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EXACT SOLUTIONS OF TIME FRACTIONAL FREE CONVECTION FLOWS OF VISCOUS FLUID OVER AN ISOTHERMAL VERTICAL PLATE WITH CAPUTO AND CAPUTO-FABRIZIO DERIVATIVES

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Abstract. The unsteady time fractional free convection flow of an incompressible Newtonian fluid over an infinite vertical plate due to an impulsive motion of the plate and constant temperature at the boundary is analyzed. The old (Caputo) and new (Caputo-Fabrizio) fractional derivative approaches have been used to develop a physical model and a comparison has been drawn between their solutions. Boundary layers equations in non-dimensional form are solved analytically by the Laplace transform technique. Exact solutions for velocity and temperature are obtained in terms of Wrights function. The expressions for rate of heat transfer in both cases are also determined. Solutions for integer order derivatives are obtained as limiting case. Numerical computations were made through software Mathcad and observed some physical aspects of fractional and material parameters are presented. It is found that the rate of heat transfer of Caputo-Fabrizio model have higher values than Caputo one as we increased the value of fractional parameter and fractional fluids tend to superpose to that of ordinary fluid.

Key words : Viscous fluid, Free convection, Vertical plate, Caputo and Caputo-Fabrizio fractional derivative, Exact solutions.
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1. Nomenclature

$Cp-$ specific heat at constant pressure
$g-$ gravitational acceleration
$Gr-$ Grashof number

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$k-$ thermal conductivity
$Pr-$ Prandtl number
$Re-$ Reynolds number
$s-$ Laplace transform parameter
$T-$ Fluid temperature
$Tw-$ Wall temperature
$T_\infty-$ Temperature far away from the plate
$u-$Velocity component along x direction
$\mu-$Dynamic viscosity
$\nu-$ Kinematic viscosity
$\theta-$ Non-dimensional temperature
α fractional parameter
$H(t)-$Heaviside unit step function

2. Introduction

Newtonian liquids depicted by Navier-Stokes conditions have been broadly examined in the writing in the course of recent decades. To a great extent, this is expected to the truth that they are moderately straightforward and their answers are advantageous. Normal convection streams past a vertical plate are indispensable in settling some mechanical also, building issues, for example, the filtration and plan of procedures, the drying of permeable materials in material ventures and sunlight based vitality gatherer. Various examinations have been accounted for in writing to comprehend the issues utilizing explanatory and numerical techniques under various limit conditions [1-21]. Recently, the fractional calculus has encountered much success in the disruption of complex dynamics. In particular, it has been proved to be a valuable tool for handling viscoelastic properties of materials. In the case of diffusion phenomena, for instance, $\alpha = 1$ corresponds to classical diffusion while for $0 < \alpha < 1$ or $\alpha > 1$ the transport phenomenon exhibits sub-diffusion, respectively, super-diffusion. Some interesting results regarding the flows of Newtonian fluids with fractional derivatives can be found in [22-24]. In the most recent, Vieru et al. [25], Nazish [26], Shakeel et al. [27], Ali et al. [28], Imran et al. [29] studied the viscous fluid with Caputo fractional derivatives under different thermal and geometric conditions. The solutions obtained with this operator are written in complex form and are expressed in terms of special functions. Moreover, the kernel of this operator is singular and difficult to handle when we used Laplace transform. In (2015) a modern definition introduced by Caputo and Fabrizio [30] with non-singular kernel and easy to use in Laplace transform. After that several researchers have shared their contribution [31-35]. Motivated by above studies, we solve the unsteady natural convection flow of viscous fluid due to an impulsive motion of the plate with constant wall temperature. The old (Caputo) and new (Caputo-Fabrizio) fractional derivative approaches have been used to develop a physical model and a comparison has been drawn between their solutions. Boundary layers equations in non-dimensional form are solved analytically by the Laplace transform technique. Exact solutions for velocity and temperature are obtained in terms of Wrights function. The expressions for rate of heat transfer in both cases are also determined. Solutions for integer order derivatives are obtained as limiting case.

3. Mathematical model of the problem

Let us consider the effect of heat transfer on unsteady boundary layer flow of an incompressible fluid with fractional derivatives past an infinite vertical flat plate situated at the plan $y = 0$. Let us suppose that, initially at $t = 0$, both the plate and fluid are stationary with constant temperature T_∞. At the beginning, the plate starts to move with constant velocity in its own plane and temperature of the plate raised to T_w. As the plate is infinite, all physical quantities are functions of y and t only. With these conditions, along with the assumption that the viscous dissipation term in the energy equation is neglected and under the usual Boussinesqs approximation on the temperature gradient, the unsteady boundary layer equations are:

$$\frac{\partial u(y,t)}{\partial t} = \nu \frac{\partial^2 u(y,t)}{\partial y^2} + g\beta(T(y,t) - T_\infty), \tag{1}$$

$$\rho C_p \frac{\partial T(y,t)}{\partial t} = k \frac{\partial^2 T(y,t)}{\partial y^2}, \tag{2}$$

the appropriate initial and boundary conditions for present work are:

$$u(y,0) = 0, \quad T(y,0) = T_\infty, \quad y \geq 0, \tag{3}$$

$$u(0,t) = U_o H(t), \;\; T(0,t) = T_w, \quad t > 0, \tag{4}$$

$$u(y,t) \to 0, \quad T(y,t) \to T_\infty, \;\; as\ y \to \infty, \tag{5}$$

$$u^{'} = \frac{ku}{\nu h}, \quad t^{'} = t\frac{gk}{nuh}, \quad y^{'} = \frac{h}{k}y, \;\; Pr_{eff} = \frac{Pr}{Re}, Pr = \frac{\mu C_p}{k},$$
$$Gr = \frac{g\beta\nu(T_w - T_\infty)}{U_o^3}, \;\; Re = \frac{\nu^2}{g}\left(\frac{h}{k}\right)^3, \;\; \theta = \frac{T - T_\infty}{T_w - T_\infty}, \tag{6}$$

into Eqs. (1)-(5) and, dropping prime notations, the set of non-dimensional fractional partial differential equations is

$$\frac{\partial u(y,t)}{\partial t} = Re \frac{\partial^2 u(y,t)}{\partial y^2} + Gr\theta(y,t), \;\; y, \; t > 0, \tag{7}$$

$$\frac{\partial\theta(y,t)}{\partial t} = \frac{1}{Pr_{eff}}\frac{\partial^2\theta(y,t)}{\partial y^2}, \quad y,\ t>0. \tag{8}$$

and then replacing the time derivative of integer order with non-integer order α, we have

$$D_t^\alpha u(y,t) = Re\frac{\partial^2 u(y,t)}{\partial y^2} + Gr\theta(y,t), \quad y,\ t>0, \tag{9}$$

$$D_t^\alpha \theta(y,t) = \frac{1}{Pr_{eff}}\frac{\partial^2\theta(y,t)}{\partial y^2}, \quad y,\ t>0. \tag{10}$$

The corresponding initial and boundary conditions are:

$$u(y,0)=0, \quad \theta(y,0)=0, \qquad y\geq 0, \tag{11}$$

$$u(0,t)=H(t), \quad \theta(0,t)=1, \quad t>0, \tag{12}$$

$$u(y,t)\to 0, \qquad \theta(y,t)\to 0, \ \ as\ y\to\infty. \tag{13}$$

4. Some basic definitions of fractional derivatives

Definition-1 The Caputo time-fractional derivative of order $\alpha\epsilon[0,1)$ is defined by

$${}^{C}D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} f'(\tau)d\tau. \tag{14}$$

The Laplace transform of Caputo time-derivative (14) is

$$L\left\{{}^{C}D_t^\alpha f(t)\right\} = s^\alpha L\left\{f(t)\right\} - s^{\alpha-1}f(0). \tag{15}$$

Definition-2 The Caputo-Fabrizio time fractional derivative of order $\alpha\in[0,1)$ is defined as,

$${}^{CF}D_t^\alpha f(t) = \frac{1}{1-\alpha}\int_0^t exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) f'(\tau)d\tau. \tag{16}$$

The Laplace transform of Caputo-Fabrizio time derivative is

$$L\left\{{}^{CF}D_t^\alpha f(t)\right\} = \frac{sL\left\{f(t)\right\} - f(0)}{(1-\alpha)s+\alpha}. \tag{17}$$

Remark It is important to point out that Caputo time-fractional derivatives and Caputo-Fabrizio time-fractional derivatives can be extended for $\alpha\to 1$. Indeed, using equations (14) and (16) and using the limit for $\alpha\to 1$, we obtain

$$\lim_{\alpha\to 1} L\left\{{}^{C}D_t^\alpha f(t)\right\} = \lim_{\alpha\to 1} {}^{CF}D_t^\alpha f(t) = sL\left\{f(t)\right\} - f(0) = L\left\{f'(t)\right\}. \tag{18}$$

As a consequence we have,

$$f'(t) = \lim_{\alpha\to 1} {}^{C}D_t^\alpha f(t) = \lim_{\alpha\to 1} {}^{CF}D_t^\alpha f(t). \tag{19}$$

5. Calculation of temperature field with Caputo derivative

The partial differential equation (10) is not coupled to the momentum Eq. (9). Consequently, we shall firstly determine the temperature field by means of Laplace transform and then, the velocity field. Applying Laplace transform [36-40] to Eq. $(11)_2$, subject to Eq. $(12)_2$ we obtain,

$$Pr_{eff}s^{\alpha}\overline{\theta}(y,s)=\frac{\partial^2\overline{\theta}(y,s)}{\partial y^2}, \tag{20}$$

where s is the transform parameter and $\overline{\theta}(y,s)$ is the Laplace transform of the function $\theta(y,t)$ which has to satisfy the conditions:

$$\overline{\theta}(y,s)=\frac{1}{s}, \qquad \overline{\theta}(y,s)\to 0, \quad as \quad y\to\infty. \tag{21}$$

The solution of Eq. (20) subject to the condition (21) is

$$\overline{\theta}(y,s)=\frac{1}{s}exp\left(-y\sqrt{Pr_{eff}s^{\alpha}}\right). \tag{22}$$

Now, applying the inverse Laplace transform to Eq. (22), using the formula $L^{-1}\left\{\frac{e^{-as^b}}{s^c}\right\}=t^{c-1}\Phi(c,-b;-at^{-b}),\ 0<b<1$, we find

$$\begin{cases}\theta(y,t)=\Phi\left(1,-\frac{\alpha}{2};-y\sqrt{Pr_{eff}}t^{-\frac{\alpha}{2}}\right), & 0<\alpha<1\\ \theta(y,t)=erfc\left(\frac{y\sqrt{Pr_{eff}}}{2\sqrt{t}}\right), & \alpha=1\end{cases} \tag{23}$$

where $\Phi(x,y;z)=\Sigma_{n=0}^{\infty}\frac{z^n}{\Gamma(n+1)\Gamma(x-ny)}$, is the Wright's function.
To measure the rate of heat transfer from plate to the fluid in terms of Nusselt number, introduce the Eq. (22) into the following relation for $\alpha\varepsilon(0,1)$

$$Nu=-\lim_{y\to 0}L^{-1}\left\{\frac{\partial\overline{T}(y,s)}{\partial y}\right\}=-L^{-1}\left\{\lim_{y\to 0}\frac{\partial\overline{T}(y,s)}{\partial y}\right\}=\sqrt{Pr_{eff}}F_{1,\alpha/2}(0,t) \tag{24}$$

respectively for $\alpha=1$

$$Nu=\sqrt{\frac{Pr_{eff}}{\pi}}. \tag{25}$$

5.1. **Calculation of temperature field with Caputo-Fabrizio derivative.** Applying the Laplace transform to Eq. (10) by using the definition Eq. (17) and initial condition $(11)_2$, we have

$$\frac{s\overline{\theta}(y,s)}{(1-\alpha)s+\alpha}=\frac{1}{Pr_{eff}}\frac{d^2\overline{\theta}(y,s)}{dy^2}. \tag{26}$$

Solution of Eq. (26) subject to conditions (21)

$$\overline{\theta}(y,s)=\frac{1}{s}exp\left(-y\sqrt{\frac{Pr_{eff}a_1s}{s+\alpha a_1}}\right), \tag{27}$$

where $a_1=\frac{1}{1-\alpha}$ with inverse Laplace transform

$$\theta(y,t)=1-\frac{2a_1Pr_{eff}}{\pi}\int_0^\infty\frac{sin(yu)}{u(a_1Pr_{eff}+u^2)}exp\left(-\frac{\alpha a_1tu^2}{a_1Pr_{eff}+u^2}\right)du, \tag{28}$$

and the Nusselt number

$$Nu=\frac{a_1Pr_{eff}e^{-\alpha a_1t}}{\sqrt{-\alpha a_1}}erf\left(-\alpha a_1t\right).$$

Temperature field for ordinary case is obtained for $\alpha\to 1$ and is given as

$$\theta(y,t)=1-\frac{2}{\pi}\int_0^\infty\frac{sin(yu)}{u}exp\left(-\frac{tu^2}{Pr_{eff}}\right)du. \tag{29}$$

Further, using the formula

$$\int_0^\infty\frac{sin(\beta u)}{u}exp\left(-\gamma u^2\right)du=\frac{\pi}{2}erf\left(\frac{\beta}{2\sqrt{\gamma}}\right), \tag{30}$$

so, Eq. (29) reduced to

$$\theta(y,t)=1-erf\left(\frac{y\sqrt{Pr_{eff}}}{2\sqrt{t}}\right)=erfc\left(\frac{y\sqrt{Pr_{eff}}}{2\sqrt{t}}\right), \tag{31}$$

respectively the Nusselt number for $\alpha\to 1$

$$Nu=\sqrt{\frac{Pr_{eff}}{\pi}}.$$

6. Calculation of velocity field with Caputo derivative

By applying the Laplace transform to Eq. (9) and using the initial condition $(11)_1$, we have

$$s^\alpha\overline{u}(y,s)=Re\frac{\partial^2\overline{u}(y,s)}{\partial y^2}+Gr\overline{\theta}(y,s), \tag{32}$$

where $\overline{u}(y,s)$ the Laplace transform of the function $u(y,t)$, which has to satisfy conditions:

$$\overline{u}(0,s)=\frac{1}{s},\quad \overline{u}(y,s)\to 0,\ as\quad y\to\infty. \tag{33}$$

The solution of the differential equation (32), subject to the conditions (33) is

$$\overline{u}(y,s)=\frac{1}{s}.exp\left(-y\sqrt{\frac{s^{\alpha}}{Re}}\right)+\frac{a}{s^{\alpha+1}}\left[exp\left(-y\sqrt{\frac{s^{\alpha}}{Re}}\right)-exp\left(-y\sqrt{Pr_{eff}s^{\alpha}}\right)\right],(34)$$

where $a=\frac{Gr}{RePr_{eff}-1}$.
By inverting Eq. (34) we find that

$$u(y,t)=\Phi\left(1,-\frac{\alpha}{2};-y\sqrt{\frac{1}{Re}}t^{-\frac{\alpha}{2}}\right)+at^{\alpha}\Phi\left(\alpha+1,-\frac{\alpha}{2};-y\sqrt{\frac{1}{Re}}t^{-\frac{\alpha}{2}}\right)-$$
$$at^{\alpha}\Phi\left(\alpha+1,-\frac{\alpha}{2};-y\sqrt{Pr_{eff}}t^{-\frac{\alpha}{2}}\right),\quad \alpha\epsilon(0,1). \tag{35}$$

6.1. **Velocity for ordinary case when $\alpha\to 1$.**

$$u(y,t)=\Phi\left(1,-\frac{1}{2};-y\sqrt{\frac{1}{Re}}t^{-\frac{1}{2}}\right)+a\Phi\left(2,-\frac{1}{2};-y\sqrt{\frac{1}{Re}}t^{-\frac{1}{2}}\right)-$$
$$a\Phi\left(2,-\frac{1}{2};-y\sqrt{Pr_{eff}}t^{-\frac{1}{2}}\right). \tag{36}$$

6.2. **Calculations of velocity field with Caputo-Fabrizio derivative.** Applying Laplace transform to Eq. (9) keeping in mind the initial condition using definition Eq. (17) and expression $\overline{\theta}(y,s)$ from Eq. (27)

$$\frac{s\overline{u}(y,s)}{(1-\alpha)s+\alpha}=Re\frac{\partial^2\overline{u}(y,s)}{\partial y^2}+\frac{Gr}{s}exp\left(-y\sqrt{\frac{Pr_{eff}a_1s}{s+\alpha a_1}}\right), \tag{37}$$

Solution of ordinary differential equation (37) subject to conditions (33) in suitable form

$$u(y,s)=\frac{1}{s}\exp\Bigg(-y\sqrt{\frac{a_1s}{\mathrm{Re}(s+\alpha a_1)}}\Bigg)+$$
$$+\frac{\mathrm{Gr}}{(1-\mathrm{RePr_{eff}})a_1}\frac{1}{s}\exp\Bigg(-y\sqrt{\frac{\mathrm{Pr_{eff}}a_1s}{(s+\alpha a_1)}}\Bigg)+$$
$$+\frac{\mathrm{Gr}\alpha}{(1-\mathrm{RePr_{eff}})}\frac{1}{s^2}\exp\Bigg(-y\sqrt{\frac{\mathrm{Pr_{eff}}a_1s}{(s+\alpha a_1)}}\Bigg)-$$
$$-\frac{\mathrm{Gr}}{(1-\mathrm{RePr}_{eff})a_1}\frac{1}{s}\exp\Bigg(-y\sqrt{\frac{a_1s}{\mathrm{Re}(s+\alpha a_1)}}\Bigg)-$$
$$-\frac{\mathrm{Gr}\alpha}{(1-\mathrm{RePr_{eff}})}\frac{1}{s^2}\exp\Bigg(-y\sqrt{\frac{a_1s}{\mathrm{Re}(s+\alpha a_1)}}\Bigg). \tag{38}$$

Now consider

$$\Phi_1\Big(y,s,\mathrm{Pr_{eff}}a_1,\alpha a_1\Big)=\frac{1}{s}\exp\bigg(-y\sqrt{\frac{\mathrm{Pr_{eff}}a_1 s}{s+\alpha a_1}}\bigg) \tag{39}$$

$$G_1\Big(y,s,\mathrm{Pr_{eff}}a_1,\alpha a_1\Big)=\frac{1}{s^2}\exp\bigg(-y\sqrt{\frac{\mathrm{Pr_{eff}}a_1 s}{s+\alpha a_1}}\bigg)=\frac{1}{s}\Phi_1\Big(y,t,\mathrm{Pr_{eff}}a_1,\alpha a_1\Big) \tag{40}$$

$$\Phi_2\Big(y,s,\frac{a_1}{\mathrm{Re}},\alpha a_1\Big)=\frac{1}{s}\exp\bigg(-y\sqrt{\frac{a_1 s}{\mathrm{Re}(s+\alpha a_1)}}\bigg) \tag{41}$$

$$G_2\Big(y,s,\frac{a_1}{\mathrm{Re}},\alpha a_1\Big)=\frac{1}{s^2}\exp\bigg(-y\sqrt{\frac{a_1 s}{\mathrm{Re}(s+\alpha a_1)}}\bigg)=\frac{1}{s}\Phi_2\Big(y,t,\frac{a_1}{\mathrm{Re}},\alpha a_1\Big) \tag{42}$$

Applying the inverse Laplace transform to the equations (39),(40),(41) and (42)

$$\Phi_1\Big(y,t,\mathrm{Pr_{eff}}a_1,\alpha a_1\Big)=1-\frac{2\mathrm{Pr_{eff}}a_1}{\pi}\int_0^\infty\frac{\sin(yx)}{(\mathrm{Pr_{eff}}a_1+x^2)}\exp\Big(-\frac{\alpha a_1 t x^2}{\mathrm{Pr_{eff}}a_1+x^2}\Big)dx \tag{43}$$

$$g_1\Big(y,t,\mathrm{Pr_{eff}}a_1,\alpha a_1\Big)=\int_0^t\Phi_1\Big(y,\tau,\mathrm{Pr_{eff}}a_1,\alpha a_1\Big)d\tau \tag{44}$$

$$\Phi_2\Big(y,t,\frac{a_1}{\mathrm{Re}},\alpha a_1\Big)=1-\frac{2a_1}{\mathrm{Re}\pi}\int_0^\infty\frac{\sin(yx)}{(\frac{a_1}{\mathrm{Re}}+x^2)}\exp\Big(-\frac{\alpha a_1 t x^2}{\frac{a_1}{\mathrm{Re}}+x^2}\Big)dx \tag{45}$$

$$g_2\Big(y,t,\frac{a_1}{\mathrm{Re}},\alpha a_1\Big)=\int_0^t\Phi_2\Big(y,\tau,\frac{a_1}{\mathrm{Re}},\alpha a_1\Big)d\tau \tag{46}$$

Applying inverse Laplace transform to equation (38) and using Eqs. (43-46) we get

$$\begin{aligned}u(y,t)=g_2\Big(y,t,\frac{a_1}{\mathrm{Re}},\alpha a_1\Big)&+\frac{\mathrm{Gr}}{(1-\mathrm{RePr_{eff}})a_1}\phi_1\Big(y,t,\mathrm{Pr_{eff}}a_1,\alpha a_1\Big)+\\&+\frac{\mathrm{Gr}\alpha}{(1-\mathrm{RePr_{eff}})}g_1\Big(y,t,\mathrm{Pr_{eff}}a_1,\alpha a_1\Big)-\\&-\frac{\mathrm{Gr}}{(1-\mathrm{RePr_{eff}})a_1}\phi_2\Big(y,t,\frac{a_1}{\mathrm{Re}},\alpha a_1\Big)-\\&-\frac{\mathrm{Gr}\alpha}{(1-\mathrm{RePr_{eff}})}g_2\Big(y,t,\frac{a_1}{\mathrm{Re}},\alpha a_1\Big).\end{aligned} \tag{47}$$

6.3. **Velocity for ordinary case when $\alpha \to 1$.**

$$u(y,t) = \Phi\left(1, -\frac{1}{2}; -y\sqrt{\frac{1}{Re}}t^{-\frac{1}{2}}\right) + a\Phi\left(2, -\frac{1}{2}; -y\sqrt{\frac{1}{Re}}t^{-\frac{1}{2}}\right) - a\Phi\left(2, -\frac{1}{2}; -y\sqrt{Pr_{eff}}t^{-\frac{1}{2}}\right). \quad (48)$$

7. Numerical discussion and results

In order to obtain some information on the fluid motion, we have made several numerical simulations using Mathcad software. The obtained results are presented in the graphs from Figs. 1-7. We are interested to analyze the influence of the fractional parameter α on the Nusselt number and on fluid velocity and to compare the flows of the ordinary fluid with flows of the fluids modeled by time-fractional Caputo and Capto-Fabrizio derivatives. Also, the influence of the time, Grashof number, Prandtl effective number and Reynolds number on the fluid velocity has been analyzed by numerical calculations and graphical illustrations. Diagrams from Figs. 1 and 2 are plotted in order to discuss the influence of the fractional parameter α and Pr_{eff} on the Nusselt number respectively. From tabular values we have observed that for fixed values of $Pr_{eff} = 0.7$ (for air) and small time, as we increased the value of fractional parameter α, the Nusselt numbers for Caput and Caputo-Fabrizio reduces. But in comparison the Nusselt number for Caputo-Fabrizio has greater values than Caputo one and an opposite behavior has been observed for large values of time. Therefore, we can enhance the rate of heat transfer of fractional models by adjusting the values of fractional parameter α. From the tabular values in Fig. 2, we pictured the influence of Pr_{eff} for fixed values of α. It is clearly seen that by increasing the value of Pr_{eff} for both fractional models, the rate of heat transfer increases. In the comparison sense the rate of Caputo-Fabrizio has great values than Caputo for small time and inverse for large values of time. To see time influence on velocity field, we have plotted Fig. 3 and observed that velocity is an increasing function of time for both fractional models. It is also observed that for different values of fractional parameter fluid velocity near the plate is maximum and then decreasing away from the plate in the free stream region. The fluid velocity of Caputo-Fabrizio slightly superposed to that of Caputo. Fig. 4 is depicted to see the impact of Pr_{eff} on fluid velocity, as expected the fluid velocity is a decreasing function of Pr_{eff}. Physically, it is due to the reason that fluids with larger Prandtl number have higher viscosity and small thermal conductivity, which makes the fluid thicker and hence causes a decrease in fluid velocity. Therefore, increasing values of the parameter Pr_{eff} leads to a slower fluid flow. Fig. 5 is presented the influence of Reynolds number and it can be seen that fluid velocity near the plate is maximum and decreases in its free stream region, as we increased the values of Reynolds number fluid velocity decreases. Grashof number impact can be seen in Fig. 6, and observed that by increasing the values of Gr fluid velocity also increases. The Grashof number approximates the ratio of the buoyancy to viscous force action on the fluid which causes the natural convection. It is due to the fact that the thermal buoyancy effects which gives rise in fluid flow. This parameter has an opposite effect than the Pr_{eff}, because increasing values of the parameter Gr leads to a faster fluid flow. To validate our analytical solutions and the results obtained through numerical techniques by Stehfest's [39] and Tzou's [40] formula are in good agreement.

8. Conclusion

Exact study of Newtonian fluid due to impulsive motion of isothermal vertical plate has been carried out. The ordinary governing equations after making dimensionless are converted in fractional partial differential equations with Caputo and Caputo-Fabrizio fractional model and then used the Laplace transform to obtain the exact solutions for temperature and velocity field. Solutions for ordinary case and the rates of heat transfer are also obtained as limiting cases. To study the physically significant of the studied problem, we have plotted some graphs and drawn some important points which are as follows.
1): Rate of heat transfer decreased as we increased the value of fractional parameter.
2): Rate of heat transfer increased by increasing the value of Pr_{eff}.
3): Fluid velocity is a decreasing function of α, Re and Pr_{eff}.
4): Fluid velocity is an increasing function of time t and Grashof number Gr.
5): Fractional fluids have greater velocities than ordinary one and velcoity of fractional fluid with Caputo-Fabrizio has slightly high velocity in comparison with Caputo one.

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α	$t = 0.5$, $\text{Pr}_{eff} = 0.7$		$t = 5$, $\text{Pr}_{eff} = 0.7$	
	${}^{C}\text{N}u$	${}^{CF}\text{N}u$	${}^{C}\text{N}u$	${}^{CF}\text{N}u$
0	0.837	0.837	0.837	0.837
0.1	0.837	0.858	0.746	0.681
0.2	0.831	0.88	0.66	0.551
0.3	0.818	0.901	0.579	0.448
0.4	0.795	0.921	0.502	0.372
0.5	0.76	0.936	0.427	0.32
0.6	0.709	0.941	0.355	0.284
0.7	0.636	0.926	0.284	0.258
0.8	0.534	0.871	0.212	0.239
0.9	0.391	0.76	0.139	0.222
1	0.195	0.195	0.062	0.062

FIGURE 1. Rate of heat transfer for different value of α

Pr_{eff}	$t=0.5,\ \alpha=0.5$		$t=5,\ \alpha=0.5$	
	^{C}Nu	^{CF}Nu	^{C}Nu	^{CF}Nu
0	0	0	0	0
1	0.909	1.119	0.511	0.382
2	1.285	1.582	0.723	0.54
3	1.574	1.938	0.885	0.661
4	1.817	2.237	1.022	0.764
5	2.032	2.501	1.142	0.854
6	2.226	2.74	1.252	0.935
7	2.404	2.96	1.352	1.01
8	2.57	3.164	1.445	1.08
9	2.726	3.356	1.533	1.146
10	2.873	3.538	1.616	1.208
11	3.013	3.71	1.695	1.267
12	3.147	3.875	1.77	1.323
13	3.276	4.033	1.842	1.377
14	3.4	4.186	1.912	1.429

FIGURE 2. Rate of heat transfer for different value of Pr_{eff}

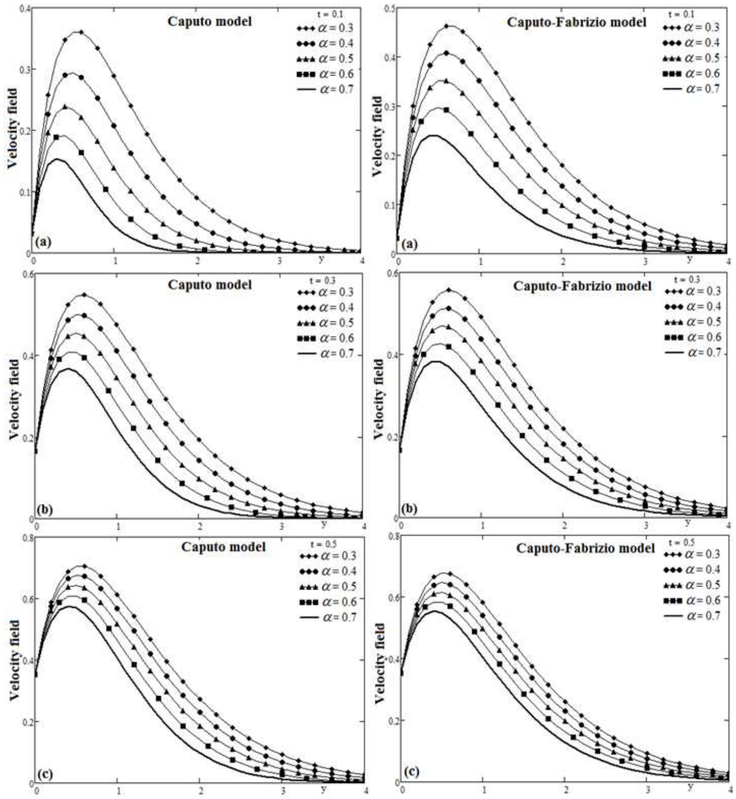

FIGURE 3. Profiles of dimensionless velocities versus y for α at $Pr_{eff} = 2$, $Re = 0.7$, $Gr = 4$ and different vales of time t.

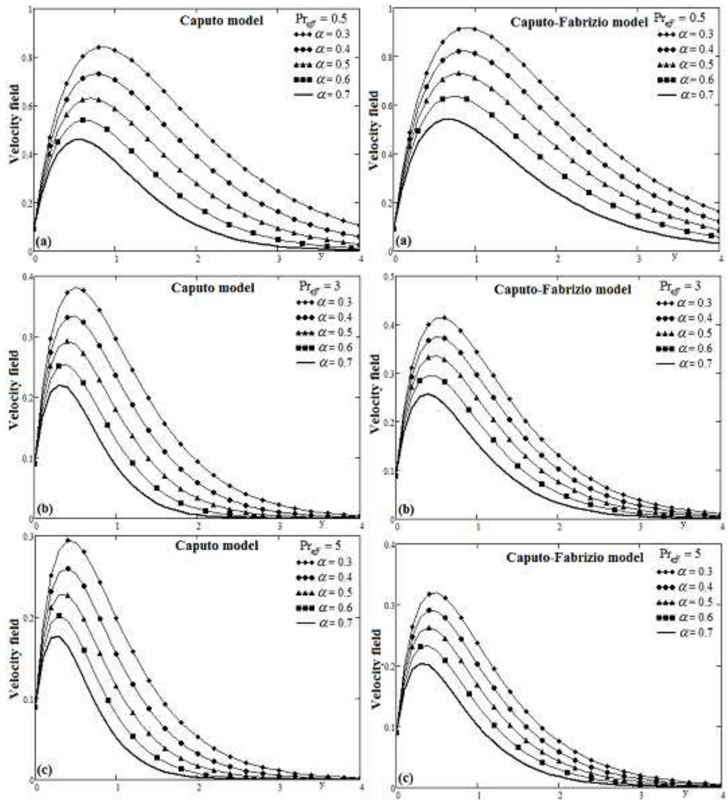

FIGURE 4. Profiles of dimensionless velocities versus y for α at $t = 2$, $Re = 0.7$, $Gr = 4$ and different vales of Pr_{eff}

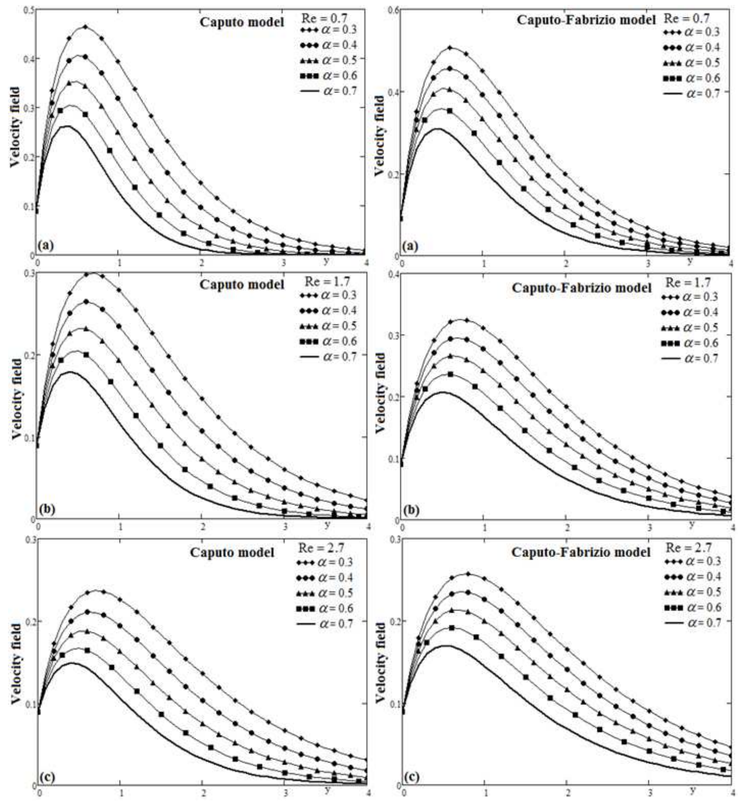

FIGURE 5. Profiles of dimensionless velocities versus y for α at $t = 2$, $Pr_{eff} = 2$, $Gr = 4$ and different vales of Re

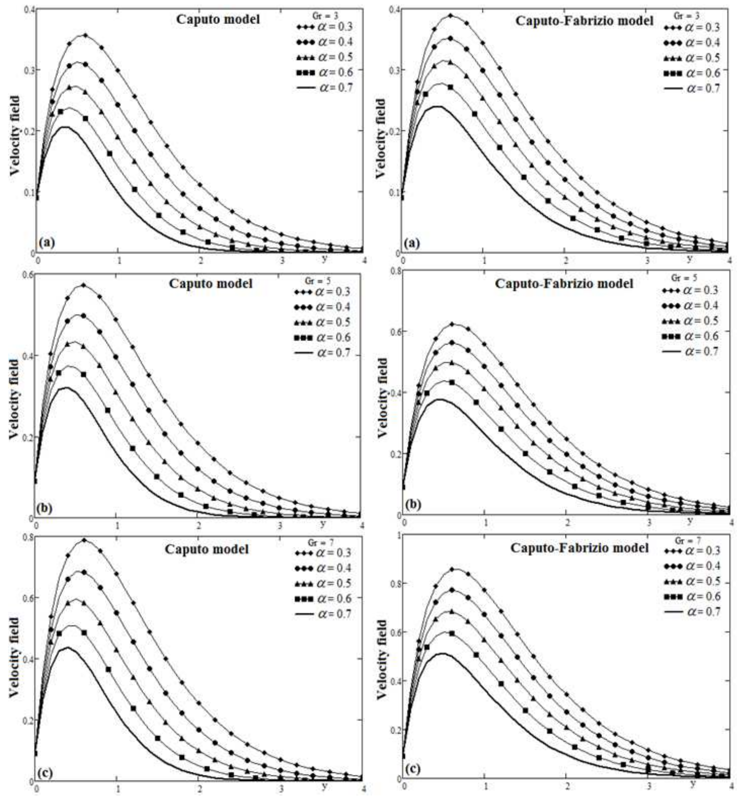

FIGURE 6. Profiles of dimensionless velocities versus y for α at $t = 2$, $Pr_{eff} = 2$, $Re = 0.7$ and different vales of Gr

y	Eq. (35)	[Stehfest's, 39]	[Tzou's, 40]	Eq. (47)	[Stehfest's, 39]	[Tzou's, 40]
0	0.354	0.354	0.354	0.354	0.354	0.354
0.5	0.301	0.301	0.301	0.301	0.301	0.301
1	0.175	0.175	0.175	0.18	0.18	0.18
1.5	0.087	0.087	0.087	0.096	0.096	0.096
2	0.04	0.04	0.04	0.048	0.048	0.048
2.5	0.017	0.017	0.017	0.024	0.024	0.024
3	0.007	$7.18\cdot10^{-3}$	$7.186\cdot10^{-3}$	0.011	0.011	0.011
3.5	0.003	$2.922\cdot10^{-3}$	$2.925\cdot10^{-3}$	$5.471\cdot10^{-3}$	$5.471\cdot10^{-3}$	$5.474\cdot10^{-3}$
4	0.001	$1.161\cdot10^{-3}$	$1.162\cdot10^{-3}$	$2.6\cdot10^{-3}$	$2.6\cdot10^{-3}$	$2.601\cdot10^{-3}$
4.5	$4.512\cdot10^{-4}$	$4.512\cdot10^{-4}$	$4.518\cdot10^{-4}$	$1.23\cdot10^{-3}$	$1.23\cdot10^{-3}$	$1.231\cdot10^{-3}$
5	$1.718\cdot10^{-4}$	$1.718\cdot10^{-4}$	$1.721\cdot10^{-4}$	$5.805\cdot10^{-4}$	$5.805\cdot10^{-4}$	$5.808\cdot10^{-4}$
5.5	$6.411\cdot10^{-5}$	$6.411\cdot10^{-5}$	$6.427\cdot10^{-5}$	$2.732\cdot10^{-4}$	$2.732\cdot10^{-4}$	$2.734\cdot10^{-4}$
6	$2.347\cdot10^{-5}$	$2.347\cdot10^{-5}$	$2.355\cdot10^{-5}$	$1.283\cdot10^{-4}$	$1.283\cdot10^{-4}$	$1.284\cdot10^{-4}$
6.5	$8.435\cdot10^{-6}$	$8.435\cdot10^{-6}$	$8.472\cdot10^{-6}$	$6.013\cdot10^{-5}$	$6.013\cdot10^{-5}$	$6.018\cdot10^{-5}$
7	$2.977\cdot10^{-6}$	$2.977\cdot10^{-6}$	$2.994\cdot10^{-6}$	$2.814\cdot10^{-5}$	$2.814\cdot10^{-5}$	$2.816\cdot10^{-5}$

FIGURE 7. Validation of obtained analytic results and results obtained through numerical inversion techniques