J. Algebra Appl. 12(5) (2013), (30 pages).
[19] B. Zelinka: *Intersection graphs of finite abelian groups*, Czech Math. J. 25(2) (1975), 171-174.