Journal of Prime Research in Mathematics Vol. **14***(2018), 13-17*

GENERALIZED ξ-RINGS

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Abstract. Let R be a ring with center $C(R)$. A ring R is called *a ξ-ring* if, for any element $x \in R$, there exists an element $y \in R$ such that $x - x^2y \in C(R)$. In Proc. Japan Acad. Sci., Ser. A – Math. (1957), Utumi describes the structure of these rings as a natural generalization of the classical strongly regular rings, that are rings for which $x = x^2y$.

In order to make up a natural connection of ξ-rings with the more general class of von Neumann regular rings, that are rings for which $x = xyx$, we introduce here the so-called *generalized ξ-rings* as those rings in which $x - xyx \in C(R)$. Several characteristic properties of this newly defined class are proved, which extend the corresponding ones established by Utumi in these Proceedings (1957).

Key words: idempotents, nipotents, regular rings, strongly regular rings, ξ-rings
AMS SUBJECT: 13B99; 16E50; 16U99.

1. Introduction and Background

Suppose R a ring having the identity element 1 which is different from the zero element 0. As usual, $Id(R)$ denotes the set of all idempotents in R, $Nil(R)$ denotes the set of all nilpotents in R with a subset $Nil_2(R)$ consisting of all nilpotents of order not exceeding 2, $J(R)$ denotes the Jacobson radical of R, and $C(R)$ denotes the center of R. Some more needed terminology and notations concerning the present subject could be found in [4] and [9], respectively.

As a common generalization of the classical *boolean* rings, it is well known that a ring R is said to be in [11] a *regular* ring (or, in the modern terminology, just *von Neumann regular*) if, for each $x \in R$, there is $y \in R$ such that the equality $x = xyx$ holds. Later on, in [1] were defined the so-called *strongly*

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regular rings in which the equality $x = x^2y$ is valid. Since it is rather obvious that strongly regular rings are free of nilpotents, they are necessarily regular; in fact, it follows routinely that $(x - xyx)^2 = 0$ provided that $x^2y = x$, which immediately forces that $x - xyx = 0$, as required. However, in regular rings there is an abundance of nilpotent elements (compare with their satisfactory generalization realized in [3]).

Generalizing the notion of strongly regular rings, Utumi introduced in [10] the class of ξ*-rings* in the sense that $x - x^2y \in C(R)$. It was shown there the fundamental property that $Nil(R) \subseteq C(R)$. This, in turn, implies that $Id(R) \subseteq C(R)$ (see, for instance, [3, Theorem 1] and [5, Lemma 2]). Moreover, in [10, Corollary] was proved also that the symmetrization $x - yx^2 \in C(R)$ is valid, as even something more, Martindale showed in [7] that the commutativity condition $xy = yx$ is always true. For some other things in this aspect, the reader should be consulted with [6] and [8], respectively.

We now intend to enlarge both regular and ξ-rings in terms of central elements. So, we come to the following key concept.

Definition 1. *We shall say that a ring R is a generalized ξ-ring if, for every $x \in R$, there exists $y \in R$ possessing the property that $x - xyx \in C(R)$.*

Concrete examples of such rings include all regular rings and some their variations; e.g., division rings and matrix rings over them. For more non-trivial examples, the interested reader can see [2] as well.

Analyzing that in the stated above definition of a regular ring we have $e = xy \in xR \cap Id(R)$ with $(1 - e)x = 0$ as well as that $f = yx \in Rx \cap Id(R)$ with $x(1 - f) = 0$, we are motivated to consider the following more general version, associated with the aforementioned ξ-rings. Thereby, we also arrive at our next basic concept.

Definition 2. *We shall say that a ring R is a left-idempotently ξ-ring if, for every $x \in R$, there exists $e \in Id(R) \cap xR$ such that $(1 - e)x \in C(R)$, and that it is a right-idempotently ξ-ring if there exists $f \in Id(R) \cap Rx$ such that $x(1 - f) \in C(R)$.*

Clearly, these two types of rings form a proper subclass of the class of generalized ξ-rings. Concrete examples of such rings also include regular rings and some their modifications. For more account information on that, we refer also to [2].

The objective of this short paper is to promote a close relationship between the considered above three ring classes. They could be viewed as a rather natural generalization of the classical regular rings. What we can currently offer is, however, systematically distributed in the next section.

2. Main Results

We begin here with the following first transversal between the afore-defined classes of ξ-rings and generalized ξ-rings. Specifically, the following criterion in terms of central nilpotent elements of order not exceeding two holds.

Theorem 1. *A generalized ξ-ring R is a ξ-ring if, and only if, $Nil_2(R) \subseteq C(R)$.*

Proof. The necessity being trivial according to the discussion quoted above, we now concentrate on the sufficiency. In doing that, given a generalized ξ-ring R equipped with the property $Nil_2(R) \subseteq C(R)$. Since in the notations above $x - xyx \in C(R)$, it must be that $x(x-xyx) = (x-xyx)x$ and that $y(x-xyx) = (x-xyx)y$. From these two equalities, we extract that $x(xy-yx)x = 0$ and $yx - xy = (yx-xy)xy + yx(yx-xy)$. Thus $x(xy-yx) \in Nil_2(R)$ and $(xy-yx)x \in Nil_2(R)$, whence by assumption it must be that $x(xy-yx) \in C(R)$ and $(xy-yx)x \in C(R)$.

Furthermore, $x(yx-xy) = x(yx-xy)xy + xyx(yx-xy) = xyx(yx-xy)$ and, by a reason of symmetry, $(yx-xy)x = (yx-xy)xyx$. Consequently, we deduce that $x(yx-xy) = xyx(yx-xy) = xy[x(yx-xy)] = [x(yx-xy)]xy = [x(yx-xy)x]y = 0$, i.e., $x(yx-xy) = 0$. Similarly, $(yx-xy)x = 0$, Substituting the last two equations in $yx - xy = (yx-xy)xy + yx(yx-xy)$, we derive that $yx - xy = 0$. Finally, one concludes that $yx = xy$ and so $x - xyx = x - x^2y \in C(R)$, as expected. □

We are now ready to proceed by proving with the second transversal, which somewhat demonstrates a left-right symmetric fulfillment under the validity of some extra real circumstances. Precisely, the following necessary and sufficient condition is fulfilled.

Proposition 2. *Suppose that R is a ring. Then R is a left-idempotently ξ-ring if, and only if, R is a right-idempotently ξ-ring.*

In addition, if $Nil_2(R) \subseteq C(R)$, then they are both ξ-rings.

Proof. "left ⇒ right". Assuming that, for any $x \in R$, there is $e \in xR \cap Id(R)$ with $(1-e)x \in C(R)$, we need to show that there is $f \in Rx \cap Id(R)$ such that $x(1-f) \in C(R)$. Writing concretely that $e = xr$ for some $r \in R$, then one checks that $f = (rx)^2 \in Rx \cap Id(R)$ such that the rotation equality $xf = ex$ holds. Likewise, $x(1-f) = x - xf = x - ex = (1-e)x \in C(R)$, as pursued. We may prove by analogy the reverse implication "right ⇒ left", concluding the general equivalence.

On the other vein, setting $(1-e)x = c \in C(R)$, one deduces that $[x(1-f)]^2 = c[x(1-f)]$ and so, by induction on the power $n \in \mathbb{N}$, we obtain that $[x(1-f)]^n = c^{n-1}[x(1-f)]$, which enables us that $x(1-f) \in Nil(R)$ provided additionally that $c \in Nil(R)$.

To establish that the left-idempotently ξ-ring R is really a ξ-ring, it is sufficiently to detect that $x(1-e) \in C(R)$. To that goal, we first observe that $ex(1-e) = x(1-e)e = 0$, so that $ex = exe$. But then $(1-e)xe = xe - ex$ and, therefore, $x(1-e) = (1-e)x - (1-e)xe \in C(R) + Nil_2(R) = C(R)$, as wanted. Analogously, it can be shown in the case of right-idempotently rings that, if there is $f \in Rx \cap Id(R)$ with $x(1-f) \in C(R)$, then $(1-f)x \in C(R)$, as desired. The second part-half now follows immediately. It is worthwhile noticing that it follows also directly using Theorem 1. □

The following consequence follows at once from the proof of the previous statement, and it is a major tool of the promised above rings connections. For completeness, we state it separately as it illustrates that some additional left-right symmetric conditions are fulfilled, too.

Corollary 3. *Let R be a ring such that $Nil_2(R) \subseteq C(R)$. Then the following two points hold:*

(1) If R is a left-idempotently ξ-ring, then there is $e \in xR \cap Id(R)$ with $x(1-e) \in C(R)$.

(2) If R is a right-idempotently ξ-ring, then there is $f \in Rx \cap Id(R)$ with $(1-f)x \in C(R)$.

As a concluding discussion, we state:

Remark 1. *As already noticed above, the inclusion $Nil(R) \subseteq C(R)$ is always true for ξ-rings. It can be, however, proved that the relation $J(R) \subseteq C(R)$ is always valid in left-idempotently (resp., right-idempotently) ξ-rings. Indeed, if $z \in J(R)$, then there is $e \in Id(R) \cap zR$ such that $(1-e)z \in C(R)$ or there is $f \in Id(R) \cap Rz$ such that $z(1-f) \in C(R)$. But since in both cases $e \in Id(R) \cap J(R) = \{0\}$ and $f \in Id(R) \cap J(R) = \{0\}$, one infers that $z \in C(R)$, as required.*

We end our work with the following:

Problem. Extend the above definitions and achievements in terms of strongly π-regular and π-regular rings, respectively. For example, there will exist $n \in \mathbb{N}$ endowed with the property that $x^n - x^{2n}y \in C(R)$ or, respectively, $x^n - x^n y x^n \in C(R)$.

Acknowledgment. The author owes his sincere thanks to the referee for the careful reading of the manuscript.

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