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INVARIANCE OF METRIC DIMENSION OF SUBDIVISION OF THE CROWN GRAPH

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Abstract. Metric dimension has applications in several areas including computer navigation, chemistry, and image processing. In this paper we give metric dimension of the crown graph $S_n, n \geq 3$, and show that it remains invariant under any of its subdivision.

Key words : metric dimension, resolving set, crown graph.
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1. Introduction

The concepts of metric dimension and resolving set were introduced by Slater in [17, 18] and studied independently by Harary and Melter in [5]. Since then the resolving sets have been widely investigated, as you can see in [2, 3, 9, 12, 16, 19, 20].

Metric dimension has applications in several different disciplines: applications of metric dimension to the navigation of robots in networks are discussed in [11], to chemistry in [4], and to image processing in [13].

A *graph* G is a pair $(V(G), E(G))$, where V is the set of vertices and E is the set of edges. A *path* from a vertex v to a vertex w is a sequence of vertices and edges that starts from v and stops at w. The number of edges in a path is the *length* of that path. A graph is said to be *connected* if there is a path between any two of its vertices. The *distance* $d(u, v)$ between two vertices u, v of a connected graph G is the length of a shortest path between them.

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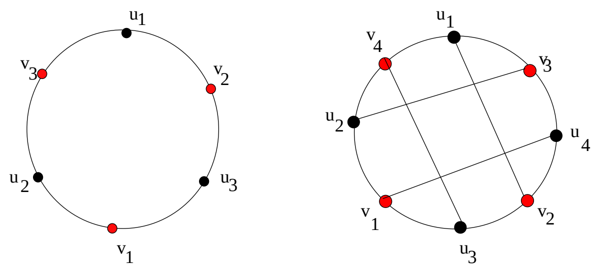
Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation of distances of v with respect to W, $r(v|_W)$, is the k-tuple $(d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a *resolving set* for G [1]. A resolving set of minimum cardinality is called a *basis* of G; the number of elements in this basis is the *metric dimension* of G, $\dim(G)$. A family $\mathcal{G}$ of connected graphs is said to have constant metric dimension if it is independent of any choice of member of that family.

The metric dimension of wheel W_n is determined by Buczkowski et al. [1], of fan f_n by Caceres et al. [3, 2], and of Jahangir graph J_{2n} by Tomescu et al. [19]. Chartrand et al. [4] proved that the family of paths P_n has the constant metric dimension 1. Javaid et al. proved in [9] that the plane graph antiprism $A_n, n \geq 5$ constitutes a family of regular graphs with constant metric dimension 3. The metric dimensions of some classes of plane graphs and convex polytopes have been studied in [8], of generalized Petersen graphs $P(n, 3)$ in [6], and of some rotationally-symmetric graphs in [7]. Murtaza et al. [15] gave some partial results about the metric dimension of the Möbius ladder, while Munir et al. gave complete results about the metric dimension of the Möbius ladder [14].

2. Crown Graph

The crown graph was introduced by Johnson in 1974 [10]. The crown graph $S_n, n \geq 3$, has $2n$ vertices and $n(n-1)$ edges. The vertex set is $V = \{u_i, v_i \mid i = 1, 2, \ldots, n\}$ and the edge set is $E = \{u_i v_j \mid i \neq j\}$. You can see S_3 and S_4 in the following figure.



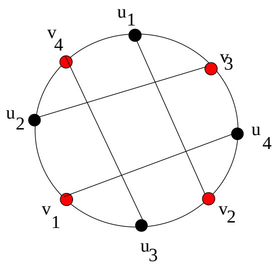
S_3 S_4

In this article, we give the metric dimension of the family of crown graphs $S_n, n \geq 3$, and show that this family is of non-constant metric dimension. Finally, we prove that its metric dimension remains invariant under any subdivision.

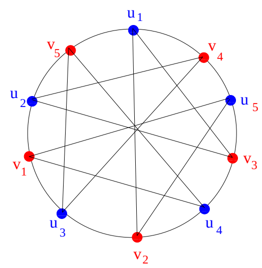
3. Main Results

Theorem 1. *The metric dimension of the crown graph S_n is $n-1$ for $n \geq 3$.*

Proof. Depending on the value of n, the proof is splitted up into two cases:
Case I. ($n = 3$) If $n = 3$, S_3 appears actually as the cycle C_6, and hence has metric dimension 2.
Case II. ($n \geq 4$) We claim that the set $W = \{u_1, u_2, \dots, u_{n-1}\}$ is a resolving set. Every coordinate of the $n-1$-tuple $r(u_i|_W), 1 \leq i \leq n-1$, is 2, except the ith coordinate, which is 0. However, every coordinate of the $n-1$-tuple $r(u_i|_W), i = n$, is 2. Also, all the coordinates of the $n-1$-tuple $r(v_i|_W), 1 \leq i \leq n-1$, is 1, except the ith coordinate, which is 3. Note that all the vectors in this case are distinct.
Now we show that no other set having number of elements less than $n-1$ can be a resolving set: If we delete any vertex from W, say u_{n-1}, then the vectors $r(u_{n-i}|_W)$ and $r(u_n|_W)$ become identical, and hence $W - \{u_{n-1}\}$ is not a resolving set. If any $(n-2)$-element set W' contains at least one vertex from the set $\{u_1, \dots, u_n\}$ and at least one vertex from the set $\{v_1, \dots, v_n\}$, then one can find two identical distance vectors. For instance, if we consider S_4 and take $W' = \{u_1, v_4\}$, then $r(v_2|_{W'}) = r(v_3|_{W'})$. Similarly, if we consider S_5 and take $W' = \{u_1, u_2, v_5\}$, then $r(u_3|_{W'}) = r(u_4|_{W'})$.



S_4



S_5

□

Now we show that the metric dimension remains invariant under any subdivision of S_n. For our convenience, we shall divide the edges into two disjoint sets, the set that contains those edges that lie on the main circle and the set that contains those edges that appear as chords of the circle. Moreover, we shall use the notation $S_n^{k,l}$ to denote that each circular edge is subdivided by k vertices and each chordal edge is subdivided by l vertices.

Theorem 2. *The metric dimension of $S_n^{k,0}$ is $n-1$ for $n \geq 7$.*

Proof. The resolving set is $W = \{u_1, u_2, \dots, u_{n-1}\}$ as all the distance vectors $r(u_i|_W), r(v_j|_W), 1 \leq i, j \leq n$, are distinct. No set with less than $n-1$ vertices is a resolving set. If we consider $W' = \{u_1, u_2, \dots, u_{n-2}\}$, then $r(u_{n-1}|_W) =$

$r(u_{n-2}|_W)$. With the similar reason, $W'' = \{v_1, v_2, \dots, v_{n-2}\}$ is not a resolving set. The cardinality of a resolving set cannot be less than $n-1$ as is confirmed from the $2n \times 2n$ distance matrix

$$D(S_n^{k,0}) = \begin{pmatrix} U & V \\ V^T & U \end{pmatrix},$$

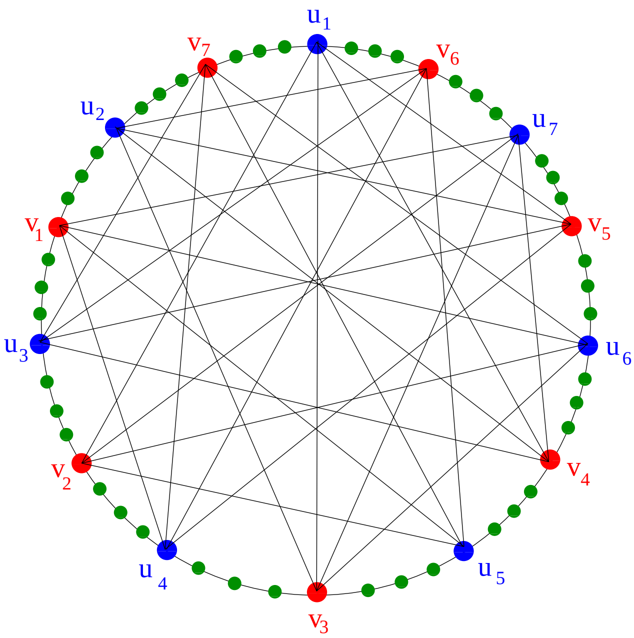
where $U = (u_{ij})$ is the $n \times n$ matrix with

$$u_{ij} = d(u_i, u_j) = d(v_i, v_j) = \begin{cases} 2, & i \neq j \\ 0, & i = j \end{cases}.$$

and $V = (v_{ij})$ is also the $n \times n$ matrix with

$$v_{ij} = d(u_i, v_j) = \begin{cases} 3, & j = i, i-1, i-2 \\ 1, & \text{otherwise} \end{cases},$$

where $v_0 = v_n$ and $v_{-1} = v_{n-1}$. For instance, see $S_7^{3,0}$ along with its distance matrix:



$S_7^{3,0}$

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	v_1	v_2	v_3	v_4	v_5	v_6	v_7
u_1	0	2	2	2	2	2	2	3	1	1	1	1	3	3
u_2	2	0	2	2	2	2	2	3	3	1	1	1	1	3
u_3	2	2	0	2	2	2	2	3	3	3	1	1	1	1
u_4	2	2	2	0	2	2	2	1	3	3	3	1	1	1
u_5	2	2	2	2	0	2	2	1	1	3	3	3	1	1
u_6	2	2	2	2	2	0	2	1	1	1	3	3	3	1
u_7	2	2	2	2	2	2	0	1	1	1	1	3	3	3
v_1	3	3	3	1	1	1	1	0	2	2	2	2	2	2
v_2	1	3	3	3	1	1	1	2	0	2	2	2	2	2
v_3	1	1	3	3	3	1	1	2	2	0	2	2	2	2
v_4	1	1	1	3	3	3	1	2	2	2	0	2	2	2
v_5	1	1	1	1	3	3	3	2	2	2	2	0	2	2
v_6	3	1	1	1	1	3	3	2	2	2	2	2	0	2
v_7	3	3	1	1	1	1	3	2	2	2	2	2	2	0

□

Proposition 3. *The metric dimension of* $S_n^{0,l}, n \geq 4$, *is 2.*

Proof. The proof is divided into two cases:
Case I. (n is odd.) If each chordal edge is subdivided by l vertices, the metric dimension of $S_n^{0,l}$ becomes independent of chordal edges for $l \geq n$.
Case II. (n is even.) In this case, again, the metric dimension of $S_n^{0,l}$ becomes independent of chordal edges for $l \geq n-1$.
It follows that the metric dimension depends only on the circular edges, and hence the case reduces to find the metric dimension of the cycle C_n, which is 2. □

Theorem 4. *The metric dimension of* $S_n^{k,k}$ *is* $n-1$ *for* $n \geq 4$.

Proof. Here, again, the resolving set is $W = \{u_1, u_2, \ldots, u_{n-1}\}$. All the remaining arguments are the same as are given in Theorem 2; the change appears only in the entries of the distance matrix $D(S_n^{k,k})$. In this case the entries of the sub matrices U and V are

$$u_{ij} = d(u_i, u_j) = d(v_i, v_j) = \begin{cases} 2k+1, & i \neq j \\ 0, & i = j \end{cases}$$

and

$$v_{ij} = d(u_i, v_j) = \begin{cases} k+1, & i \neq j \\ 3k+3, & i = j \end{cases}.$$

□

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