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SOLVING DIFFERENTIAL EQUATIONS BY WAVELET TRANSFORM METHOD BASED ON THE MOTHER WAVELETS & DIFFERENTIAL INVARIANTS

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Abstract. Nowadays, wavelets have been widely used in various fields of science and technology. Meanwhile, the wavelet transforms and the generation of new Mother wavelets are noteworthy. In this paper, we generate new Mother wavelets and analyze the differential equations by using of their corresponding wavelet transforms. This method by Mother wavelets and the corresponding wavelet transforms produces analytical solutions for PDEs and ODEs.

Key words: Wavelet, Mother wavelet, Wavelet transform, Differential invariant, Degree reduction
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1. Introduction

In mathematics, the combined techniques have been appeared very successful for analyzing problems. In this paper, we propose a new method based on the wavelet transforms and differential invariants. The first is related to the wavelet theory, and the latter uses Lie groups for analyzing differential equations. Hence, we begin by introducing a history of Lie groups.

At the end of the nineteenth century, Sophus Lie introduced the notion of Lie group in order to study the solutions of ordinary differential equations (ODEs). He showed that the order of an ordinary differential equation can

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be reduced by one if it is invariant under one-parameter Lie group of point transformations. This observation unified and extended the available integration techniques. Lie devoted the remainder of his mathematical career to developing these continuous groups that have now an impact on many areas of mathematically based sciences. The applications of Lie groups to differential equations and systems were mainly established by Lie and Emmy Noether and then advocated by lie Cartan [2]. Such Lie groups are invertible point transforms of both the dependent and independent variables of the differential equations. The symmetry group method provides ultimate tools for analyzing differential equations and the important points are to understand and to construct solutions of differential equations. Many examples of applications of Lie groups in the theory of differential equations were discussed in the literature, the most important of them are: reduction of order of ODEs, construction of invariant solutions, mapping solutions to other solutions and the detection of linearizing transforms (for some other applications of Lie symmetries see [1], [2]).

The wavelets are important functions in the functional and harmonic analysis. The first wavelet was introduced by the David Hilbert 's Ph.D. student, Alfréd Haar (Hungarian Mathematician) in 1909 [5]. Nowadays, wavelets have numerous applications in many fields of science and technology, such as seismology, image processing, signal processing, coding theory, biosciences, financial mathematics, fractals and so on [1]. In the many of applications of wavelets for solving differential equations, the numerical solutions in special conditions were obtained. Meanwhile, wavelets with two or more variables are very important. In this paper, we make some new Mother wavelets with two variables, these wavelets depend on the differential invariants of differential equations. Therefore, we can use their correspondent transforms for solving differential equations. This method called Wavelet Transform Method (WTM) based on the Mother wavelets. We will show the performance of WTM with an example. Then, the obtained solutions by WTM will be compared with other methods such as the Lie symmetry method.

The remainder of the paper organized as follows. In section 2, we recall some needed preliminaries to construct the differential invariants, Mother wavelets and wavelet transforms. In section 3, the wavelet transform method based on the Mother wavelets is proposed. In sections 4, WTM is demonstrated by an example. Finally, the conclusions and future works are presented.

2. Preliminaries

In this section, we recall some needed results to construct differential invariants, the Mother wavelets, and their transforms. First, we remember the

Lie symmetry method that can be applied for obtaining the differential invariants and reducting the order of PDEs. After that, the wavelets and their transforms are discussed. The related definitions and theorems are considered.

2.1. **The Lie symmetry method.** In this section, we recall some needed concepts and results from the Lie theory and the general procedure for determining symmetries for any system of partial differential equations (PDEs) (see [8] and [9]). To begin, let us consider

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l, \tag{1}$$

as the general case of a nonlinear system of partial differential equations of order nth in p independent and q dependent variables involving $x = (x^1, \cdots, x^p)$, $u = (u^1, \cdots, u^q)$ and the derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n. We consider a one-parameter Lie group of infinitesimal transforms acting on the independent and dependent variables of the system (1):

$$(\tilde{x}^i, \tilde{u}^j) = (x^i, u^j) + s(\xi^i, \eta^j) + O(s^2), \quad i = 1 \cdots, p, \ j = 1 \cdots, q,$$

where s is the parameter of the transform and ξ^i, η^j are the infinitesimals of the transforms for the independent and dependent variables, respectively. The infinitesimal generator $\mathbf{v}$ associated with this group of transforms can be written as $\mathbf{v} = \sum_{i=1}^p \xi^i \partial_{x^i} + \sum_{j=1}^q \eta^j \partial_{u^j}$, So a symmetry of a differential equation is a transform which maps solutions of the equation to other solutions. The invariance of the system (1) under the infinitesimal transforms leads to the invariance conditions (Theorem 2.36 of [9]):

$$\mathrm{Pr}^{(n)}\mathbf{v}\big[\Delta_\nu(x, u^{(n)})\big] = 0, \quad \Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l,$$

where $\mathrm{Pr}^{(n)}$ is called the n^{th} order prolongation of the infinitesimal generator given by $\mathrm{Pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^q \sum_J \phi_J^\alpha(x, u^{(n)})\partial_{u_J^\alpha}$, where $J = (j_1, \cdots, j_k)$, $1 \leq j_k \leq p$, $1 \leq k \leq n$ and the sum is over all J's of order $0 < \#J \leq n$. If $\#J = k$, the coefficient ϕ_J^α of $\partial_{u_J^\alpha}$ will only depend on k-th and lower order derivatives of u, and $\phi_\alpha^J(x, u^{(n)}) = D_J(\phi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha) + \sum_{i=1}^p \xi^i u_{J,i}^\alpha$, where $u_i^\alpha := \partial u^\alpha / \partial x^i$ and $u_{J,i}^\alpha := \partial u_J^\alpha / \partial x^i$.

In fact, these infinitesimal symmetries form a Lie algebra under the usual Lie bracket. The first useful result of symmetry group methods is to construct new solutions from known solutions. The second is when a nonlinear system of differential equations admits infinite symmetries, so it is possible to transform it to a linear system. Here, the symmetry group method will be applied to the PDE to be connected directly to some order differential equations. To do this, a particular linear combinations of infinitesimals are considered and their corresponding invariants are determined.

For every generator vector field, we can determine differential invariants corresponding to vector field by solving the following characteristics system

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\phi}$$

and reduce the order of PDE by expressing PDE in the coordinates (x, t, u). Those coordinates will be constructed by searching for independent invariants (y, v) corresponding to the infinitesimal generator. Hence by using the chain rule, the expression of the equation in the new coordinate allows us to the reduced equation. For more informations and examples, see [10].

2.2. **The wavelets.** In the mathematics and other scientific fields, the wavelets are very important functions. In this section, we introduce wavelets as functions belong to $L^2(\mathbb{R}^2)$ (the space of squared integrable functions with integral norm).

Definition 1. *A Function $\psi \in L^2(\mathbb{R}^2)$ is called a wavelet, if it satisfies in the follow admissible condition*

$$C_\psi = \int_{\mathbb{R}^2} \frac{|F(\psi)(\omega)|^2 d\omega}{|\omega|^2} > 0$$

where $F(\psi)(\omega)$ is the Fourier transform of wavelet ψ and as follows

$$F(\psi)(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp(-ix.\omega)\psi(x) d\omega$$

and C_ψ is called wavelet coefficient of ψ. Note that $\omega = (\omega_1, \omega_2)$ and $x = (x_1, x_2)$ belong to $\mathbb{R}^2$. For more details and examples, see [2].

Definition 2. *The wavelet ψ is called Mother wavelet, if it satisfies in the following properties*

$$\int_{\mathbb{R}^2} \psi(x) dx = 0, \tag{2}$$

$$\int_{\mathbb{R}^2} |\psi(x)|^2 dx = 1, \tag{3}$$

$$\lim_{|\omega| \to \infty} F(\psi(\omega)) = 0 \tag{4}$$

Indeed, the first property equivalents to $C_\psi > 0$, that is admissible condition for the Mother wavelet ψ. For more details see [5].

In fact, the Mother wavelets have admissible condition, n-zero moments and exponential decay properties. The Mother wavelet have two parameters:

the translation parameter $b = (b_1, b_2)$ and the scaling parameter $a > 0$. The Mother wavelet corresponding to (a, b) as follows

$$\psi_{a,b}(x) = \psi(\frac{x-b}{a}) = \psi(\frac{x_1 - b_1}{a}, \frac{x_2 - b_2}{a})$$

Definition 3. *The wavelet transform corresponding to the Mother wavelet ψ for function $f \in L^2(\mathbb{R}^2)$ with parameters (a, b) is defined as follow*

$$W_\psi(f)(a,b) = \frac{1}{|a|.C_\psi} \int_{\mathbb{R}^2} \psi_{a,b}(x).f(x)dx$$

Thus, wavelet transform depends on wavelet ψ, function f, and parameters (a, b).

Theorem 1. *The wavelet transform is an operator from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ that satisfies in following properties:*

1. *Linearity:* $W_\psi[\alpha f(x) + \beta g(x)] = \alpha W_\psi[f(x)] + \beta W_\psi[g(x)]$,
2. *Translation:* $W_\psi[f(x-k)] = W(a, b-k), k \in \mathbb{R}^2$,
3. *Scaling:* $W_\psi[\frac{1}{\sqrt{s}} f(\frac{x}{s})] = W(\frac{a}{s}, \frac{b}{s})$,
4. *Wavelet shifting:* $W_{\psi(x-k)}[f(x)] = W(a, b+ak)$,
5. *Linear combination:* $W_{\alpha\psi_1 + \beta\psi_2}[f(x)] = \alpha W_{\psi_1}[f(x)] + \beta W_{\psi_2}[f(x)]$,
6. *Wavelet scaling:* $W_{\frac{\psi(x/s)}{\sqrt{(|s|)}}}[f(x)] = W(as, b)$.

Proof. for proof and more details, see [5]. □

In fact, the wavelet transforms corresponding Mother wavelets are isometries. Therefore, In smooth manifold M, if $W(M)$ is a collection of wavelet transforms of M, then $W(M)$ is a Lie subgroup of $I(M)$ (the isometry group of M) [2].

The admissible condition implies that the wavelet transform is invertible. On the other hands, since the wavelet transform is an isometry, Thus it is invertible. The inversion formula for the wavelet transform $W_\psi(f)$ is

$$f(x) = f(x_1, x_2) = \frac{1}{C_\psi} \int_{\mathbb{R}^+ \times \mathbb{R}^n} W_\psi f(a,b)\, \psi_{a,b}(x) \frac{da\, db_1\, db_2}{a^3}$$

So, by inversion formula (also called the synthesis formula), the function $f(x)$ corresponds to the wavelet transform $W_\psi(f)$ will be obtained.

3. The wavelet transform method

The wavelet tranform method (WTM) have the following steps:

1 . Apply equivalence algorithms (for example, the Lie symmetry method) on the differential equation, and obtain differential invariants.

2 . Build the suitable Mother wavelets based on the differential invariants.

3 . Multiply the Mother wavelet in both sides of the equation and take the wavelet transform. Solve the reduced differential equation and obtain the wavelet transform.

4 . By the inversion formula, calculate the analytic solution.

In follow, some WTM formulas are proposed. First, we remember the Bonnet theorem, for proof and more details see [3].

Theorem 2 (*The Bonnet theorem*)**.** *If f, g are continuous functions on $[a, b]$ such that g does not change its sign on $[a, b]$, then there exist $c \in [a, b]$*

$$\int_a^b f(x).g(x)dx = f(c)\int_a^b g(x)dx.$$

Note that, all functions such as wavelets and solutions are smooth and Mother wavelets (based on their constructions) are compactly supported. Therefore, there exist $[a, b]$ in support and $c \in [a, b]$ that satisfies in the Bonnet theorem.

Theorem 3. *Assume $\Delta_\nu(x, t, u^{(m)}) = 0$ is m-th order differential equation with two independent variables (x, t) and ψ is a Mother wavelet based on the differential invariants (with t is constant and x is variable), we have:*

i) $W_\psi(\partial_t u)(x, t) = \frac{d}{dt}W_\psi(u)(x, t)$,

ii) $W_\psi(\partial_t^n u)(x, t) = \frac{d^n}{dt^n}W_\psi(u)(x, t)$,

iii) $W_\psi(\partial_x u)(x, t) = (\frac{\partial\psi}{\partial x}/\psi)W_\psi(u)(x, t)$,

iv) $W_\psi(\partial_x^n u)(x, t) = (-1)^n(\frac{\partial^n\psi}{\partial x^n}/\psi)W_\psi(u)(x, t)$.

Proof. We consider $a = 1, b = 0$ and prove (i)-(iv).

i) We have

$$\begin{aligned} W_\psi(\partial_t u)(x, t) &= \frac{1}{\sqrt{c_\psi}}\int u_t\psi dx = \frac{1}{\sqrt{c_\psi}}\int \lim_{h\to 0}\frac{u(x, t+h) - u(x, t)}{h}\psi dx \\ &= \lim_{h\to 0}\frac{1}{h}\{\frac{1}{\sqrt{c_\psi}}\int u(x, t+h)\psi dx - \frac{1}{\sqrt{c_\psi}}\int u(x, t)\psi dx\} \\ &= \lim_{h\to 0}\frac{\tilde{u}(x, t+h) - \tilde{u}(x, t)}{h} = \frac{d}{dt}\tilde{u}(x, t). \end{aligned}$$

where $\tilde{u}(x, t) = W_\psi(x, t)$.

ii) By following the induced above procedure (according to the derivation order of t (i.e. n)), we have

$$\begin{aligned} W_\psi(\partial_t^n u)(x, t) &= \frac{1}{\sqrt{c_\psi}}\int u_t^{(n)}\psi dx = \frac{1}{\sqrt{c_\psi}}\frac{d}{dt}\int\frac{\partial^{n-1}u}{\partial t^{n-1}}\psi dx \\ &= \frac{d}{dt}\{\frac{d}{dt}\int\frac{\partial^{n-2}u}{\partial t^{n-2}}\psi dx\} = ... = \frac{d^n}{dt^n}\tilde{u}(x, t). \end{aligned}$$

iii) We know that

$$W_\psi(\partial_x u)(x,t) = \frac{1}{\sqrt{c_\psi}}\int u_x\psi dx = (\frac{1}{\sqrt{c_\psi}}.\psi.u]_{-\infty}^{+\infty} - \frac{1}{\sqrt{c_\psi}}\int u\frac{\partial\psi}{\partial x}dx$$
$$= -\big(\frac{\frac{\partial\psi}{\partial x}}{\psi}\big)(c)\tilde{u}(x,t).$$

where (for caculating the integral) we use the integral by part with $u=\psi, dv=\frac{\partial u}{\partial x}$, also assume $\lim_{x,t\to\infty} u(x,t)=0$ and finally, we use the Bonnet theorem with

$$f(x) = \frac{\frac{\partial\psi}{\partial x}}{\psi},\ \ g(x) = u(x,t)\psi(x,t)$$

(assume $supp(g)=A.B$, such that A,B are closed intervals in $\mathbb{R}$) at $c\in A$.
iv) By following inducely above procedure (according to the derivation order of x), we get

$$W_\psi(\frac{\partial^n u}{\partial x^n})(x,t) = (\frac{1}{\sqrt{c_\psi}}.\psi\frac{d^{n-1}u}{dx^{n-1}}]_{-\infty}^{+\infty} - (\frac{1}{\sqrt{c_\psi}}.\frac{\partial\psi}{\partial x}.\frac{d^{n-2}u}{dx^{n-2}}]_{-\infty}^{+\infty}$$
$$+ ... + (-1)^n\big(\frac{\frac{\partial^n\psi}{\partial x^n}}{\psi}\big)(c)\tilde{u}(x,t).$$

hereafter, if

$$\lim_{x,t\to\infty} u(x,t) = \lim_{x,t\to\infty} u_x(x,t) = ... = \lim_{x,t\to\infty} u_x^{(n-1)}(x,t) = 0,$$

by the Bonnet theorem with

$$f(x) = \frac{\frac{\partial^n\psi}{\partial x^n}}{\psi},\ \ g(x) = u(x,t)\psi(x,t)$$

(assume $supp(g)=A.B$, such that A,B are closed intervals in $\mathbb{R}$) at $c\in A$, formula (iv) will be obtained. □

In fact, we take the wavelet transform from every side of $\Delta_\nu(x,t,u^{(m)})=0$ by the assumption $t=cte, x=variable, a=1, b=0$, then solve the reduced equation according to $\tilde{u}(x,t)$ and its t-derivations, obtain $\tilde{u}(x,t)$, here after, for given Mother wavelet $\psi(x,t)$ and obtained wavelet transform $\tilde{u}(x,t)$, we calculate $u(x,t)$ from the below formula ($1D$-inversion formula)

$$u(x,t) = \int_{\mathbb{R}} \tilde{u}(x,t)\psi(x,t)dx \tag{5}$$

$u(x,t)$ is desired analytic solution and Differential equation is solved by WTM according to ψ based on the differential invariants. In the following section, for example, we apply WTM on the heat equation.

4. Example

Here, we demonstrate WTM by example, we implement WTM on the heat equation and obtain solutions. Finally, WTM results will be proposed. First, we apply the Lie symmetry method on the heat equation $u_t = u_{xx}$ and obtain symmetry groups, vector fields and differential invariants (for more details and calculations, see [9]). The results of Lie symmetry method proposed in the following table:

Table 1. The Lie symmetry method: Symmetry groups, Vector fields, Dimension of Lie groups, Differential invariants

Symmetry groups	**Vector Field**	dim(g)	**Differential invariants**
Translation with factor (c)	$c\partial_x + \partial_t$	2	$x - ct, u$
Scaling with factor (a)	$x\partial_x + 2t\partial_t + 2au\partial_u,$	3	$(x/\sqrt{t}), (u/t^a)$
Galilean boost	$2t\partial_x - xu\partial_u$	2	$t, u\exp(x^2/4t)$

In table 1, the symmetry groups are translation, scaling and Galilean boost (respectively). In table 2, we offer the adequate Mother wavelet for every differential invariant and symmetry group.

Table 2. The Mother wavelets: symmetry groups, differetial invariants and Mother wavelets

Symmetry groups	**Differential Invariants**	**Mother wavelets**
Translation	$x - ct, u$	$(4/5)\exp(-(x^2+t^2)/2)\sin(\pi(x-ct)/2)$ $(4/5)\exp(-(x^2+t^2)/2)\cos(\pi(x-ct)/2)$
Scaling	$(x/t), (x/\sqrt{t}), (u/t^a)$	$\exp(-(x^2+15t^2)/20)\cos(x/t)\sin(x/t)$
Galilean boost	$t, u\exp(x^2/4t)$	$\exp(-(x^2+15t^2)/20)\cos(x/t)\sin(x/t)$

By computation, it can be seen that the offered functions have properties $(2) - (4)$ of the Mother wavelets. Figures 1-3 show the graphs of Mother wavelets and some properties are clear from these figures.

Now, we apply WTM on the heat equation. First, consider the Mother wavelet

$$\psi_1 := (4/5)\exp(-\frac{x^2+t^2}{2})\sin(\frac{\pi}{2}(x-2t))$$

obviously, $supp(\psi_1) = \{(x,t) \in [-\pi,\pi].[-\pi,\pi] \,|\, (x-2t) \neq 2k\} = A.B$. Then, by multiplying both sides of the heat equation in ψ_1 and taking the wavelet transform, we have

$$\frac{d}{dt}\tilde{u}(x,t) = (\frac{\partial^2}{\partial x^2}\psi_1)\tilde{u}(x,t)$$

by calculating the second derivation of ψ_1 and using of the Bonnet theorem, there exists $c \in A$ such that

$$\frac{d}{dt}\tilde{u}(x,t) = -(c^2 - \pi c - 3.5)\tilde{u}(x,t)$$

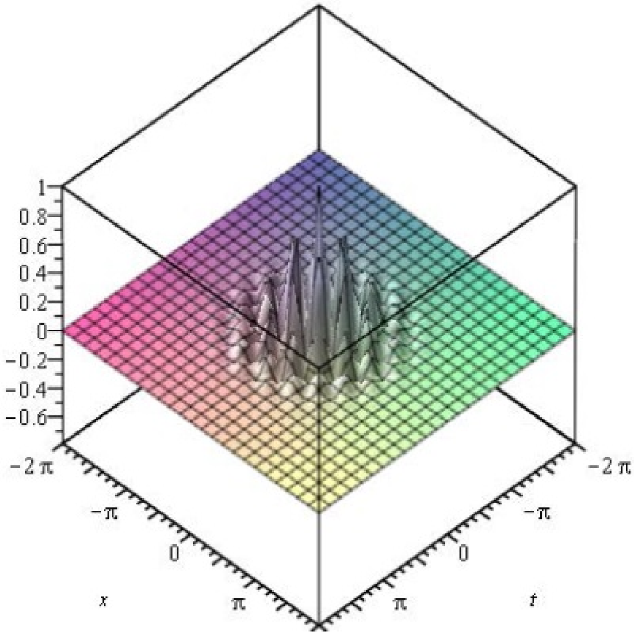

FIGURE 1. The graph of ψ_1

therefore

$$\tilde{u}(x,t) = F(x)\exp(-(c^2 - \pi c - 3.5)t) + K$$

where K is constant at $\mathbb{R}$ and $F(x)$ is a function of x that according to the initial or boundary conditions will be determined. Thus, the general analytic solution from (5) is

$$u(x,t) = (4/5)\int \big\{F(x)\exp(-(c^2-\pi c-3.5)t)\exp(-\frac{x^2+t^2}{2})\sin(\frac{\pi}{2}(x-2t))\big\}dx$$

Second, consider the Mother wavelet

$$\psi_2 := (4/5)\exp(-\frac{x^2+t^2}{2})\cos(\frac{\pi}{2}(x-2t))$$

Obviously, $supp(\psi_2) = \{(x,t) \in [-\pi,\pi].[-\pi,\pi] \,|\, (x-2t) \neq 2k+1\} = A.B$. Then by multiplying the both sides of heat equation in ψ_2 and taking the wavelet transform, we get

$$\frac{d}{dt}\tilde{u}(x,t) = (\frac{\partial^2}{\partial x^2}\psi_2)\tilde{u}(x,t)$$

by calculating the second derivation of ψ_2 and using of the Bonnet theorem, there exists $c \in A$ such that

$$\frac{d}{dt}\tilde{u}(x,t) = -(c^2 + \pi c - 3.5)\tilde{u}(x,t)$$

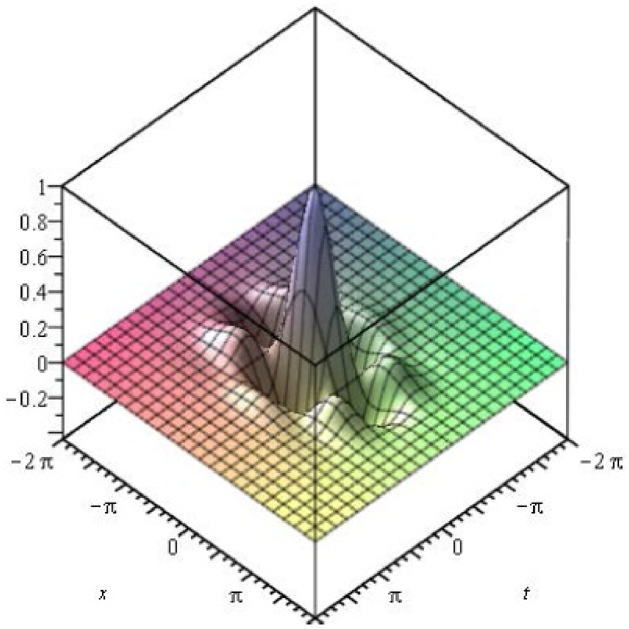

FIGURE 2. The graph of ψ_2

therefore

$$\tilde{u}(x,t) = F(x)\exp(-(c^2+\pi c-3.5)t)+K$$

where K is a real constant and $F(x)$ is a real function that can be determined according to the initial-boundary conditions. Thus, according to (5), the general analytic solution is the following

$$u(x,t) = \int \big\{F(x)\exp(-(c^2+\pi c-3.5)t)\exp(-\frac{x^2+t^2}{2})\cos(\frac{\pi}{2}(x-2t))\big\}dx$$

Third, let us ψ_3 as follows

$$\psi_3 := \exp(-\frac{x^2+15t^2}{20})\cos(\frac{x}{t})\sin(\frac{x}{t})$$

Obviously, $supp(\psi_3) = \{(x,t) \in [-2\pi, 2\pi].[\frac{-\pi}{2}, \frac{\pi}{2}] \mid \frac{x}{t} \neq \frac{k\pi}{2}\} = A.B$. After taking the wavelet transform from the both sides of heat equation under ψ_3, we have

$$\frac{d}{dt}\tilde{u}(x,t) = (\frac{\partial^2}{\partial x^2}\psi_3)\tilde{u}(x,t)$$

Now, with calculate the second derivation ψ_3 and use of the Bonnet theorem, there exist $c \in A$ such that

$$\frac{d}{dt}\tilde{u}(x,t) = -\frac{c^2t^2-40ct-10t^2-400}{100t^2}\tilde{u}(x,t)$$

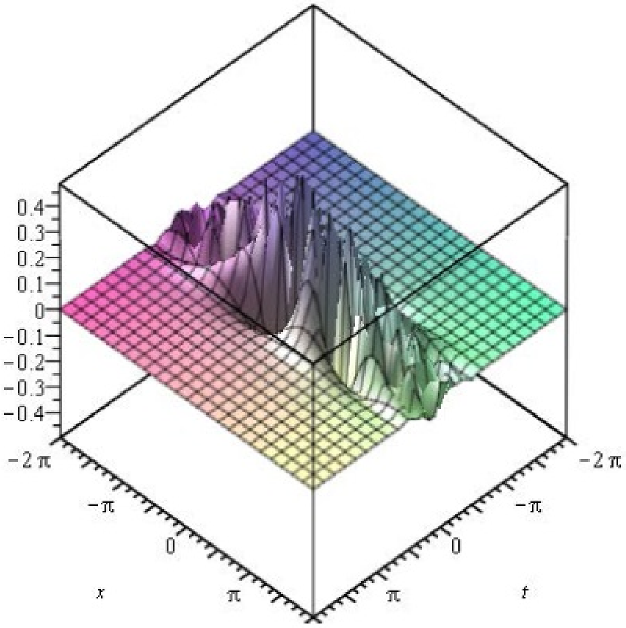

FIGURE 3. The graph of ψ_3

therefore

$$\tilde{u}(x,t) = F(x)\exp(-\int\{\frac{c^2t^2 - 40ct - 10t^2 - 400}{100t^2}\}dt) + K$$

By a little calculation, we get

$$\tilde{u}(x,t) = F(x)t^{-\frac{2}{5}c}\exp(\frac{c^2t^2 - 10t^2 + 400}{100}) + K$$

where K and $F(x)$ are constant at $\mathbb{R}$ and the function of x can be determined based on according to the initial or boundary conditions. Therefore from (5), the general analytic solution as follows

$$u(x,t) = \int\{F(x)t^{-\frac{2}{5}c}\exp(\frac{c^2t^2 - 10t^2 + 400}{100})\exp(-\frac{x^2 + 15t^2}{20})\cos(\frac{x}{t})\sin(\frac{x}{t})\}dx.$$

We can calculate such complicated integrals with numerical methods like approximation by taylor (around every points) or maclaurin (around zero) serie exapnsion of exponential & trigonometric functions, Thus by approximating these integrals, final solution are obtained. The results of wavelet transform method on the heat equation are presented in table 3.

Table 3. WTM results: the Mother wavelet, the wavelet tranform, the analytic solution

M.W	**The wavelet tranform**	**The analytic solution**
ψ_1	$F(x)\exp(-(c^2-\pi c-3.5)t)+K$	$\int\{F(x)\exp(-(c^2-\pi c-3.5)t).\psi_1\}dx$
ψ_2	$F(x)\exp(-(c^2+\pi c-3.5)t)+K$	$\int\{F(x)\exp(-(c^2+\pi c-3.5)t).\psi_2\}dx$
ψ_3	$F(x)t^{-\frac{2}{5}c}\exp(\frac{c^2t^2-10t^2+400}{100})+K$	$\int\{F(x)t^{-\frac{2}{5}c}\exp(\frac{c^2t^2-10t^2+400}{100}).\psi_3\}dx$

Note that, in comparision with WTM based on the quasi-wavelets [13], by Mother wavelet ψ_1,ψ_2, we conclude that $A(c)=\frac{\pi}{c}$, therefore $c=1.38,-4.47$, but $c\in[-\pi,\pi]$ so $c=1.38$ and $u(x,t)=F(x)\exp(-\frac{\pi^2}{4}t)+K$ (for more details and information about WTM based on the quasi-wavelets, see [13]).

5. Conclusions & future works

In this paper, we present a new method based on the Mother wavelets and the corresponding wavelet transforms. We used the findings of symmetry methods such as the Lie's symmetry method to construct favorable Mother wavelets. Then, we found the related wavelet transforms and affect them on both sides of the differential equation. Because of the integral nature of these transforms, the differential equation has been reduced. We solve the resulting reduced equation and get the solution. As we have seen, this method is very dependent on the presentation of a suitable Mother wavelet. Therefore, the basic and primary step is finding the appropriate Mother wavelet that is based on the differential invariants of the differential equation. In the future, by implementing this algorithm on various differential equations, we seek to find the appropriate Mother wavelet for each equation and each symmetry group. We also try to generalize this method to higher dimensions and more independent variables.

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