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A NEW PARADIGM FOR INCREASING THE CONTINUITY OF SUBDIVISION SCHEMES

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Abstract. Subdivision schemes having high continuity are always required for designing of smooth curves and surfaces. In this paper, we present a paradigm to generate a family of binary approximating subdivision schemes with high continuity based on probability distribution. The analysis and convexity preservation of some members of the family are also presented. Subdivision schemes give skewed behavior on convex data due to probability parameter.

Key words : Binary, approximating subdivision schemes, binomial probability distribution, convexity preservation.
AMS SUBJECT : 65D17, 65D15, 65D10, 65C20.

1. Introduction

Subdivision schemes are important tools for generation of smooth curves and surfaces in Computer Aided Geometric Design (CAGD). Developing new subdivision schemes for curve and surface design has its own importance.

Initially, de Rham [17] and Chaikin [18] are regarded as the pioneers. They developed the corner cutting schemes which generate C^1 limit curve. This algorithm was one of the first refinement algorithms to generate curves. Now a days wide variety of approximating and interpolating schemes have been proposed in the literature which possess shape parameters. Shape parameters are used in subdivision schemes to control the shapes.

In 2008, Siddiqi and Ahmad [5, 6] offered five and six points approximating subdivision schemes having C^4 and C^6 limit curves respectively when the shape parameter $w = \frac{1}{4}$. Next year, Mustafa et al. [7] presented 6-point

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binary approximating subdivision schemes which have C^6 continuity for the shape parameter $-1.6 < w < 1.3$. In 2012, Ghaffar and Mustafa [9] presented a family of even point ternary approximating subdivision schemes.
In 2014, Zheng et al. [19] presented a technique to increase the continuity of any subdivision scheme. They increase the continuity by multiplying the symbol with $\left(\frac{1+z}{2}\right)^k$ factor, after multiplication they get C^{k+n} continuous subdivision schemes. But this technique has following demerits, when they multiply symbol with factor then complexity of subdivision schemes are increased, support of subdivision schemes are increased and the mathematical computation of subdivision schemes are also increased.
Convexity preserving is a hot topic in curve design. A lot of work have been published on this topic during the past decades. Tan et al. [20] discussed the convexity preservation of five point binary approximating subdivision schemes. Siddiqi and Noreen [21] presented convexity preservation of six point ternary interpolating subdivision schemes.

1.1. **Contribution and findings.** The study of probability distribution is the analysis of certain games of chance, and it has found applications in most branches of science and engineering. Probability distribution have two types, discrete and continuous. Binomial probability distribution is a type of discrete distribution that was introduced by J. Bernoulli [22]. In this paper, first time Binomial distribution has been used to introduced subdivision schemes. The effects of probability parameter of subdivision schemes have been discussed. Furthermore, we have following contribution and findings:

- A new family of binary approximating subdivision schemes for curve design with high continuity.
- The maximum continuity, Hölder regularity and degree of generation can be obtained at probability parameter $p = \frac{1}{2}$.
- The primal and dual subdivision schemes can be obtained at different values of the probability parameter.
- Subdivision scheme give skewed behavior on convex data due to probability parameter.

The paper is organized as follows. In Section 2, we present construction of family of binary approximating subdivision schemes. Analysis of the proposed family is presented in Section 3. Convexity preservation is discussed in Section 4. In Section 5, we present the comparison and applications of proposed family of subdivision schemes. Conclusions are drawn in Sections 6.

2. Construction of family of subdivision schemes

In this section, we will elaborate our technique for the generation of new family of schemes. For this, we consider 4-point scheme (B-Spline of degree 5) [1] given by

$$\begin{aligned} f_{2i}^{k+1} &= \frac{6}{32}f_{i-1}^k + \frac{20}{32}f_i^k + \frac{6}{32}f_{i+1}^k, \\ f_{2i+1}^{k+1} &= \frac{1}{32}f_{i-1}^k + \frac{15}{32}f_i^k + \frac{15}{32}f_{i+1}^k + \frac{1}{32}f_{i+2}^k. \end{aligned} \tag{1}$$

The Laurent polynomial of scheme (1) is

$$K(z) = \frac{1}{32}\left(1 + 6z + 15z^2 + 20z^3 + 15z^4 + 6z^5 + z^6\right). \tag{2}$$

Binomial probability generating function [2] is given as

$$A_n(z) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} z^i, \quad \text{where} \quad i = 0, 1, 2, \cdots, n. \tag{3}$$

n is the number of trails, p is the probability of success, $q = 1 - p$ is the probability of failure with $p + q = 1$. The general symbol of new family of binary approximating subdivision schemes is defined as

$$M_{p_n}(z) = K(z)A_n(z),$$

where $K(z)$ defined in (2) and $A_n(z)$ defined in (3). Simplest form of $M_{p_n}(z)$ is

$$M_{p_n}(z) = \frac{1}{32}(1+z)^6 \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} z^i. \tag{4}$$

After substituting different values of n in (4), we obtain the symbols of family members with $p \in (0, 1)$. Table 1 gives the complexity and mask of proposed family of approximating subdivision schemes.

3. Analysis of 6-point scheme

Aim of this section is to present the complete analysis of one family member M_{p_5} of proposed family of subdivision schemes. Here we present some important properties such as continuity analysis, Hölder regularity, degree of generation and degree of reproduction. We also discuss the convexity preservation of M_{p_5}.

Table 1. *Shows the mask of family of binary approximating subdivision schemes corresponding to different values of n, here m shows complexity of the schemes (i.e. 4-, 5-, ...-point schemes).*

n	m	Mask
1	4	$M_{p_1}=\frac{1}{32}[(1-p),(6-5p),(15-9p),(20-5p),(15+5p),(6+9p),(1+5p),p]$
2	5	$M_{p_2}=\frac{1}{32}[(1-p)^2,(6-10p+4p^2),(15-18p+4p^2),(20-10p-4p^2),(15+10p-10p^2),(6+18p-4p^2),(1+10p+4p^2),(2p+4p^2),p^2]$
3	5	$M_{p_3}=\frac{1}{32}[(1-p)^3,(6-15p+12p^2-3p^3),(15-27p+12p^2),(20-15p-12p^2+8p^3),(15+15p-30p^2+6p^3),(6+27p-12p^2-6p^3),(1+15p+12p^2-8p^3),(3+12p^2),(3p^2+3p^3),p^3]$
4	6	$M_{p_4}=\frac{1}{32}[(1-p)^4,(6-20p+24p^2-12p^3+2p^4),(15-36p+24p^2-3p^4),(20-20p-24p^2+32p^3-8p^4),(15+20p-60p^2+24p^3+2p^4),(6+36p-24p^2-24p^3+12p^4),(1+20p+24p^2-32p^3+2p^4),(4p+24p^2-8p^4),(6p^2+12p^3-3p^4),(4p^3+2p^4),p^4]$
5	6	$M_{p_5}=\frac{1}{32}[p^5,(p^5+5p^4),(-5p^5+10p^4+10p^3),(-5p^5-15p^4+30p^3+10p^2),(10p^5-40p^4+40p^2+5p),(10p^5+10p^4-80p^3+40p^2+25p+1),(-10p^5+60p^4-60p^3-40p^2+45p+6),(-10p^5+10p^4+60p^3-100p^2+25p+15),(5p^5-40p^4+80p^3-40p^2-25p+20),(5p^5-15p^4+40p^2-45p+15),(-p^5+10p^4-30p^3+40p^2-25p+6),(-p^5+5p^4-10p^3+10p^2-5p+1)]$

After substituting $n=5$ in (4), we get the Laurent polynomial $M_{p_5}(z)$ of 6-point binary approximating subdivision scheme

$$\begin{aligned}
M_{p_5}(z) \;=\; & \frac{1}{32z^5}\Big[p^5+(p^5+5p^4)z+(-5p^5+10p^4+10p^3)z^2+(-5p^5 \\
& -15p^4+30p^3+10p^2)z^3+(10p^5-40p^4+40p^2+5p)z^4 \\
& +(10p^5+10p^4-80p^3+40p^2+25p+1)z^5+(-10p^5+60p^4 \\
& -60p^3-40p^2+45p+6)z^6+(-10p^5+10p^4+60p^3 \\
& -100p^2+25p+15)z^7+(5p^5-40p^4+80p^3-15p^4+40p^2 \\
& -45p+15)z^9+(-p^5+10p^4-30p^3+40p^2-25p+6)z^{10} \\
& +(-p^5-40p^2-25p+20)z^8+(5p^5+5p^4-10p^3+10p^2 \\
& -5p+1)z^{11}\Big]. \qquad (5)
\end{aligned}$$

The scheme corresponding to the symbol (5) is

$$\begin{aligned}
f_{2i}^{k+1} &= \frac{1}{32}\Big[(p^5+5p^4)f_{i-2}^k+(p^5-15p^4+30p^3+10p^2)f_{i-1}^k+(10p^5\\
&\quad +10p^4-80p^3+40p^2+25p+1)f_i^k+(-10p^5+10p^4+60p^3\\
&\quad -100p^2+25p+15)f_{i+1}^k+(5p^5-15p^4+40p^2-45p+15)f_{i+2}^k\\
&\quad +(-p^5+5p^4-10p^3+10p^2-5p+1)f_{i+3}^k\Big],\\
f_{2i+1}^{k+1} &= \frac{1}{32}\Big[p^5f_{i-2}^k+(-5p^5+10p^4+10p^3)f_{i-1}^k+(10p^5-40p^4+40p^2\\
&\quad +5p)f_i^k+(-10p^5+60p^4-60p^3-40p^2+45p+6)f_{i+1}^k+(5p^5\\
&\quad -40p^4+80p^3-40p^2-25p+20)f_{i+2}^k+(-p^5+10p^4-30p^3\\
&\quad +40p^2-25p+6)f_{i+3}^k\Big]. \qquad (6)
\end{aligned}$$

Theorem 1. *The 6-point binary approximating subdivision scheme* (6) *is* C^5 *continuous.*

Proof. From (5), we have

$$M_{p_5}(z)=\left(\frac{1+z}{2}\right)^5 b_5(z), \quad \text{with} \quad b_5(z)=(1+z)c_5(z),$$

where

$$\begin{aligned}
c_5(z) &= \frac{1}{z^5}\Big[p^5+(5p^4-5p^5)z+(10p^5-20p^4+10p^3)z^2+(-10p^5\\
&\quad -30p^3+10p^2+30p^4)z^3+(5p^5-20p^4+30p^3-20p^2+5p)z^4\\
&\quad +(-p^5+5p^4-10p^3+10p^2-5p+1)z^5\Big].
\end{aligned}$$

To prove C^5 continuity of scheme M_{p_5}, we need to show that the difference scheme S_{b_5} is convergent. For this we have to show that S_{c_5} is contractive. For contractiveness of S_{c_5}, we see that

$$\begin{aligned}
||S_{c_5}||_\infty &= \max\{16p^5-40p^4+40p^3-20p^2+5p, -16p^5+40p^4-40p^3\\
&\quad +20p^2-5p+1\}.
\end{aligned}$$

The condition for $||S_{c_5}||_\infty<1$ is $p\in(0,1)$. Therefore by [3], if $||S_{c_5}||_\infty<1$, then $c_5(z)$ is contractive and $b_5(z)$ is convergent. If $b_5(z)$ is convergent then scheme (6) is C^5 continuous. □

Remark 1. *The 6-point binary approximating subdivision scheme* (6) *is* C^9 *continuous at* $p=\frac{1}{2}$. *Here we see that at* $p=\frac{1}{2}$, *our proposed family converts to B-splines of degree n+5.*

3.1. **Generation and reproduction analysis.** The subdivision scheme with symbol $M_{p_n}(z)$ reproduces polynomials of degree d with respect to the parameterizations $\tau = M^{'}_{p_n}(1)/2$ if and only if

$$M^{(k)}_{p_n}(-1) = 0, \quad M^{(k)}_{p_n}(1) = 2\prod_{j=0}^{k-1}(\tau - j), \quad k = 0, 1, ..., d. \tag{7}$$

Polynomial reproduction of degree d requires polynomial generation of degree d [3].

Theorem 2. *The degree of generation of 6-point binary approximating subdivision scheme* (6) *is 5.*

Proof. From (5), we can re-write the Laurent polynomial as

$$M_{p_5}(z) = (1+z)^{5+1} b_{p_5}(z),$$

where

$$\begin{aligned} b_{p_5}(z) \quad = \quad & \frac{1}{32z^5}\Big[p^5 + (-5p^5 + 5p^4)z + (10p^5 - 20p^4 + 10p^2)z^3 + (-10p^5 \\ & +30p^4 - 30p^3 + 10p^2)z^3 + (5p^5 - 20p^4 + 30p^3 - 20p^2 + 5p)z^4 \\ & +(-p^5 + 5p^4 - 10p^3 + 10p^2 - 5p + 1)z^5\Big]. \end{aligned}$$

Hence degree of generation is 5. □

Theorem 3. *The 6-point binary approximating subdivision scheme* (6) *has linear reproduction with parametrization depends on value of p.*

Proof. By taking the first derivative of $M_{p_5}(z)$ with respect to z defined in (5). After putting $z = 1$, we get $(M_{p_5})^{'}(1) = 6 - 10p$. This implies that $\tau = \frac{6-10p}{2}$, for different values of p the scheme corresponding to the symbol $M_{p_5}(z)$ has dual as well as primal parametrization. We can easily verify that the first and second derivatives of $M_{p_5}(z)$ at $z = -1$ are equal to 0. Further, we can also verify (7) for $k = 0$ and 1. This complete the proof. □

In Table 2, we summarize the results of important properties of some family members of proposed family of binary approximating subdivision schemes. It is observed that high order continuity and generation degree can be achieved at $p = \frac{1}{2}$.

4. Convexity preservation

In this section, we will show the convexity preservation of the scheme M_{p_5}.

Theorem 4. *Suppose that the initial control points $\{p^0_i\}_{i\in\mathbb{Z}}$ are strictly convex, i.e. $d^0_i > 0$, $\forall\ i \in \mathbb{Z}$. Let $d^k_i = 2^{2k+1}(f^k_{i-1} - 2f^k_i + f^k_{i+1})$ be defined as second*

Table 2. *Summary of the results of important properties of proposed family of schemes, here n, M_{p_n}, GD, RD, OC, and HR, represent trial number, proposed schemes, degree of generation, degree of reproduction, order of continuity and Hölder regularity of proposed schemes at $p=\frac{1}{2}$ respectively.*

n	M_{p_n}	GD	RD	OC	$OC_{\frac{1}{2}}$	$GD_{\frac{1}{2}}$	$HR_{\frac{1}{2}}$
1	M_{p_1}	5	1	C^5	C^5	6	6
2	M_{p_2}	5	1	C^5	C^6	7	7
3	M_{p_3}	5	1	C^5	C^7	8	8
4	M_{p_4}	5	1	C^5	C^8	9	9
5	M_{p_5}	5	1	C^5	C^9	10	10

order differences, $r_i^k=\frac{d_{i+1}^k}{d_i^k}$, $R^k=max\{r_i^k,\frac{1}{r_i^k}\}$, $\forall \quad k\geq 0,\ k\in\mathbb{Z}, i\in\mathbb{Z}$, *the parameter* p *satisfies* $0<p<1$ *and* $\lambda\in\mathbb{R}$, *where* $0<\lambda\leq\frac{4p^5-20p^3+20p^2-5p+2}{4p^5-20p^3+20p^2-5p-2}$, *and* $\frac{1}{\lambda}\leq R^0\leq\lambda$. *If*

$$d_i^k>0,\quad \frac{1}{\lambda}\leq R^k\leq\lambda,\forall\quad k\geq 0, k\in\mathbb{Z}, i\in\mathbb{Z}, \tag{8}$$

then 6-point binary approximating subdivision scheme (6) *is convexity preserving.*

Proof. Using second order divided difference formula,

$$d_i^{k+1}=2^{2k+1}(f_{i-1}^{k+1}-2f_i^{k+1}+f_{i+1}^{k+1}). \tag{9}$$

By using (9), the scheme (6) become

$$\begin{aligned} d_{2i}^{k+1} &= \frac{1}{4}\Big[p^5 d_{i-2}^k+(-4p^5+10p^3)d_{i-1}^k+(6p^5-30p^3+20p^2+5p)d_i^k \\ &\quad +(-4p^5+30p^3-40p^2+10p+4)d_{i+1}^k+(p^5-10p^3+20p^2 \\ &\quad -15p+4)d_{i+2}^k\Big]. \end{aligned} \tag{10}$$

$$\begin{aligned} d_{2i+1}^{k+1} &= \frac{1}{4}\Big[(-p^5+5p^4)d_{i-1}^k+(4p^5-20p^4+10p^3+10p^2)d_i^k+(-6p^5 \\ &\quad +30p^4-30p^3-10p^2+15p+1)d_{i+1}^k+(4p^5-20p^4+30p^3-10p^2 \\ &\quad -10p+6)d_{i+2}^k+(-p^5+5p^4-10p^3+10p^2-5p+1)d_{i+3}^k\Big]. \end{aligned} \tag{11}$$

We use mathematical induction to prove $d_i^k>0$ and $\frac{1}{\lambda}\leq R^k\leq\lambda$. When $k=0$, it is obvious from statement $d_i^0>0$ and $\frac{1}{\lambda}\leq R^0\leq\lambda$. Suppose $d_i^k>0$ and $\frac{1}{\lambda}\leq R^k\leq\lambda$ is true for k. Now we prove it for $k+1$. By re-writing equation (10) as

$$\begin{aligned} d_{2i}^{k+1} &= \frac{d_i^k}{4}\Big[(-8p^5+40p^3-40p^2+10p+4)\lambda^2-(-8p^5+40p^3-40p^2 \\ &\quad +10p-4)\lambda\Big], \end{aligned} \tag{12}$$

$d_{2i}^{k+1}>0$, for $0<\lambda\leq\frac{4p^5-20p^3+20p^2-5p+2}{4p^5-20p^3+20p^2-5p-2}$, with $p\in(0,1)$. Now consider equation (11) to show $d_{2i+1}^{k+1}>0$.

$$\begin{aligned} d_{2i+1}^{k+1} &= \frac{d_i^k}{4}\Big[(-p^5+5p^4)\frac{d_{i-1}^k}{d_i^k}+(4p^5-20p^4+10p^3+10p^2)\frac{d_i^k}{d_i^k}+(-6p^5 \\ &\quad +30p^4-30p^3-10p^2+15p+1)\frac{d_{i+1}^k}{d_i^k}+(4p^5-20p^4+30p^3-10p^2 \\ &\quad -10p+6)\frac{d_{i+2}^k}{d_i^k}+(-p^5+5p^4-10p^3+10p^2-5p+1)\frac{d_{i+3}^k}{d_i^k}\Big], \end{aligned}$$

$$\begin{aligned} d_{2i+1}^{k+1} &= \frac{d_i^k}{4\lambda}\Big[(-8p^5+40p^4-40p^3+10p+2)+(8p^5-40p^4+40p^3 \\ &\quad -10p+6)\lambda\Big]. \end{aligned} \tag{13}$$

$d_{2i+1}^{k+1}>0$ for $0<\lambda\leq\frac{4p^5-20p^4+20p^3-5p-1}{4p^5-20p^4+20p^3-5p+3}$ with $p\in(0,1)$.
Thus $d_i^{k+1}>0$, $\forall\ k\geq 0$, $k\in\mathbb{Z}$, $i\in\mathbb{Z}$. To prove $\frac{1}{\lambda}\leq R^{k+1}\leq\lambda$, $\forall\quad k\geq 0$ it suffices to prove that

$$\frac{1}{\lambda}<r_i^{k+1}<\lambda, \quad \forall i\in\mathbb{Z}.\forall k\geq 0, k\in\mathbb{Z}, i\in\mathbb{Z}.$$

Since we know that $r_{2i}^{k+1}=\frac{d_{2i+1}^{k+1}}{d_{2i}^{k+1}}$,

$$r_{2i}^{k+1}-\lambda=\frac{d_{2i+1}^{k+1}}{d_{2i}^{k+1}}-\lambda=\frac{d_{2i+1}^{k+1}-\lambda d_{2i}^{k+1}}{d_{2i}^{k+1}}=\frac{A}{B}.$$

Let $r_{2i}^{k+1}-\lambda=\frac{A}{B}$. From (10), we have already proved that $d_{2i}^{k+1}>0$ so, $B>0$ and the numerator satisfies,

$$\begin{aligned} A &= r_{i-2}^k(-p^5+5p^4+\lambda 4p^5-\lambda 10p^3)+r_{i-1}^k r_{i-2}^k(4p^5-20p4 \\ &\quad +10p^3+10p^2-\lambda 6p^5+\lambda 30p^3-\lambda 20p^2-\lambda 5p)+r_{i-2}^k r_{i-1}^k r_i^k(-6p^5+30p^4 \\ &\quad -30p^3-10p^2+15p+1+\lambda 4p^5-\lambda 30p^3+\lambda 40p^2-\lambda 10p-4\lambda) \\ &\quad +r_{i-2}^k r_{i-1}^k r_i^k r_{i+1}^k(4p^5-20p^4+30p^3-10p^2-10p+6-p^5\lambda+\lambda 10p^3 \\ &\quad -\lambda 20p^2+\lambda 15p-4\lambda)-p^5\lambda, \end{aligned}$$

$$\begin{aligned} A &= -p^5\lambda + r^k_{i-2}(-p^5 + 5p^4 + \lambda 4p^5 - \lambda 10p^3) + r^k_{i-1}r^k_{i-2}(4p^5 - 20p^4 \\ &\quad +10p^3 + 10p^2 - \lambda 6p^5 + \lambda 30p^3 - \lambda 20p^2 - \lambda 5p) + \Big\{(-6p^5 + 30p^4 \\ &\quad -30p^3 - 10p^2 + 15p + 1 + \lambda 4p^5 - \lambda 30p^3 + \lambda 40p^2 - \lambda 10p - 4\lambda) \\ &\quad +\frac{1}{\lambda}(4p^5 - 20p^4 + 30p^3 - 10p^2 - 10p + 6_\lambda p^5 + \lambda 10p^3 - \lambda 20p^2 \\ &\quad +\lambda 15p - 4\lambda)\Big\} r^k_{i-2}r^k_{i-1}r^k_i, \end{aligned}$$

$$\begin{aligned} A &= -p^5\lambda + s^k_{i-2}(-p^5 + 5p^4 + \lambda 4p^5 - \lambda 10p^3) + s^k_{i-1}s^k_{i-2}(4p^5 - 20p4 \\ &\quad +10p^3 + 10p^2 - \lambda 6p^5 + \lambda 30p^3 - \lambda 20p^2 - \lambda 5p) + \Big\{(-\lambda 6p^5 + \lambda 30p^4 \\ &\quad -\lambda 30p^3 - \lambda 10p^2 + \lambda 15p + \lambda + \lambda^2 4p^5 - \lambda^2 30p^3 + \lambda^2 40p^2 - \lambda^2 10p \\ &\quad -4\lambda^2) + 4p^5 - 20p^4 + 30p^3 - 10p^2 - 10p + 6_\lambda p^5 + \lambda 10p^3 - \lambda 20p^2 \\ &\quad +\lambda 15p - 4\lambda\Big\} r^k_{i-2}r^k_{i-1}, \end{aligned}$$

$$\begin{aligned} A &= -p^5\lambda - p^5\lambda + \lambda 5p^4 + \lambda^2 4p^5 - \lambda^2 10p^3 + 4p^5 - 20p^4 + 10p^3 + 10p^2 \\ &\quad -\lambda 6p^5 + \lambda 30p^3 - \lambda 20p^2 - \lambda 5p - \lambda 6p^5 + \lambda 30p^4 - \lambda 30p^3 - \lambda 10p^2 \\ &\quad +\lambda 15p + \lambda + \lambda^2 4p^5 - \lambda^2 30p^3 + \lambda^2 40p^2 - \lambda^2 10p - 4\lambda^2 + 4p^5 - 20p^4 \\ &\quad +30p^3 - 10p^2 - 10p + 6_\lambda p^5 + \lambda 10p^3 - \lambda 20p^2 + \lambda 15p - 4\lambda, \end{aligned}$$

$$\begin{aligned} A &= \lambda^2\big(8p^5 - 40p^3 + 40p^2 - 10p - 4\big) + \lambda\big(-15p^5 + 35p^4 + 10p^3 - 50p^2 \\ &\quad +10p - 3) + (8p^5 - 40p^4 + 40p^3 - 10p + 6\big). \end{aligned} \tag{14}$$

This implies that $A < 0$, for $0 < \lambda \leq \frac{4p^5-20p^4+20p^3-5p-1}{4p^5-20p^4+20p^3-5p+3}$.
Thus $r^{k+1}_{2i} - \lambda = \frac{A}{B} < 0$. Therefore $r^{k+1}_{2i} < \lambda$, $\forall i \in \mathbb{Z}$.
Similarly

$$\frac{1}{r^{k+1}_{2i}} - \lambda = \frac{d^{k+1}_{2i}}{d^{k+1}_{2i+1}} - \lambda = \frac{d^{k+1}_{2i} - \lambda d^{k+1}_{2i+1}}{d^{k+1}_{2i+1}} = \frac{C}{D}.$$

From (11), we have already proved that $d^{k+1}_{2i+1} > 0$ so, $D > 0$ and the numerator satisfies,

$$\begin{aligned} C &= p^5 + r^k_{i-2}(-4p^5 + 10p^3 + \lambda p^5 - \lambda 5p^4) + r^k_{i-1}r^k_{i-2}(6p^5 - 30p^3 + 20p^2 \\ &\quad +5p - \lambda 4p^5 + \lambda 20p^4 - \lambda 10p^3 - \lambda 10p62) + r^k_{i-2}r^k_{i-1}r^k_i(-4p^5 + 30p^3 \\ &\quad -40p^2 + 10p + 4 + \lambda 6p^5 - \lambda 30p^4 + \lambda 30p^3 + \lambda 10p^2 - 15\lambda p - \lambda) \\ &\quad +r^k_{i-2}r^k_{i-1}r^k_i r^k_{i+1}(2p^5 - 5p^4 + 10p^2 - 10p + 3 - 4p^5\lambda + \lambda 20p^4 - \lambda 30p^3 \\ &\quad +\lambda 10p^2 + 10\lambda p - 6\lambda), \end{aligned}$$

$$\begin{aligned} C &= p^5 + r^k_{i-2}(-4p^5 + 10p^3 + \lambda p^5 - \lambda 5p^4) + r^k_{i-1}r^k_{i-2}(6p^5 - 30p^3 \\ &\quad +20p^2 + 5p - \lambda 4p^5 + \lambda 20p^4 - \lambda 10p^3 - \lambda 10p^2) + (-4\lambda p^5 + 30\lambda p^3 \\ &\quad -40\lambda p^2 + 10p\lambda + 4\lambda + \lambda^2 6p^5 - \lambda^2 30p^4 + \lambda^2 30p^3 + \lambda^2 10p^2 - 15\lambda^2 p \\ &\quad -\lambda^2 + 2p^5 - 5p^4 + 10p^2 - 10p + 3 - 4p^5\lambda + \lambda 20p^4 - \lambda 30p^3 \\ &\quad +\lambda 10p^2 + 10\lambda p - 6\lambda)r^k_{i-2}r^k_{i-1}r^k_i, \end{aligned}$$

$$\begin{aligned} C &= p^5 + r^k_{i-2}(-4p^5 + 10p^3 + \lambda p^5 - \lambda 5p^4) + r^k_{i-1}r^k_{i-2}(4\lambda p^5 - 30\lambda p^3 \\ &\quad +20\lambda p^2 + 5\lambda p - 4\lambda - 4\lambda^2 p^5 + \lambda 20p^4 - 10\lambda^2 p^3 - 10\lambda^2 p^2 - 4\lambda p^5 \\ &\quad +30\lambda p^3 - \lambda 40p^2 + \lambda 10p + 4\lambda + 6\lambda^2 p^5 - \lambda^2 30p^4 + \lambda^2 30p^3 + \lambda^2 10p^2 \\ &\quad -15\lambda^2 p - \lambda^2 + 2p^5 - 5p^4 + 10p^2 - 10p + 3 - 4\lambda p^5 + \lambda 20p^4 - \lambda 30p^3 \\ &\quad +\lambda 10p^2 + 10\lambda p - 6\lambda), \end{aligned}$$

$$\begin{aligned} C &= \lambda^2(3p^5 - 15p^4 + 20p^3 - 15p - 1) + \lambda(-6p^5 + 20p^4 - 20p^3 - 10p^2 \\ &\quad +25p - 6) + (3p^5 - 5p^4 + 10p^2 - 10p + 3). \end{aligned} \tag{15}$$

This implies that $C < 0$ for $0 < \lambda \leq \frac{4p^5-20p^4+20p^3-5p-1}{4p^5-20p^4+20p^3-5p+3}$.
Thus $\frac{1}{r^{k+1}_{2i}} - \lambda = \frac{C}{D} < 0$. Combining (14) and (15), it can be written as

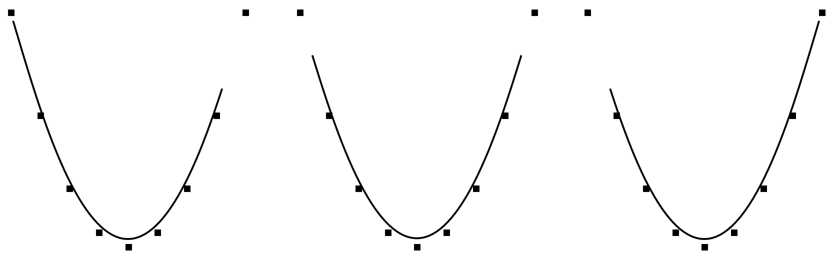
$$\frac{1}{\lambda} \leq r^{k+1}_{2i} \leq \lambda, \quad \forall i \in \mathbb{Z}.$$

Similarly we can show that

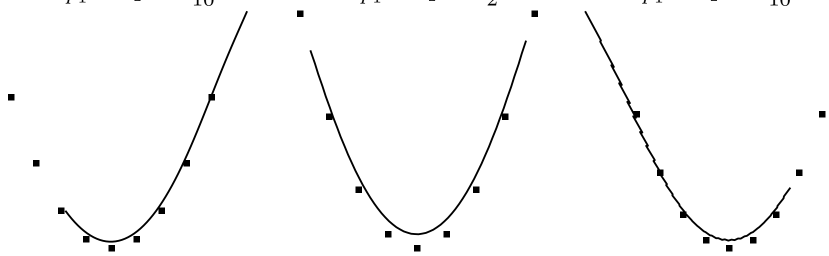
$$\frac{1}{\lambda} \leq r^{k+1}_{2i+1} \leq \lambda, \quad \forall i \in \mathbb{Z}.$$

Hence $d^k_i > 0$ and $\frac{1}{\lambda} \leq R^k \leq \lambda$ is satisfied for $k+1$. Hence proposed scheme M_{p_5} preserves convexity. This completes the proof. □

It is clear from Figure 1 that if initial control points are strictly convex. Then the limit curves generated by scheme corresponding to M_{p_1} show negatively, normal and positively skewed behavior on convex data for $p < \frac{1}{2}$, $p = \frac{1}{2}$ and $p > \frac{1}{2}$ respectively. But limit curves generated by scheme corresponding to M_{p_5} show positive, normal and negatively skewed behavior on convex data for $p < \frac{1}{2}$, $p = \frac{1}{2}$ and $p > \frac{1}{2}$ respectively.



(**a**) negatively skewed M_{p_1} at $p = \frac{2}{10}$ (**b**) normal M_{p_1} at $p = \frac{1}{2}$ (**c**) positively skewed M_{p_1} at $p = \frac{9}{10}$



(**d**) positively skewed M_{p_5} at $p = \frac{1}{10}$ (**e**) normal M_{p_5} at $p = \frac{1}{2}$ (**f**) negatively skewed M_{p_5} at $p = \frac{8}{10}$

FIGURE 1. (**a**) − (**c**) *and* (**d**) − (**f**) *present limit curves for convex data produced by the schemes corresponding to* M_{p_1} *and* M_{p_5} *respectively.*

5. Comparison and applications

In this section, we will present the comparison and applications of proposed family of schemes.

5.1. **Comparison of continuity analysis.** Here we will present the comparison of continuity analysis of proposed family of schemes with existing parametric subdivision schemes. We also show that our proposed family of binary approximating subdivision schemes gives higher continuity as compare to existing parametric subdivision schemes. Table 3 present the comparison of proposed family of approximating subdivision schemes.

5.2. **Applications.** Here we will discussed visual performance of proposed family of subdivision schemes. The control polygons are drawn by doted line and the smooth curves obtained by our proposed schemes by full line. Figures 2 present limit curves for close polygons. Figures 3 present comparison of limit curves for close polygons produced by the schemes corresponding to M_{p_1} and M_{p_5} at $p = \frac{1}{2}$.

Table 3. Shows the comparison of continuity analysis of existing parametric subdivision schemes and proposed family of schemes. E, OC, $C.P$, M_{p_n} and $OC_{\frac{1}{2}}$ denote the existing schemes, continuity of existing schemes, Convexity preservation, proposed family of schemes and continuity of proposed schemes at $p = \frac{1}{2}$ respectively.

E	OC	C.P of E	M_{p_n}	$OC_{\frac{1}{2}}$	C.P of
4-point [16]	C^2	No	M_{p_1}	C^5	No
4-point [11]	C^4	No	M_{p_1}	C^5	No
4-point [12]	C^5 at $w = \frac{1}{4}$	No	M_{p_1}	C^5	No
5-point [5]	C^4 at $w = \frac{1}{4}$	No	M_{p_3}	C^7	No
5-point [13]	C^5 at $w = \frac{1}{128}$	No	M_{p_3}	C^7	No
5-point [10]	C^3 at $\alpha = 1, \beta = 32$	Yes	M_{p_3}	C^7	No
6-point[9]	C^7 at $w = \frac{1}{12}$	No	M_{p_5}	C^9	Yes
6-point[7]	C^8 if $w \in (-1.6, 1.3)$	No	M_{p_5}	C^9	Yes
6-point[8]	C^3	No	M_{p_5}	C^9	Yes
6-point[14]	C^2	No	M_{p_5}	C^9	Yes
6-point[15]	C^2	No	M_{p_5}	C^9	Yes
6-point[6]	C^6 at $w = \frac{1}{4}$	No	M_{p_5}	C^9	Yes

6. Conclusion

In this paper, we have presented a technique to increase the continuity of binary approximating subdivision schemes and generate a new family of approximating subdivision schemes. We present the complete analysis of some members of the family of schemes. We also analyzed that our proposed family gives high continuity when probability parameter $p = \frac{1}{2}$ as compare to the existing subdivision schemes. At all other values of p the results of subdivision schemes remain same. Visual performances of proposed family of subdivision schemes are also discussed. Subdivision schemes show skewed behavior on convex data for different values of p.

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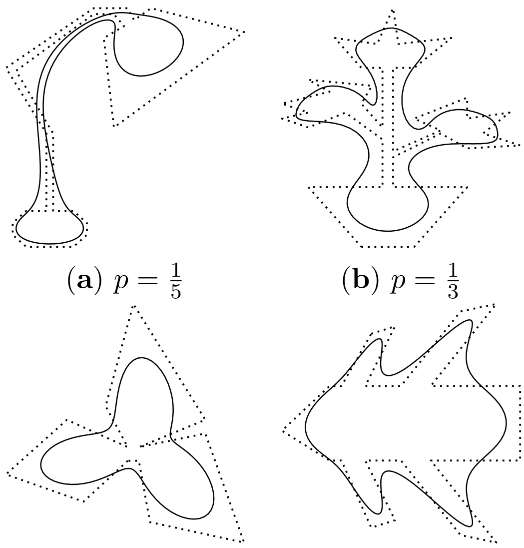
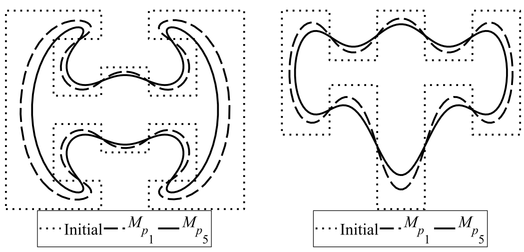


(**a**) $p = \frac{1}{5}$ (**b**) $p = \frac{1}{3}$ (**c**) $p = \frac{1}{2}$

(**d**) $p = \frac{4}{10}$ (**e**) $p = \frac{6}{10}$ (**f**) $p = \frac{9}{10}$

FIGURE 2. $(\mathbf{a}) - (\mathbf{c})$ *and* $(\mathbf{d}) - (\mathbf{f})$ *present limit curves for close polygons produced by scheme corresponding to* M_{p_5} *and* M_{p_1} *respectively.*



(**a**) (**b**) (**c**)

FIGURE 3. $(\mathbf{a}), (\mathbf{b})$ *and* $(\mathbf{c})$ *present limit curves for close polygons produced by scheme corresponding to* M_{p_1} *and* M_{p_5} *at* $p = \frac{1}{2}$.

REFERENCES

[1] J. M. Lane and R. F. Riesenfeld: *A theoretical development for the computer generation and display of piecewise polynomial surfaces*, IEEE Transactions on Pattern Analysis and Machine Intelligence, 2(1), (1980), 35-46.

[2] C. Walck: *Handbook on statistical distributions for experimentalists*, University of Stockholm, 2007.

[3] K. Hormann: *Subdivision schemes for geometric modelling. a hands-on tutorial*, Dolomites Research Notes on Approximation, ISSN 2035-6803, Publisher: University of Verona, 2012.

[4] O. Rioul: *Simple regularity criteria for subdivision schemes*, SIAM Journal on Mathematical Analysis, 23(6), (1992), 1544-1576.

[5] S. S. Siddiqi and N. Ahmad: *A new five-point approximating subdivision scheme*, International Journal of Computer Mathematics, 85(1), (2008), 65-72.

[6] S. S. Siddiqi and N. Ahmad: *A C^6 approximating subdivision scheme*, Applied Mathematics Letters, 21(7), (2008), 722-728.

[7] G. Mustafa, F. Khan and A. Ghaffar: *The m-point approximating subdivision scheme*, Lobachevskii Journal of Mathematics, 30(2), (2009), 138-145.

[8] G. Mustafa, F. Khan and M. S. Hashmi: *A generalized proof of the smoothness of 6-point interpolatory scheme*, Journal of Information and Computing Science, 5(4), (2010), 299-304.

[9] A. Ghaffar and G. Mustafa: *A family of even-point ternary approximating schemes*, ISRN Applied Mathematics, 2012, 14 pages.

[10] J. Tan, X. Zhuang and L. Zhang: *A new four-point shape-preserving C^3 subdivision scheme*, Computer Aided Geometric Design, 31(1), (2014), 57-62.

[11] Z. Hongxing: *A class of four-point subdivision schemes with two parameters and its properties*, Journal of Computer Aided Design and Computer Graphics, 16(8), (2004), 1140-1145.

[12] A. Ghaffar, G. Mustafa and K. Qin: *The 4-point a-ary approximating subdivision scheme*, Open Journal of Applied Sciences, 3(1), (2013), 106-111.

[13] H. Cao and J. Tan: *A binary five-point relaxation subdivision scheme*, Journal of Information and Computational Science, 10(18), (2013), 5903-5910.

[14] L. Romani: *From approximating subdivision schemes for exponential splines to high-performance interpolating algorithms*, Journal of Computational and Applied Mathematics, 224, 1 (2009), 383-396.

[15] K. P. Ko, B. G. Lee and G. J. Yoon: *A study on the mask of interpolatory symmetric subdivision schemes*, Applied Mathematics and Computation, 187, 2 (2007), 609-621.

[16] N. Dyn, M. S. Floater and K. Hormann: *A C^2 four-point subdivision scheme with fourth order accuracy and its extensions*, Mathematical Methods for Curves and Surfaces: Troms2004, Nashboro Press, Brentwood, 128, 1 (2005), 145-156.

[17] G. de Rham: *Sur une Courbe Plane*, Journal de Mathmatiques Pures et Appliques, 35, 1 (1956), 25-42.

[18] G. M. Chaikin: *An algorithm for high speed curve generation*, Computer Graphics and Image Processing, 3 (1974), 346-349.

[19] H. C. Zheng, S. P. Huang, F. Guo and G. H. Peng: *Designing multi-parameter curve subdivision schemes with high continuity*, Applied Mathematics and Computation, 243 (2014), 197-208.

[20] J. Tan, Y. Yao, H. Cao and L. Zhang: *Convexity preservation of five-point binary subdivision scheme with a parameter*, Applied Mathematics and Computation, 245 (2014), 279-288.

[21] S. S. Siddiqi and T. Noreen: *Convexity preservation of six point C^2 interpolating subdivision scheme*, Applied Mathematics and Computation, 265 (2015), 936-944.

[22] J. Bernoulli: *Ars Conjectandi*, Opus Posthumum. Accedit Tractatus de Seriebus infinitis, et Epistola Gallice scripta de ludo Pilae recticularis. Impensis Thurnisiorum, Fratrum, Basel, 1713.