A NEW PARADIGM FOR INCREASING THE CONTINUITY OF SUBDIVISION SCHEMES

GHULAM MUSTAFA¹, MUHAMMAD ASGHAR², MADIHA NAVEED³

ABSTRACT. Subdivision schemes having high continuity are always required for designing of smooth curves and surfaces. In this paper, we present a paradigm to generate a family of binary approximating subdivision schemes with high continuity based on probability distribution. The analysis and convexity preservation of some members of the family are also presented. Subdivision schemes give skewed behavior on convex data due to probability parameter.

Key words: Binary, approximating subdivision schemes, binomial probability distribution, convexity preservation.


1. Introduction

Subdivision schemes are important tools for generation of smooth curves and surfaces in Computer Aided Geometric Design (CAGD). Developing new subdivision schemes for curve and surface design has its own importance.

Initially, de Rham [17] and Chaikin [18] are regarded as the pioneers. They developed the corner cutting schemes which generate $C^1$ limit curve. This algorithm was one of the first refinement algorithms to generate curves. Now a days wide variety of approximating and interpolating schemes have been proposed in the literature which possess shape parameters. Shape parameters are used in subdivision schemes to control the shapes.

In 2008, Siddiqi and Ahmad [5, 6] offered five and six points approximating subdivision schemes having $C^4$ and $C^6$ limit curves respectively when the shape parameter $w = \frac{1}{4}$. Next year, Mustafa et al. [7] presented 6-point

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binary approximating subdivision schemes which have $C^6$ continuity for the shape parameter $-1.6 < w < 1.3$. In 2012, Ghaffar and Mustafa [9] presented a family of even point ternary approximating subdivision schemes. In 2014, Zheng et al. [19] presented a technique to increase the continuity of any subdivision scheme. They increase the continuity by multiplying the symbol with $(\frac{1+2}{2})^k$ factor, after multiplication they get $C^{k+n}$ continuous subdivision schemes. But this technique has following demerits, when they multiply symbol with factor then complexity of subdivision schemes are increased, support of subdivision schemes are increased and the mathematical computation of subdivision schemes are also increased.

Convexity preserving is a hot topic in curve design. A lot of work have been published on this topic during the past decades. Tan et al. [20] discussed the convexity preservation of five point binary approximating subdivision schemes. Siddiqi and Noreen [21] presented convexity preservation of six point ternary interpolating subdivision schemes.

1.1. Contribution and findings. The study of probability distribution is the analysis of certain games of chance, and it has found applications in most branches of science and engineering. Probability distribution have two types, discrete and continuous. Binomial probability distribution is a type of discrete distribution that was introduced by J. Bernoulli [22]. In this paper, first time Binomial distribution has been used to introduced subdivision schemes. The effects of probability parameter of subdivision schemes have been discussed. Furthermore, we have following contribution and findings:

- A new family of binary approximating subdivision schemes for curve design with high continuity.
- The maximum continuity, Hölder regularity and degree of generation can be obtained at probability parameter $p = \frac{1}{2}$.
- The primal and dual subdivision schemes can be obtained at different values of the probability parameter.
- Subdivision scheme give skewed behavior on convex data due to probability parameter.

The paper is organized as follows. In Section 2, we present construction of family of binary approximating subdivision schemes. Analysis of the proposed family is presented in Section 3. Convexity preservation is discussed in Section 4. In Section 5, we present the comparison and applications of proposed family of subdivision schemes. Conclusions are drawn in Sections 6.
2. Construction of family of subdivision schemes

In this section, we will elaborate our technique for the generation of new family of schemes. For this, we consider 4-point scheme (B-Spline of degree 5) [1] given by

\[
f_{2i}^{k+1} = \frac{6}{32} f_{i-1}^k + \frac{20}{32} f_i^k + \frac{6}{32} f_{i+1}^k, \tag{1}
\]

\[
f_{2i+1}^{k+1} = \frac{1}{32} f_{i-1}^k + \frac{15}{32} f_i^k + \frac{15}{32} f_{i+1}^k + \frac{1}{32} f_{i+2}^k.
\]

The Laurent polynomial of scheme (1) is

\[
K(z) = \frac{1}{32} (1 + 6z + 15z^2 + 20z^3 + 15z^4 + 6z^5 + z^6). \tag{2}
\]

Binomial probability generating function [2] is given as

\[
A_n(z) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} z^i, \quad \text{where} \quad i = 0, 1, 2, \ldots, n. \tag{3}
\]

\(n\) is the number of trails, \(p\) is the probability of success, \(q = 1 - p\) is the probability of failure with \(p + q = 1\). The general symbol of new family of binary approximating subdivision schemes is defined as

\[
M_{pn}(z) = K(z) A_n(z),
\]

where \(K(z)\) defined in (2) and \(A_n(z)\) defined in (3). Simplest form of \(M_{pn}(z)\) is

\[
M_{pn}(z) = \frac{1}{32} (1 + z)^6 \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} z^i. \tag{4}
\]

After substituting different values of \(n\) in (4), we obtain the symbols of family members with \(p \in (0, 1)\). Table 1 gives the complexity and mask of proposed family of approximating subdivision schemes.

3. Analysis of 6-point scheme

Aim of this section is to present the complete analysis of one family member \(M_{pn}\) of proposed family of subdivision schemes. Here we present some important properties such as continuity analysis, Hölder regularity, degree of generation and degree of reproduction. We also discuss the convexity preservation of \(M_{pn}\).
Table 1. Shows the mask of family of binary approximating subdivision schemes corresponding to different values of $n$, here $m$ shows complexity of the schemes (i.e. 4-, 5-, . . . -point schemes).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>Mask</th>
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<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>$M_{p_1} = \frac{1}{16}[(1 - p), (6 - 5p), (15 - 9p), (20 - 5p), (15 + 5p), (6 + 9p), (1 + 5p), p]$</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>$M_{p_2} = \frac{1}{16}[(1 - p)^2, (6 - 10p + 4p^2), (15 - 18p + 4p^2), (20 - 10p - 4p^2), (15 + 10p - 10p^2), (6 + 18p - 4p^2), (1 + 10p + 4p^2), (2p + 4p^2), p^2]$</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>$M_{p_5} = \frac{1}{16}[(1 - p)^5, (p^5 + 5p^4), (-5p^5 + 10p^4 + 10p^3), (-5p^5 - 15p^4 + 30p^3 + 10p^2), (10p^5 - 40p^4 + 40p^3 + 10p^2), (10p^5 + 10p^4 - 80p^3 + 40p^2 + 25 + 1), (-10p^5 + 60p^4 - 10p^3 + 40p^2 + 45 + 6), (-10p^5 + 10p^4 + 60p^3 - 100p^2 + 25p + 15), (5p^5 - 40p^4 + 80p^3 - 100p^2 - 25p + 20), (5p^5 - 15p^4 + 40p^2 - 45p + 15), (-p^5 + 10p^4 - 30p^3 + 40p^2 - 25p + 6), (-p^5 + 5p^4 - 10p^3 + 10p^2 - 5p + 1)]$</td>
</tr>
</tbody>
</table>

After substituting $n = 5$ in (4), we get the Laurent polynomial $M_{p_5}(z)$ of 6-point binary approximating subdivision scheme

$$M_{p_5}(z) = \frac{1}{32z^5} \left[ p^5 + (p^5 + 5p^4)z + (-5p^5 + 10p^4 + 10p^3)z^2 + (-5p^5 - 15p^4 + 30p^3 + 10p^2)z^3 + (10p^5 - 40p^4 + 40p^3 + 5p)z^4 + (10p^5 + 10p^4 - 80p^3 + 40p^2 + 25 + 1)z^5 + (-10p^5 + 60p^4 - 10p^3 + 40p^2 + 45 + 6)z^6 + (10p^5 + 10p^4 + 60p^3 - 100p^2 + 25p + 15)z^7 + (5p^5 - 40p^4 + 80p^3 - 100p^2 - 25p + 20)z^8 + (-p^5 + 10p^4 - 30p^3 + 40p^2 - 25p + 6)z^9 + (-p^5 - 40p^2 - 25p + 20)z^{10} + (5p^5 + 5p^4 - 10p^3 + 10p^2 - 5p + 1)z^{11} \right]$$

(5)
The scheme corresponding to the symbol (5) is

$$f_{2i}^{k+1} = \frac{1}{32} \left[ (p^5 + 5p^4)f_{i-2}^k + (p^5 - 15p^4 + 30p^3 + 10p^2)f_{i-1}^k + (10p^5 + 10p^4 - 80p^3 + 40p^2 + 25p + 1) f_i^k + (-10p^5 + 10p^4 + 60p^3 - 100p^2 + 25p + 15) f_{i+1}^k + (5p^5 - 15p^4 + 40p^2 - 45p + 15) f_{i+2}^k + (-p^5 + 5p^4 - 10p^3 + 10p^2 - 5p + 1) f_{i+3}^k \right].$$

$$f_{2i+1}^{k+1} = \frac{1}{32} \left[ p^5 f_{i-2}^k + (-p^5 + 10p^4 + 10p^3) f_{i-1}^k + (10p^5 - 40p^4 + 40p^2 + 5p) f_i^k + (-10p^5 + 60p^4 - 60p^3 - 40p^2 + 45p + 6) f_{i+1}^k + (5p^5 - 40p^4 + 80p^3 - 40p^2 - 25p + 20) f_{i+2}^k + (-p^5 + 10p^4 - 30p^3 + 40p^2 - 25p + 6) f_{i+3}^k \right].$$

(6)

**Theorem 1.** The 6-point binary approximating subdivision scheme (6) is $C^5$ continuous.

**Proof.** From (5), we have

$$M_{p_5}(z) = \left( \frac{1 + z}{2} \right)^5 b_5(z), \text{ with } b_5(z) = (1 + z)c_5(z),$$

where

$$c_5(z) = \frac{1}{2^5} \left[ z^5 + (5p^4 - 5p^5)z + (10p^5 - 20p^4 + 10p^3)z^2 + (-10p^5 - 30p^3 + 10p^2 + 30p^4)z^3 + (5p^5 - 20p^4 + 30p^3 - 10p^2 + 5p)z^4 + (-p^5 + 5p^4 - 10p^3 + 10p^2 - 5p + 1)z^5 \right].$$

To prove $C^5$ continuity of scheme $M_{p_5}$, we need to show that the difference scheme $S_{b_5}$ is convergent. For this we have to show that $S_{c_5}$ is contractive. For contractiveness of $S_{c_5}$, we see that

$$||S_{c_5}||_\infty = \max \{16p^5 - 40p^4 + 40p^3 - 20p^2 + 5p, -16p^5 + 40p^4 - 40p^3 + 20p^2 - 5p + 1\}.$$

The condition for $||S_{c_5}||_\infty < 1$ is $p \in (0, 1)$. Therefore by [3], if $||S_{c_5}||_\infty < 1$, then $c_5(z)$ is contractive and $b_5(z)$ is convergent. If $b_5(z)$ is convergent then scheme (6) is $C^5$ continuous.

**Remark 1.** The 6-point binary approximating subdivision scheme (6) is $C^9$ continuous at $p = \frac{1}{2}$. Here we see that at $p = \frac{1}{2}$, our proposed family converts to B-splines of degree $n+5$. 

3.1. Generation and reproduction analysis. The subdivision scheme with symbol $M_{pn}(z)$ reproduces polynomials of degree $d$ with respect to the parameterizations $\tau = M_{pn}'(1)/2$ if and only if

$$M_{pn}^{(k)}(-1) = 0, \quad M_{pn}^{(k)}(1) = 2 \prod_{j=0}^{k-1} (\tau - j), \quad k = 0, 1, ..., d. \quad (7)$$

Polynomial reproduction of degree $d$ requires polynomial generation of degree $d$ [3].

**Theorem 2.** The degree of generation of 6-point binary approximating subdivision scheme (6) is 5.

**Proof.** From (5), we can re-write the Laurent polynomial as

$$M_{p5}(z) = (1 + z)^{5+1}b_{p5}(z),$$

where

$$b_{p5}(z) = \frac{1}{32z^5} \left[ p^5 + (5p^5 + 5p^4)z + (10p^5 - 20p^4 + 10p^2)z^3 + (-10p^5 + 30p^4 + 30p^3 - 20p^2 + 5p)z^4 + (-p^5 + 5p^4 - 10p^3 + 10p^2 - 5p + 1)z^5 \right].$$

Hence degree of generation is 5. \qed

**Theorem 3.** The 6-point binary approximating subdivision scheme (6) has linear reproduction with parametrization depends on value of $p$.

**Proof.** By taking the first derivative of $M_{p5}(z)$ with respect to $z$ defined in (5). After putting $z = 1$, we get $(M_{p5})'(1) = 6 - 10p$. This implies that $\tau = \frac{6 - 10p}{2}$, for different values of $p$ the scheme corresponding to the symbol $M_{p5}(z)$ has dual as well as primal parametrization. We can easily verify that the first and second derivatives of $M_{p5}(z)$ at $z = -1$ are equal to 0. Further, we can also verify (7) for $k = 0$ and 1. This complete the proof. \qed

In Table 2, we summarize the results of important properties of some family members of proposed family of binary approximating subdivision schemes. It is observed that high order continuity and generation degree can be achieved at $p = \frac{1}{2}$.

4. Convexity preservation

In this section, we will show the convexity preservation of the scheme $M_{p5}.$

**Theorem 4.** Suppose that the initial control points $\{p_i^0\}_{i \in \mathbb{Z}}$ are strictly convex, i.e. $d_i^0 > 0, \forall i \in \mathbb{Z}$. Let $d_i^k = 2^{2k+1}(f_{i-1}^k - 2f_i^k + f_{i+1}^k)$ be defined as second
Using second order divided difference formula, order differences, \( r_i^k = \frac{d_i^{k+1}}{d_i^{k}} \), \( R^k = \max \{r_i^k, \frac{1}{r_i^k}\} \), \( k \geq 0 \), \( k \in \mathbb{Z}, i \in \mathbb{Z} \), the parameter \( p \) satisfies \( 0 < p < 1 \) and \( \lambda \in \mathbb{R} \), where \( 0 < \lambda \leq \frac{4p^5 - 20p^4 + 20p^2 - 5p + 2}{4p^3 - 20p^2 + 20p^2 - 5p} \) and \( \frac{1}{\lambda} \leq R^0 \leq \lambda \). If \( d_i^k > 0 \), \( \frac{1}{\lambda} \leq R^k \leq \lambda, \forall \ k \geq 0, k \in \mathbb{Z}, i \in \mathbb{Z} \), then 6-point binary approximating subdivision scheme (6) is convexity preserving.

**Proof.** Using second order divided difference formula,

\[
d_i^{k+1} = 2^{2k+1}(f_{i-1}^{k+1} - 2f_i^{k+1} + f_{i+1}^{k+1}).
\]

By using (9), the scheme (6) become

\[
d_{2i}^{k+1} = \frac{1}{4}\left[p^5d_{i-2}^k + (-4p^5 + 10p^3)d_{i-1}^k + (6p^5 - 30p^3 + 20p^2 + 5p)d_i^k + (-4p^5 + 30p^3 - 40p^2 + 10p + 4)d_{i+1}^k + (p^5 - 10p^3 + 20p^2 - 15p + 4)d_{i+2}^k\right].
\]

\[
d_{2i+1}^{k+1} = \frac{1}{4}\left[(-p^5 + 5p^4)d_{i-1}^k + (4p^5 - 20p^4 + 10p^3 + 10p^2)d_i^k + (-6p^5 + 30p^3 - 10p^2 + 15p + 1)d_{i+1}^k + (4p^5 - 20p^4 + 30p^3 - 10p^2 - 10p + 6)d_{i+2}^k + (p^5 - 5p^4 + 10p^3 + 10p^2 - 5p + 1)d_{i+3}^k\right].
\]

We use mathematical induction to prove \( d_i^k > 0 \) and \( \frac{1}{\lambda} \leq R^k \leq \lambda \). When \( k = 0 \), it is obvious from statement \( d_i^0 > 0 \) and \( \frac{1}{\lambda} \leq R^0 \leq \lambda \). Suppose \( d_i^k > 0 \) and \( \frac{1}{\lambda} \leq R^k \leq \lambda \) is true for \( k \). Now we prove it for \( k + 1 \). By re-writing equation (10) as

**Table 2. Summary of the results of important properties of proposed family of schemes, here \( n \), \( M_{p_n} \), GD, RD, OC, and \( HR \), represent trial number, proposed schemes, degree of generation, degree of reproduction, order of continuity and Hölder regularity of proposed schemes at \( p = \frac{1}{2} \) respectively.**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( M_{p_n} )</th>
<th>GD</th>
<th>RD</th>
<th>OC</th>
<th>( OC_{\frac{3}{2}} )</th>
<th>( GD_{\frac{3}{2}} )</th>
<th>( HR_{\frac{3}{2}} )</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>1</td>
<td>( C^0 )</td>
<td>( C^0 )</td>
<td>6</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>1</td>
<td>( C^0 )</td>
<td>( C^0 )</td>
<td>7</td>
<td>7</td>
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</tr>
<tr>
<td>3</td>
<td>5</td>
<td>1</td>
<td>( C^0 )</td>
<td>( C^0 )</td>
<td>8</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>1</td>
<td>( C^0 )</td>
<td>( C^0 )</td>
<td>9</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>1</td>
<td>( C^0 )</td>
<td>( C^0 )</td>
<td>10</td>
<td>10</td>
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</tr>
</tbody>
</table>
\[ d^{k+1}_{2i} = \frac{d^k}{4} \left[ (-8p^5 + 40p^3 - 40p^2 + 10p + 4)\lambda^2 - (-8p^5 + 40p^3 - 40p^2 + 10p - 4)\lambda \right], \quad (12) \]

\[ d^{k+1}_{2i} > 0, \text{ for } 0 < \lambda \leq \frac{4p^5 - 20p^3 + 20p^2 - 5p + 2}{4p^5 - 20p^3 + 20p^2 - 5p - 2}, \text{ with } p \in (0, 1). \]

Now consider equation (11) to show \( d^{k+1}_{2i+1} > 0 \).

\[ d^{k+1}_{2i+1} = \frac{d^k}{4} \left[ (-p^5 + 5p^4) \frac{d_{k-1}}{d_i} + (4p^5 - 20p^4 + 10p^3 + 10p^2) \frac{d_{k+1}}{d_i} + (-6p^5 \right. \]
\[ + 30p^4 - 30p^3 - 10p^2 + 15p + 1) \frac{d_{k+1}}{d_i} + (4p^5 - 20p^4 + 30p^3 - 10p^2 \right. \]
\[ - 10p + 6) \frac{d_{k+2}}{d_i} + (-p^5 + 5p^4 - 10p^3 + 10p^2 - 5p + 1) \frac{d_{k+3}}{d_i} \right]. \]

\[ d^{k+1}_{2i+1} > 0 \text{ for } 0 < \lambda \leq \frac{4p^5 - 20p^3 + 20p^2 - 5p - 1}{4p^5 - 20p^3 + 20p^2 - 5p + 3}, \text{ with } p \in (0, 1). \]

Thus \( d^{k+1}_{i} > 0, \forall \ k \geq 0, \ k \in \mathbb{Z}, \ i \in \mathbb{Z} \). To prove \( \frac{1}{\lambda} \leq R^{k+1} \leq \lambda, \forall \ k \geq 0 \) it suffices to prove that

\[ \frac{1}{\lambda} < r^{k+1}_i < \lambda, \ \forall i \in \mathbb{Z}, \forall k \geq 0, \ k \in \mathbb{Z}, \ i \in \mathbb{Z}. \]

Since we know that \( r^{k+1}_{2i} = \frac{d^{k+1}_{2i+1}}{d^{k+1}_{2i+}}, \)

\[ r^{k+1}_{2i} - \lambda = \frac{d^{k+1}_{2i+1}}{d^{k+1}_{2i+}} - \lambda = \frac{d^{k+1}_{2i+1} - \lambda d^{k+1}_{2i+1}}{d^{k+1}_{2i+}} = \frac{A}{B}. \]

Let \( r^{k+1}_{2i} - \lambda = \frac{A}{B} \). From (10), we have already proved that \( d^{k+1}_{2i} > 0 \) so, \( B > 0 \) and the numerator satisfies,

\[ A = r^{k+1}_{i-2}(-p^5 + 5p^4 + \lambda 4p^5 - \lambda 10p^3) + r^{k+1}_{i-1}r^{k+1}_{i-2}(4p^5 - 20p^4 \]
\[ + 10p^3 + 10p^2 - \lambda 6p^5 + \lambda 30p^3 - \lambda 20p^2 - \lambda 5p) + r^{k+1}_{i-2}r^{k+1}_{i-1}r^{k+1}_{i}(-6p^5 + 30p^4 \]
\[ - 30p^3 - 10p^2 + 15p + 1 + \lambda 4p^5 - \lambda 30p^3 + \lambda 40p^2 - \lambda 10p - 4\lambda \)
\[ + r^{k+1}_{i-2}r^{k+1}_{i-1}r^{k+1}_{i}(-6p^5 + 20p^4 + 30p^3 - 10p^2 - 10p + 6 - p^5 \lambda + \lambda 10p^3 \]
\[- \lambda 20p^2 + \lambda 15p - 4 \lambda) - p^5 \lambda, \]
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\[ A = -p^5 \lambda + r^k_{i-2}(-p^5 + 5p^4 + \lambda 4p^5 - \lambda 10p^3) + r^k_{i-1} r^k_{i-2}(4p^5 - 20p^4 + 10p^3 + 10p^2 - \lambda 6p^5 + \lambda 30p^3 - \lambda 20p^2 - \lambda 5p) + \left\{ (-6p^5 + 30p^4 - 30p^3 - 10p^2 + 15p + 1 + \lambda 4p^5 - \lambda 30p^3 + \lambda 40p^2 - \lambda 10p - 4\lambda) + \frac{1}{\lambda} (4p^5 - 20p^4 + 30p^3 - 10p^2 - 10p + 6 \lambda p^5 + \lambda 10p^3 - \lambda 20p^2 + \lambda 15p - 4\lambda) \right\} r^k_{i-2} r^k_{i-1} r^k_i, \]

\[ A = -p^5 \lambda + s^k_{i-2}(-p^5 + 5p^4 + \lambda 4p^5 - \lambda 10p^3) + s^k_{i-1} s^k_{i-2}(4p^5 - 20p^4 + 10p^3 + 10p^2 - \lambda 6p^5 + \lambda 30p^3 - \lambda 20p^2 - \lambda 5p) + \left\{ (-\lambda 6p^5 + \lambda 30p^4 - \lambda 30p^3 - \lambda 10p^2 + \lambda 15p + \lambda^2 4p^5 - \lambda^2 30p^3 + \lambda^2 40p^2 - \lambda^2 10p - 4\lambda^2) + 4p^5 - 20p^4 + 30p^3 - 10p^2 - 10p + 6 \lambda p^5 + \lambda 10p^3 - \lambda 20p^2 + \lambda 15p - 4\lambda, \right\} \]

\[ A = \lambda^2 (8p^5 - 40p^3 + 40p^2 - 10p - 4) + \lambda (-15p^5 + 35p^4 + 10p^3 - 50p^2 + 10p - 3) + (8p^5 - 40p^4 + 40p^3 - 10p + 6). \quad (14) \]

This implies that \( A < 0, \) for \( 0 < \lambda \leq \frac{4p^5 - 20p^4 + 20p^3 - 5p - 1}{4p^5 - 20p^4 + 20p^3 - 5p + 3}. \)

Thus \( r^{k+1}_{2i} - \lambda = \frac{4}{D} < 0. \) Therefore \( r^{k+1}_{2i} < \lambda, \forall i \in Z. \)

Similarly

\[ \frac{1}{r^{k+1}_{2i}} - \lambda = \frac{d^{k+1}_{2i+1}}{d^{k+1}_{2i+1}} - \lambda = \frac{d^{k+1}_{2i+1} - \lambda d^{k+1}_{2i+1}}{d^{k+1}_{2i+1}} = \frac{C}{D}. \]

From (11), we have already proved that \( d^{k+1}_{2i+1} > 0 \) so, \( D > 0 \) and the numerator satisfies,

\[ C = p^5 + r^k_{i-2}(-4p^5 + 10p^3 + \lambda p^5 - \lambda 5p^4) + r^k_{i-1} r^k_{i-2}(6p^5 - 30p^3 + 20p^2 + 5p - \lambda 4p^5 + \lambda 20p^4 - \lambda 10p^3 - \lambda 10p^3 + \lambda 6p^5 - \lambda 30p^3 + \lambda 40p^2 - \lambda 10p - 4\lambda) + r^k_{i-2} r^k_{i-1} (2p^5 - 5p^4 + 10p^2 - 10p + 3 - 4p^5 + 30p^3 + \lambda 10p^2 + 10\lambda p - 6\lambda), \]
Thus $p > M$ and

$$C = p^5 + r_{i-2}^k(-4p^5 + 10p^3 + \lambda p^5 - \lambda 5p^4) + r_{i-1}^k r_{i-2}^k(6p^5 - 30p^3 + 20p^2 + 5p - \lambda 4p^5 + \lambda 20p^4 - \lambda 10p^3 - \lambda 10p^2) + (-4\lambda p^5 + 30\lambda p^3 - 40\lambda p^2 + 10p\lambda + 4\lambda + \lambda^2 6p^5 - \lambda^2 30p^4 + \lambda^2 30p^3 + \lambda^2 10p^2 - 15\lambda^2 p - \lambda^2 + 2p^5 - 5p^4 + 10p^2 - 10p + 3 - 4p^5 \lambda + \lambda 20p^4 - \lambda 30p^3 + \lambda 10p^2 + 10\lambda p - 6\lambda) r_{i-2}^k r_{i-1}^k r_i^k,$$

$$C = p^5 + r_{i-2}^k(-4p^5 + 10p^3 + \lambda p^5 - \lambda 5p^4) + r_{i-1}^k r_{i-2}^k(4\lambda p^5 - 30\lambda p^3 + 20\lambda p^2 + 5\lambda p - 4\lambda - 4\lambda^2 p^5 + \lambda 20p^4 - 10\lambda^2 p^3 - 10\lambda^2 p^2 - 4\lambda p^5 + 30\lambda p^3 - \lambda 40p^2 + \lambda 10p + 4\lambda + 6\lambda^2 p^5 - \lambda^2 30p^4 + \lambda^2 30p^3 + \lambda^2 10p^2 - 15\lambda^2 p - \lambda^2 + 2p^5 - 5p^4 + 10p^2 - 10p + 3 - 4p^5 \lambda + \lambda 20p^4 - \lambda 30p^3 + \lambda 10p^2 + 10\lambda p - 6\lambda),$$

$$C = \lambda^2(3p^5 - 15p^4 + 20p^3 - 15p - 1) + \lambda(-6p^5 + 20p^4 - 20p^3 - 10p^2 + 25p - 6) + (3p^5 - 5p^4 + 10p^2 - 10p + 3).$$

This implies that $C < 0$ for $0 < \lambda \leq \frac{4p^5 - 20p^4 + 20p^3 - 5p - 1}{4p^5 - 20p^4 + 20p^3 - 5p + 3}$. Thus $\frac{1}{\lambda} - \lambda = \frac{C}{D} < 0$. Combining (14) and (15), it can be written as

$$\frac{1}{\lambda} \leq r_{2i+1}^k \leq \lambda, \quad \forall i \in \mathbb{Z}.$$ 

Similarly we can show that

$$\frac{1}{\lambda} \leq r_{2i+1}^k \leq \lambda, \quad \forall i \in \mathbb{Z}.$$ 

Hence $d_i^k > 0$ and $\frac{1}{\lambda} \leq R_k \leq \lambda$ is satisfied for $k + 1$. Hence proposed scheme $M_{p_i}$ preserves convexity. This completes the proof. 

It is clear from Figure 1 that if initial control points are strictly convex. Then the limit curves generated by scheme corresponding to $M_{p_1}$ show negatively, normal and positively skewed behavior on convex data for $p < \frac{1}{2}, p = \frac{1}{2}$ and $p > \frac{1}{2}$ respectively. But limit curves generated by scheme corresponding to $M_{p_2}$ show positive, normal and negatively skewed behavior on convex data for $p < \frac{1}{2}, p = \frac{1}{2}$ and $p > \frac{1}{2}$ respectively.
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Figure 1. (a) – (c) and (d) – (f) present limit curves for convex data produced by the schemes corresponding to $M_{p_1}$ and $M_{p_5}$ respectively.

5. Comparison and Applications

In this section, we will present the comparison and applications of proposed family of schemes.

5.1. Comparison of Continuity Analysis. Here we will present the comparison of continuity analysis of proposed family of schemes with existing parametric subdivision schemes. We also show that our proposed family of binary approximating subdivision schemes gives higher continuity as compare to existing parametric subdivision schemes. Table 3 present the comparison of proposed family of approximating subdivision schemes.

5.2. Applications. Here we will discussed visual performance of proposed family of subdivision schemes. The control polygons are drawn by doted line and the smooth curves obtained by our proposed schemes by full line. Figures 2 present limit curves for close polygons. Figures 3 present comparison of limit curves for close polygons produced by the schemes corresponding to $M_{p_1}$ and $M_{p_5}$ at $p = \frac{1}{2}$. 
Table 3. Shows the comparison of continuity analysis of existing parametric subdivision schemes and proposed family of schemes. $E$, $OC$, $C.P$, $Mp_n$ and $OC_{\frac{1}{2}}$ denote the existing schemes, continuity of existing schemes, Convexity preservation, proposed family of schemes and continuity of proposed schemes at $p = \frac{1}{2}$ respectively.

<table>
<thead>
<tr>
<th>E</th>
<th>$OC$</th>
<th>C.P of E</th>
<th>$Mp_n$</th>
<th>$OC_{\frac{1}{2}}$</th>
<th>C.P of</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-point [16]</td>
<td>$C^2$</td>
<td>No</td>
<td>$Mp_1$</td>
<td>$C^6$</td>
<td>No</td>
</tr>
<tr>
<td>4-point [11]</td>
<td>$C^4$</td>
<td>No</td>
<td>$Mp_1$</td>
<td>$C^5$</td>
<td>No</td>
</tr>
<tr>
<td>4-point [12]</td>
<td>$C^5$ at $w = \frac{1}{4}$</td>
<td>No</td>
<td>$Mp_1$</td>
<td>$C^5$</td>
<td>No</td>
</tr>
<tr>
<td>5-point [5]</td>
<td>$C^4$ at $w = \frac{1}{4}$</td>
<td>No</td>
<td>$Mp_3$</td>
<td>$C^7$</td>
<td>No</td>
</tr>
<tr>
<td>5-point [13]</td>
<td>$C^5$ at $w = \frac{125}{128}$</td>
<td>No</td>
<td>$Mp_3$</td>
<td>$C^7$</td>
<td>No</td>
</tr>
<tr>
<td>5-point [10]</td>
<td>$C^4$ at $\alpha = 1, \beta = 32$</td>
<td>Yes</td>
<td>$Mp_3$</td>
<td>$C^7$</td>
<td>No</td>
</tr>
<tr>
<td>6-point [9]</td>
<td>$C^7$ at $w = \frac{1}{12}$</td>
<td>No</td>
<td>$Mp_3$</td>
<td>$C^9$</td>
<td>Yes</td>
</tr>
<tr>
<td>6-point [7]</td>
<td>$C^8$ if $w \in (-1.6, 1.3)$</td>
<td>No</td>
<td>$Mp_5$</td>
<td>$C^9$</td>
<td>Yes</td>
</tr>
<tr>
<td>6-point [8]</td>
<td>$C^3$</td>
<td>No</td>
<td>$Mp_5$</td>
<td>$C^9$</td>
<td>Yes</td>
</tr>
<tr>
<td>6-point [14]</td>
<td>$C^2$</td>
<td>No</td>
<td>$Mp_5$</td>
<td>$C^9$</td>
<td>Yes</td>
</tr>
<tr>
<td>6-point [15]</td>
<td>$C^2$</td>
<td>No</td>
<td>$Mp_5$</td>
<td>$C^9$</td>
<td>Yes</td>
</tr>
<tr>
<td>6-point [6]</td>
<td>$C^6$ at $w = \frac{1}{4}$</td>
<td>No</td>
<td>$Mp_3$</td>
<td>$C^9$</td>
<td>Yes</td>
</tr>
</tbody>
</table>

6. Conclusion

In this paper, we have presented a technique to increase the continuity of binary approximating subdivision schemes and generate a new family of approximating subdivision schemes. We present the complete analysis of some members of the family of schemes. We also analyzed that our proposed family gives high continuity when probability parameter $p = \frac{1}{2}$ as compare to the existing subdivision schemes. At all other values of $p$ the results of subdivision schemes remain same. Visual performances of proposed family of subdivision schemes are also discussed. Subdivision schemes show skewed behavior on convex data for different values of $p$.

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This work is supported by National Research Program for Universities (NRPU) P. No. 3183, Pakistan.
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\[
\begin{align*}
(a) \, p &= \frac{1}{5} \\
(b) \, p &= \frac{1}{3} \\
(c) \, p &= \frac{1}{2} \\
(d) \, p &= \frac{4}{10} \\
(e) \, p &= \frac{6}{10} \\
(f) \, p &= \frac{9}{10}
\end{align*}
\]

**Figure 2.** (a) – (c) and (d) – (f) present limit curves for close polygons produced by scheme corresponding to \( M_{p_5} \) and \( M_{p_1} \) respectively.

\[
\begin{align*}
(a) \\
(b) \\
(c)
\end{align*}
\]

**Figure 3.** (a), (b) and (c) present limit curves for close polygons produced by scheme corresponding to \( M_{p_3} \) and \( M_{p_5} \) at \( p = \frac{1}{7} \).

**References**
