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CONNECTIVE ECCENTRICITY INDEX OF CERTAIN PATH-THORN GRAPHS

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Abstract. Let G be a simple connected graph with $V(G)$ and $E(G)$ as the vertex set and edge set respectively. A topological index is a numeric quantity by which we can characterize the whole structure of a molecular graph or a network to predict the physical or chemical activities of the involved chemical compounds in the molecular graph or network. The connective eccentricity index of the graph G is defined as $\xi^{ce}(G) = \sum\limits_{v \in G} \frac{d(v)}{e(v)}$, where $d(v)$ and $e(v)$ denote the degree and eccentricity of the vertex $v \in G$ respectively. In this paper, we compute the connective eccentricity index of the various families of the path-thorn graphs and present the obtained results with the help of suitable mathematical expressions consisting on various summations. More precisely, the computed results are general extensions of the some known results.

Key words : Distance-based index; Eccentricity; Path-thorn graph.
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1. Introduction

Topological indices are numerical quantaties of the graph which characterise the whole graph and these are isomorphic under the operation of graph isomorphism. Topological indices have found applications in a sapcific area of mathematical chemistry which is known as chemical graph theory. The combination of information science, mathematics, and chemistry leads to a new subject called cheminformatics. It studies the quantitative structure-property relationship (QSPR) and the quantitative structure-activity relationship (QSAR) that are used to predict the biological activities in the chemical compounds of

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the underlying molecular graph. There are some main classes of topological indices such as degree-based, distance-based and polynomial related indices of the graphs.

For the first time in 1947, Harold Wiener made the use of a topological index in chemistry while working on the boiling point of paraffin which later named the Wiener index, see [23]. He introduced the notion of path number of the graph as the sum of distance between any two carbon atoms in molecules. The Wiener index is equal to the count of all shortest distances in the graph. After the Wiener index, the theory of topological indices is started. In the progressive studies of indices, various topological indices have been introduced by different chemists and mathematicians. A large numbers of such indices depend on degrees of vertices and some others depend on distance property of the vertices. The distance-based index such as the total eccentricity index, eccentric connectivity index, eccentric distance sum index, average eccentricity index, reformed eccentric connectivity index, adjacent eccentric distance sum index and superaugmented eccentric connectivity index are studied in [7, 8, 15, 18, 20, 17, 10, 14, 16, 21, 22, 24]. The connective eccentricity index (CEI) is most familiar among the distance-based indices which is defined by Gupta, Singh and Madan, see [9].

In order to explore the potential of the CEI in predicting biological activity, the authors used non-peptide N-benzimidazole derivatives to investigate the predictability of the CEI with respect to antihypertensive activity. It is noted that the results obtained by using CEI were better than the corresponding values obtained by using Balanan's mean square distance index and accuracy of prediction was found to be about 80 percent in the active range for details, see [9]. De [3] reported some bounds for CEI in term of some graph invariants such as maximum and minimum degree, radius, diameter, first Zagreb index and first Zagreb eccentricity index, etc. Ashrafi et al. [1, 2] determined the closed formulas for the CEI of nanotubes and nanotori.

Ghorbani and Malekjani [12] computed the CEI of an infinite family of fullerenes. Yu and Feng in [25] derived some lower or upper bounds for the CEI of graphs in term of several graph invariants such as independence number, radius, vertex connectivity, the numbers of vertices with eccentricity 1 and investigated the maximal and minimal value of CEI among all n-vertex graphs with fixed numbers of pendent vertices. Nilanjan, Pal and Nayeem have also studied CEI on some graph operations. Nilanjan presented some bounds for this CEI in terms of different graph invariants [4, 5]. In [11], Ghorbani calculated some bounds of CEI and explicit expression for this index for two infinite classes of dendrimers. Nilanjan et al. compute the CEI of complete thorny graph, bipartite thorny graph, star thorny graph, cycle thorny graph and path thorny graph in [6].

In this paper, we compute the results related to CEI of complete path-thorn graph, complete bipartite path-thorn graph, star path-thorn graph, cycle path-thorn graph and path path-thorn graph. In general, these constructed graphs and their mathematical expressions with respect to CEI are the extensions of some known results. The rest of the paper is organised as; Section 2 contains the basic definitions related to graphs theoretic concepts. Mainly thorny graph and path-thorn graph are distinguished. Section 3 contains the main results of this paper.

2. Preliminaries

Let G be a simple connected graph with $E(G)$ and $V(G)$ as the edge set and vertex set respectively. We also let m and n be the number of vertices and edges of the given graph G. For a vertex $v \in V(G)$, $d(v)$ denote the degree of vertex v. For vertices $u, v \in V(G)$, the distance is denoted by $d(u, v)$ and is defined as the length of a shortest path between vertices u and v in G. The eccentricity of a vertex v is the maximum distance from v to any other vertex of graph G and is denoted by $e(v)$. The diameter of a graph G is the maximum eccentricity of any vertex in the graph and is denoted by $\mathrm{diam}(G)$.

Definition 2.1. Let G be a graph then the total eccentricity index, eccentric connectivity index, eccentric distance sum index, and superaugmented eccentric connectivity index are defined as

$$\xi(G) = \sum_{v \in G} e(v), \quad \xi^c(G) = \sum_{v \in G} e(v)d(v), \quad \xi^d(G) = \sum_{v \in G} e(v)D(v), \quad \text{and}$$

$\xi^{ac}(G) = \sum_{v \in G} \frac{M(v)}{e(v)}$ respectively, where $d(v)$ denotes the degree, $e(v)$ shows eccentricity and $M(v)$ presents the product of degrees of all neighbors of vertex v of the graph G and $D(v) = \sum_{u \in G} d(u, v)$.

Definition 2.2. Let G be a graph then connective eccentricity index of graph G is defined as $\xi^{ce}(G) = \sum_{v \in G} \frac{d(v)}{e(v)}$, where $d(v)$ and $e(v)$ denote the degree and eccentricity of the vertex.

Let G be a graph with vertex set $\{v_i : 1 \leq i \leq n\}$ and $\{p_i : 1 \leq i \leq n\}$ be the set of positive integers then the thorn graph of any graph G denoted by $G^*_{\{p_1, p_2, \ldots, p_n\}}$ is obtained by attaching p_i pendant vertices to v_i for each i. This notation is given by Gutman, for detail see [13]. Now, we extend this definition and define path-thorn graph. Let $\{v_1, v_2, ..., v_n\}$ be the vertex set of G and $\{p_1^q, p_2^q, ..., p_n^q\}$ be the set of positive numbers which present the number of paths such that the length of each path is fixed by q. The path-thorn graph of the graph G is denoted by $G^*_{\{p_1^q, p_2^q, ..., p_n^q\}}$ and obtained by attaching p_i^q paths to the vertices v_i for each i.

Let K_ν be a complete graph with ν vertices. The complete path-thorn graph K_n^* is obtained by attaching p_i^q paths each of length q to the vertex v_i of K_ν, where $i = 1, 2, ..., \nu$. Let K_{ν_1,ν_2} be the complete bipartite graph with $\nu_1 + \nu_2$ vertices such that ν_1 vertices have degree ν_2, ν_2 vertices have degree ν_1 and eccentricity of each vertex is 2. Therefore, connective eccentricity index (CEI) of complete bipartite K_{ν_1,ν_2} is $\nu_1\nu_2$. Let $K^*_{m,n}$ be the complete bipartite path-thorn graph obtained by attaching p_i^q number of paths each of length q to each vertex of K_{ν_1,ν_2}, where $1 \leq i \leq \nu_1 + \nu_2$. Similarly, we obtain path path-thorn graph P_n^*, cycle path-thorn graph C_n^* and star path-thorn graph $S_n^* = K_{1,(n-1)}$ from path P_ν, cycle C_ν and star $S_\nu = K_{1,(\nu-1)}$ by joining p_i^q paths each of length q to each vertex v_i of P_ν, C_ν and $S_\nu = K_{1,(\nu-1)}$ respectively, where $1 \leq i \leq \nu$.

3. Main Results

In this section, we present the main results of the connective eccentricity index (CEI) for the various families of the path-thorn graphs such as complete path-thorn, complete bipartite path-thorn, star path-thorn, cycle path-thorn and path path-thorn graphs.

Theorem 3.1. Let K_ν be a complete graph and K_n^* be its complete path-thorn graph. Then, CEI of complete path-thorn graph is given by

$$\xi^{ce}(K_n^*) = \frac{\nu(\nu-1)}{q+1} + [\frac{3q+2}{(q+1)(2q+1)} + \sum_{j=1}^{q-1} \frac{2}{q+1+j}] \sum_{i=1}^{\nu} p_i^q,$$

where $\nu \geq 3$, $q \geq 1$, $p_i \geq 1$ for $1 \leq i \leq \nu$ and $n = \nu + q \sum_{i=1}^{\nu} p_i$.

Proof. Let K_ν be a complete graph with vertices v_i for $i = 1, 2, 3, ..., \nu$ and the complete path-thorn graph K_n^* is obtained by attaching p_i^q path-thorn each of length q to the vertex v_i of K_ν. The vertices of newly attached path-thorn are v_{ij}^t for $i = 1, 2, 3, ..., \nu$, $j = 1, 2, 3, ..., q$ and $t = 1, 2, 3, ...p_i^q$. The degrees and eccentricities of the vertices of K_n^* are $d_{k_n^*}(v_i) = \nu - 1 + p_i^q$, $e_{k_n^*}(v_i) = q + 1$, $d_{k_n^*}(v_{ij}^t) = 2$, $d_{k_n^*}(v_{iq}^t) = 1$ and $e_{k_n^*}(v_{ij}^t) = q + 1 + j$, where $i = 1, 2, 3, ..., \nu$, $j = 1, 2, 3...., q-1$ and $t = 1, 2, ...p_i^q$.

$$\xi^{ce}(K_n^*) = \sum_{i=1}^{n} \frac{d(v_i)}{e(v_i)} = \sum_{i=1}^{\nu} \frac{d(v_i)}{e(v_i)} + \sum_{i=1}^{\nu}\sum_{j=1}^{q-1}\sum_{t=1}^{p_i^q} \frac{d(v_{ij}^t)}{e(v_{ij}^t)} + \sum_{i=1}^{\nu}\sum_{t=1}^{p_i^q} \frac{d(v_{iq}^t)}{e(v_{iq}^t)}$$

$$= \sum_{i=1}^{\nu} \frac{\nu - 1 + p_i^q}{q+1} + \sum_{i=1}^{\nu}\sum_{j=1}^{q-1}\sum_{t=1}^{p_i^q} \frac{2}{q+1+j} + \sum_{i=1}^{\nu}\sum_{t=1}^{p_i^q} \frac{1}{2q+1}$$

$$= \frac{\nu(\nu-1)}{q+1} + \sum_{i=1}^{\nu} \frac{p_i^q}{q+1} + \sum_{j=1}^{q-1} \frac{2}{q+1+j} \sum_{i=1}^{\nu} p_i^q + \frac{1}{2q+1} \sum_{i=1}^{\nu} p_i^q$$

$$= \frac{\nu(\nu-1)}{q+1} + \frac{1}{q+1} \sum_{i=1}^{\nu} p_i^q + \sum_{j=1}^{q-1} \frac{2}{q+1+j} \sum_{i=1}^{\nu} p_i^q + \frac{1}{2q+1} \sum_{i=1}^{\nu} p_i^q$$

$$= \frac{\nu(\nu-1)}{q+1} + [\frac{1}{q+1} + \sum_{j=1}^{q-1} \frac{2}{q+1+j} + \frac{1}{2q+1}] \sum_{i=1}^{\nu} p_i^q$$

$$\xi^{ce}(K_n^*) = \frac{\nu(\nu-1)}{q+1} + [\frac{3q+2}{(q+1)(2q+1)} + \sum_{j=1}^{q-1} \frac{2}{q+1+j}] \sum_{i=1}^{\nu} p_i^q.$$

Theorem 3.2. Let K_{ν_1,ν_2} be a complete bipartite graph and $K_{m,n}^*$ be its complete bipartite path-thorn graph. Then, CEI of $K_{m,n}^*$ is given by

$$\xi^{ce}(K_{m,n}^*) = \frac{2\nu_1\nu_2}{(q+2)} + [\sum_{i=1}^{\nu_1} p_i^q + \sum_{i=1}^{\nu_2} p_i^{*q}] \frac{(3q+4)}{(q+2)(2q+2)} + 2\sum_{j=1}^{q-1} \frac{1}{q+2+j} \sum_{i=1}^{\nu_1} p_i^q +$$
$$2\sum_{j=1}^{q-1} \frac{1}{q+2+j} \sum_{i=1}^{\nu_2} p_i^{*q},$$

where $\nu_1, \nu_2 \geq 2$, $q \geq 1$, $p_i \geq 1$ for $1 \leq i \leq \nu_1 + \nu_2$ and $|V(K_{m,n}^*)| = (\nu_1 + \nu_2) + q \sum_{i=1}^{\nu} p_i = m + n$.

Proof. Let $\{v_1, v_2, v_3, .., v_{\nu_1}, u_1, u_2, u_3, ..., u_{\nu_2}\}$ be the vertex set of graph K_{ν_1,ν_2}. The path-thorn p_i^q and p_i^{*q} each of length q are attached to v_i and u_i respectively to get $K_{m,n}^*$ for $i = 1, 2, ..., \nu_1$ and $i = 1, 2, .., \nu_2$ respectively. The vertices of the newly attached path-thorn are v_{ij}^t for $i = 1, 2, 3, ..., \nu_1$, $j = 1, 2, 3, ..., q$, $t = 1, 2, 3, ..., p_i^q$ and u_{ij}^t for $i = 1, 2, 3, ..., \nu_2$, where $j = 1, 2, 3, ..., q$, $t = 1, 2, 3, ..., p_i^{*q}$. The eccentricities and degrees of the vertices of the graph $K_{m,n}^*$ are given by $d(vi) = \nu_2 + p_i^q$, $e(v_i) = q + 2$, $e(u_i) = q + 2$, $d(v_i) = \nu_2 + p_i^q$, $d(u_i) = \nu_1 + p_i^{*q}$, $e(v_{ij}^t) = q + 2 + j$, $e(u_{ij}^t) = q + 2 + j$, for $v_i, i = 1, 2, ..., \nu_1$, $j = 1, 2, ..., q$, $t = 1, 2, ..., p_i^q$, and for $u_i, i = 1, 2, ..., \nu_2$, $j = 1, 2, ..., q$, $t = 1, 2, ..., p_i^{*q}$, $d(v_{ij}^t) = 2$, $d(u_{ij}^t) = 2$, $i = 1, 2, ..., \nu_2$, $j =$

$1,2,...,q-1,\ d(v_{iq}^t)=1,\ d(u_{iq}^t)=1$, for vertices of v path-thorn are $t=1,2,...,p_i^q$ and for vertices of u thorn paths are $t=1,2,...,p_i^{*q}$.

$$\xi^{ce}(K_{m,n}^*)=\sum_{i=1}^{\nu_1}\frac{d(v_i)}{e(v_i)}+\sum_{i=1}^{\nu_2}\frac{d(u_i)}{e(u_i)}+\sum_{i=1}^{\nu_1}\sum_{j=1}^{q-1}\sum_{t=1}^{p_i}\frac{d(v_{ij}^t)}{e(v_{ij}^t)}+\sum_{i=1}^{\nu_2}\sum_{j=1}^{q-1}\sum_{t=1}^{p_i^*}\frac{d(u_{ij}^t)}{e(u_{ij}^t)}$$

$$+\sum_{i=1}^{\nu_1}\sum_{t=1}^{pi}\frac{d(v_{iq}^t)}{e(v_{iq}^t)}+\sum_{i=1}^{\nu_2}\sum_{t=1}^{p_i^*}\frac{d(u_{iq}^t)}{e(u_{iq}^t)}$$

$$=\sum_{i=1}^{\nu_1}\frac{\nu_2+p_i^q}{q+2}+\sum_{i=1}^{\nu_2}\frac{\nu_1+p_i^{*q}}{q+2}+\sum_{i=1}^{nu_1}\sum_{j=1}^{q-1}\sum_{t=1}^{p_i^q}\frac{2}{q+q2+j}+\sum_{i=1}^{nu_2}\sum_{j=1}^{q-1}\sum_{t=1}^{p_i^{*q}}\frac{2}{q+2+j}$$

$$+\frac{1}{2q+2}\sum_{i=1}^{nu_2}p_i^q+\frac{1}{2q+2}\sum_{i=1}^{\nu_2}p_i^{*q}$$

$$=\frac{(\nu_1)(\nu_2)}{q+2}+\frac{(\nu_1)(\nu_2)}{q+2}+\sum_{i=1}^{\nu_1}\frac{p_i^q}{q+2}+\sum_{i=1}^{\nu_2}\frac{p_i^{*q}}{q+2}+\sum_{j=1}^{q-1}\frac{2}{q+2+j}\sum_{i=1}^{\nu_1}p_i^q$$

$$+\sum_{j=1}^{q-1}\frac{2}{q+2+j}\sum_{i=1}^{\nu_2}p_i^{*q}+\frac{1}{2q+2}\sum_{i=1}^{\nu_1}p_i^q+\frac{1}{2q+2}\sum_{i=1}^{\nu_2}p_i^{*q}$$

$$=\frac{2(\nu_1)(\nu_2)}{(q+2)}+[\frac{1}{q+2}+\frac{1}{2q+2}+2\sum_{j=1}^{q-1}\frac{1}{q+2+j}]\sum_{i=1}^{\nu_1}p_i^q+[\frac{1}{q+2}+\frac{1}{2q+2}$$

$$+2\sum_{j=1}^{q-1}\frac{1}{q+2+j}]\sum_{i=1}^{\nu_2}p_i^{*q}$$

$$\xi^{ce}(K_{m,n}^*)=\frac{2(\nu_1)(\nu_2)}{(q+2)}+[\sum_{i=1}^{\nu_1}p_i^q+\sum_{i=1}^{\nu_2}p_i^{*q}]\frac{(3q+4)}{(q+2)(2q+2)}+2\sum_{j=1}^{q-1}\frac{1}{q+2+j}\sum_{i=1}^{\nu_1}p_i^q$$
$$+2\sum_{j=1}^{q-1}\frac{1}{q+2+j}\sum_{i=1}^{\nu_2}p_i^{*q}.$$

Theorem 3.3. Let S_ν be a star graph and $\xi^{ce}(S_n^*)$ be its star path-thorn graph. Then, CEI of $\xi^{ce}(S_n^*)$ is given by

$$\xi^{ce}(S_n^*)=\frac{(\nu-1)(2q+3)}{(q+1)(q+2)}+\frac{7T}{12}+\frac{5p_1^q}{6}+\sum_{j=1}^{q-1}\frac{2p_1^q}{q+1+j}+\sum_{j=1}^{q-1}\frac{2T}{q+2+j}.$$

where $q\geq 1$, $p_i\geq 1$ and p_1^q is attached to central vertex of S_ν. $|V(S_n^*)|=\nu+q\sum_{i=1}^{\nu}p_i$.

Proof. Let S_ν be star graph and star path-thorn graph S_n^* obtained by attaching path-thorns p_i^q of length q to each vertex $(\nu_i), i = 1, 2, 3, ..., \nu$ of S_ν . The eccentricities and degrees of the vertices are

$e(v_1) = q + 1,\ d(v_1) = \nu - 1 + p_1^q,\ e(v_i) = q + 2,\ d(v_i) = 1 + p_i^q,\ i = 2, 3, ..., n,\ e(u_{1j}^t) = q+1+j,\ j = 1, 2, 3, ..., q,\ d(u_{1j}^t) = 2,\ t = 1, 2, 3, ..., p_i^q,\ j = 1, 2, 3, ..., q-1,\ d(u_{1q}^t) = 1,\ t = 1, 2, 3, ..., p_i^q,\ d(u_{ij}^t) = q+2+jj = 1, 2, 3, ..., q$

$i = 1, 2, 3, ..., \nu,\ t = 1, 2, 3, ..., p_i^q,\ d(u_{ij}^t) = 2,\ j = 1, 2, 3, ..., q-1,\ i = 1, 2, 3, ..., \nu,\ t = 1, 2, 3, ..., p_i^q\ \ d(u_{iq}^t) = 1,\ i = 1, 2, 3, ..., \nu\ t = 1, 2, 3, ..., p_i^q.$

$$\xi^{ce}(S_n^*) = \frac{d(v_1)}{e(v_1)} + \sum_{i=2}^{\nu} \frac{d(v_i)}{ev(_i)} + \sum_{t=1}^{P_i^q}\sum_{j=1}^{q-1} \frac{d(u_{1j}^t)}{e(u_{1j}^t)} + \sum_{t=1}^{p_i^q} \frac{d(u_{1q}^t)}{e(u_{1q}^t)} +$$

$$\sum_{t=2}^{p_i^q}\sum_{j=1}^{q-1}\sum_{i=2}^{\nu} \frac{d(u_{ij}^t)}{e(u_{ij}^t)} + \sum_{t=2}^{p_i^q}\sum_{i=2}^{\nu} \frac{d(u_{iq}^t)}{e(u_{iq}^t)}$$

$$= \frac{\nu - 1 + p_1^q}{q+1} + \sum_{i=2}^{\nu} \frac{1 + p_i^q}{q+2} + \sum_{t=1}^{p_1^q}\sum_{j=1}^{q-1} \frac{2}{q+1+j} + \sum_{t=1}^{p_1^q} \frac{1}{2q+1} + \sum_{j=1}^{q-1}\sum_{t=1}^{p_i^q}\sum_{i=2}^{\nu} \frac{2}{q+2+j} +$$

$$\sum_{i=2}^{\nu}\sum_{t=2}^{p_i^q} \frac{1}{2q+2}$$

$$\xi^{ce}(S_n^*) = \frac{(\nu-1)(2q+3)}{(q+1)(q+2)} + \frac{7T}{12} + \frac{5p_1^q}{6} + \sum_{j=1}^{q-1} \frac{2p_1^q}{q+1+j} + \sum_{j=1}^{q-1} \frac{2T}{q+2+j}.$$

Theorem 3.4. Let C_ν be a cycle graph with odd vertices and $(C_n^*$ be its cycle path-thorn graph. Then, CEI of cycle path-thorn Graph is given by

$$\xi^{ce}(C_n^*) = \frac{4\nu}{n+2q-1} + \frac{4(\nu+3q-1)T}{(\nu+4q-1)(\nu+2q-1)} + \sum_{j=1}^{q-1} \frac{4T}{\nu+2q+2j-1}.$$

where $q \geq 1$, $p_i \geq 1$ and $|V(C_n^*)| = \nu + q\sum_{i=1}^{\nu} p_i$.

Proof. Let C_ν be a cycle graph with odd vertices and $(C_n^*$ be its cycle path-thorn graph obtained by attaching path-thorns p_i^q of length q to each vertex of C_ν. The eccentricity and degrees of new cycle path thorn graph are $(v_i) = \frac{\nu+2q-1}{2},\ d(v_i) = 2+p_i^q,\ i = 1, 2, 3, ..., \nu,\ e(u_{ij}^t) = \frac{\nu+2q+2j-1}{2},\ j = 1, 2, 3, ..., q-1,\ \ i = 1, 2, 3, ..., n,\ \ t = 1, 2, 3, ..., p_i^q,\ \ d(u_{ij}^t) = 2,\ \ j = 1, 2, 3, ..., q-1\ \ i = 1, 2, 3, ..., \nu,\ t = 1, 2, 3, ..., p_i^q,\ \ d(u_{iq}^t) = 1,\ \ e(u_{iq}^t) = frac\nu + 4q - 12,\ i = 1, 2, 3, ..., \nu,\ \ t = 1, 2, 3, ..., p_i^q.$

$$\xi^{ce}(C_n^*) = \sum_{i=1}^{\nu} \frac{d(v_i)}{e(v_i)} + \sum_{i=1}^{\nu}\sum_{t=1}^{p_i^q}\sum_{j=1}^{q-1} \frac{d(u_{ij}^t)}{e(u_{ij}^t)} + \sum_{i=1}^{\nu}\sum_{t=1}^{p_i^q} \frac{d(u_{iq}^t)}{e(u_{iq}^t)}$$

$$= \sum_{i=1}^{\nu} \frac{2+p_i^q}{\frac{\nu+2q-1}{2}} + \sum_{j=1}^{q-1} \frac{2}{\frac{\nu+2q+2j-1}{2}} \sum_{i=1}^{\nu} p_i^q + \frac{1}{\frac{\nu+4q-1}{2}} \sum_{i=1}^{\nu} p_i^q$$

$$= \sum_{i=1}^{\nu} \frac{4+2p_i^q}{\nu+2q-1} + \sum_{j=1}^{q-1} \frac{4}{\nu+2q+2j-1} \sum_{i=1}^{\nu} p_i^q + \frac{2}{\nu+4q-1} \sum_{i=1}^{\nu} p_i^q$$

$$= \frac{4\nu}{n+2q-1} + \sum_{i=1}^{\nu} \frac{2p_i^q}{\nu+2q-1} + [\frac{2}{\nu+4q-1}] \sum_{i=1}^{\nu} p_i^q + \sum_{j=1}^{q-1} \frac{4}{\nu+2q+2j-1} \sum_{i=1}^{\nu} p_i^q$$

$$= \frac{4\nu}{n+2q-1} + \frac{2T}{\nu+2q-1} + \frac{2T}{\nu+4q-1} + \sum_{j=1}^{q-1} \frac{4T}{\nu+2q+2j-1}$$

$$\xi^{ce}(C_n^*) = \frac{4\nu}{n+2q-1} + \frac{4(\nu+3q-1)T}{(\nu+4q-1)(\nu+2q-1)} + \sum_{j=1}^{q-1} \frac{4T}{\nu+2q+2j-1}$$

Theorem 3.5. Let C_ν be a cycle graph with even vertices and (C_n^* be its cycle path-thorn graph. Then, CEI of Cycle path-thorn graph is given by

$$\xi^{ce}(C_n^*) = \frac{4\nu}{\nu+2q} + \frac{4T(\nu+3q)}{(\nu+2q)(n+4q)} + \sum_{j=1}^{q-1} \frac{4T}{\nu+4j}$$

where $q \geq 1$, $p_i \geq 1$ and $|V(C_n^*)| = \nu + q\sum_{i=1}^{\nu} p_i$.

Proof. Let C_ν be a cycle graph with odd vertices and (C_n^* be its cycle path-thorn graph obtained by attaching path-thorns p_i^q of length q to each vertex of C_ν.The eccentricities and degrees of new cycle path thorn graph are $C_n^* e(v_i) = \frac{\nu+2q}{2}$,$d(v_i) = 2 + P_i^q$, $i = 1,2,3,...,\nu$, $e(v_{ij}^t) = \frac{\nu+4q}{2}$, $j = 1,2,3,...,q-1$, $i = 1,2,3,...,\nu$, $t = 1,2,3,...,P_i^q$, $d(u_{ij}^t) = 2$, $j = 1,2,3,...,q-1$, $i = 1,2,3,...,\nu$, $t = 1,2,3,...,P_i^q$, $d(u_{iq}^t) = 1$, $e(u_{iq}^t) = \frac{\nu+4q}{2}$, $i = 1,2,3,...,\nu, t = 1,2,3,...,P_i^q$.

$$\xi^{ce}(C_n^*) = \sum_{i=1}^{\nu} \frac{d(v_i)}{e(v_i)} + \sum_{i=1}^{\nu}\sum_{t=1}^{P_i^q}\sum_{j=1}^{q-1} \frac{d(u_{ij}^t)}{e(u_{ij}^t)} + \sum_{i=1}^{\nu}\sum_{t=1}^{P_i^q} \frac{d(u_{iq}^t)}{e(u_{iq}^t)}$$

$$= \sum_{i=1}^{\nu} \frac{2+P_i^q}{\frac{\nu+2q}{2}} + \sum_{j=1}^{q-1} \frac{2}{\frac{\nu+4j}{2}} \sum_{i=1}^{\nu} P_i^q + \frac{1}{\frac{\nu+4q}{2}} \sum_{i=1}^{n} P_i^q$$

$$= \frac{4\nu}{n+2q} + \frac{2T}{\nu+2q} + \frac{2T}{\nu+4q} + \sum_{j=1}^{q-1} \frac{4T}{\nu+4j}$$

$$\xi^{ce}(C_n^*) = \frac{4\nu}{\nu+2q} + \frac{4T(\nu+3q)}{(\nu+2q)(n+4q)} + \sum_{j=1}^{q-1} \frac{4T}{\nu+4j}.$$

Theorem 3.6. Let P_ν be path graph with even vertices and P_m^* be its path path-thorn graph. Then, its CEI Graph is given by

$$\xi^{ce}(P_m^*) = \frac{2+P_{\nu/2}^q + P_{\nu/2}^{'q}}{\nu+q-1} + \sum_{i=1}^{[\nu/2]-1} \frac{4+P_i^q+P_i^{'q}}{q+i-1+\nu/2} +$$

$$\sum_{i-1}^{\nu/2}\sum_{j=1}^{q-1} \frac{4+P_i^q+P_i^{'q}}{2j+i-1+\nu/2} + \sum_{i=1}^{\nu/2} \frac{P_i^q+P_i^{'q}}{2q+i-1+\nu/2}$$

where $q \geq 1$, $p_i \geq 1$ and $|V(P_n^*)| = \nu + q\sum_{i=1}^{\nu} p_i$.

Proof. Let P_m be the paths of even vertices s.t $m \geq 2$ $\{v_{\nu/2}^{'}, v_{\nu-1/2}^{'}, ..., v_2^{'}, v_1^{'}, v_1, v_2, ..., v_{\nu/2}\}$ we attached path-thon P_i^q and $P_i^{'q}$ of length q to all vertices of P_ν.The degrees and eccentricities of P_m^* are $d(v_{\nu/2}) = 1 + P_{\nu/2}^q, d(v_{\nu/2}^{'}) = 1 + P_{\nu/2}^{'q}$ $e(v_{\nu/2}) = \nu - 1 + q, e(v_{\nu/2}^{'}) = \nu - 1 + q$ and $d(v_i) = 2+P_i^q, d(v_i^{'}) = 2+P_i^{'q}, e(v_i) = q-1+i+\nu/2, d(v_i^{'}) = q-1+i+\nu/2, i = 1, 2, ..., [\nu/2]-1, d(v_{ij}^t) = 2, d(v_{ij}^{'t}) = 2\, e(v_{ij}^t) = 2j+i-1+\nu/2\, e(v_{ij}^{'t}) = 2j+i-1+\nu/2 \; for \; i = 1,2,3,...,[\nu/2]-1, j = 1,2,3,...,q-1 \; and \; d(v_{iq}^t) = 1,\; d(v_{iq}^{'t}) = 1\, e(v_{iq}^t) = 2q+i-1+\nu/2,\; e(v_{iq}^{'t}) = 2q+i-1+\nu/2$

$$\xi^{ce}(P_m^*) = \frac{d(v_{\nu/2})}{e(v_{m/2})} + \frac{d(v_{\nu/2}^{'})}{e(v_{\nu/2}^{'})} + \sum_{i=1}^{[\nu/2]-1} \frac{d(v_i)}{e(v_i)} + \sum_{i=1}^{[\nu/2]-1} \frac{d(v_i^{'})}{e(v_i^{'})}$$

$$+ \sum_{i=1}^{\nu/2}\sum_{j=1}^{q-1}\sum_{t=1}^{P_i^q} \frac{d(v_{ij}^t)}{e(v_{ij}^t)} + \sum_{i=1}^{\nu/2}\sum_{j=1}^{q-1}\sum_{t=1}^{P_i^{'q}} \frac{d(v_{ij}^{'t})}{e(v_{ij}^{'t})}$$

$$+ \sum_{i=1}^{\nu/2}\sum_{t=1}^{P_i^q} \frac{d(v_{iq}^t)}{e(v_{iq}^t)} + \sum_{i=1}^{\nu/2}\sum_{t=1}^{P_i^{'q}} \frac{d(v_{iq}^{'t})}{e(v_{iq}^{'t})}$$

$$= \frac{1+P_{\nu/2}^q}{\nu-1+q} + \frac{1+P_{\nu/2}^{'q}}{\nu-1+q} + \sum_{i=1}^{[\nu/2]-1} \frac{2+P_i^q}{q-1+i+\nu/2}$$

$$+\sum_{i=1}^{[\nu/2]-1}\frac{2+P_i^{'q}}{q-1+i+\nu/2}+\sum_{i=1}^{\nu/2}\sum_{j=1}^{q-1}\sum_{t=1}^{P_i^q}\frac{2}{2j+i-1+\nu/2}$$

$$+\sum_{i=1}^{\nu/2}\sum_{j=1}^{q-1}\sum_{t=1}^{P_i^{'q}}\frac{2}{2j+i-1+\nu/2}+\sum_{i=1}^{\nu/2}\sum_{t=1}^{P_i^q}\frac{1}{2q+i-1+\nu/2}$$

$$+\sum_{i=1}^{\nu/2}\sum_{t=1}^{P_i^{'q}}\frac{1}{2q+i-1+\nu/2}$$

$$\xi^{ce}(P_m^*)=\frac{2+P_{\nu/2}^q+P_{\nu/2}^{'q}}{\nu-1+q}+\sum_{i=1}^{[\nu/2]-1}\frac{4+P_i^q+P_i^{'q}}{q-1+i+\nu/2}$$

$$+\sum_{i=1}^{\nu}\sum_{j=1}^{q-1}\frac{4+P_i^q+P_i^{'q}}{i-1+2j+[\nu/2]}+\sum_{i=1}^{\nu}\frac{P_i^q+P_i^{'q}}{i-1+2j+[\nu/2]}$$

Theorem 3.7. Let P_ν be path graph with odd vertices and P_m^* be its path path-thorn graph. Then, its CEI is given by

$$\xi^{ce}(P_m^*)=\frac{2+P_o^q}{q+[\nu-1]/2}+\frac{2+P_{\nu/2}^q\ +P_{\nu/2}^{'q}}{i-1+q+[\nu+1]/2}+\sum_{i=1}^{[\nu/2]-1}\frac{4+P_i^q\ +P_i^{'q}}{q-1+i+[\nu+1]/2}$$

$$+\sum_{i=1}^{\nu/2}\sum_{j=1}^{q-1}\frac{2P_i^q+2P_i^{'q}}{2j+i-1+[\nu+1]/2}+\sum_{i=1}^{\nu/2}\frac{P_i^q+P_i^{'}}{2q+i-1+[\nu+1]/2}.$$

where $q\geq 1$, $p_i\geq 0$ and $|V(P_n^*)|=\nu+q\sum\limits_{i=1}^{\nu}p_i$

Proof. Let path consist of $\nu-1$ vertices and there is one vertex fix named v_o that is the central vertex. Path will be of the form $\{v_{\nu/2}^{'},v_{\nu-1/2}^{'},...,v_2^{'},v_1^{'},v_o,v_1,v_2,...,v_{\nu/2}\}$. We attached P_o^q path-thorn to central vertex P_i^q to $v_{\nu/2}$, and $P_i^{'q}$ to $v_{\nu/2}^{'}$. The eccentricities and degrees of path-thorn graph are $d(v_o)=2+P_o$, $e(v_o)=q+[\nu-1/2]$, $d(v_{\nu/2})=1+P_{/2}^q$, $d(v_{\nu/2}^{'})=1+P_{\nu/2}^{'q}$, $e(v_{\nu/2})=i-1+q+[\nu+1]/2$, $e(v_{\nu/2}^{'})=i-1+q+[\nu+1]/2$, $d(v_i)=2+P_i^q$, $d(v_i^{'})=2+P_i^{'q}, e(v_i)=q-1+i+[\nu+1]/2$, $e(v_i^{'})=q-1+i+[\nu+1]/2$, $e(v^t)_{ij}=2j+i-1+[\nu+1]/2$, $e(v^{'t})_{ij}=2j+i-1+[\nu+1]/2$, $d(v^t)_{ij}=2$, $d(v^{'t})_{ij}=2\ for\ i=1,2,3,...,\nu/2,\ j=1,2,3,...,q-1,\ t=1,2,...,P_i\ and\ e(v^t)_{iq}=2q+i-1+[\nu+1]/2,\ e(v^{'t})_{iq}=2q+i-1+[\nu+1]/2,\ d(v^t)_{iq}=1,\ d(v^{'t})_{iq}=1,\ i=1,2,3,...,\nu/2,\ t=1,2,3,...,P_i$

$$\xi^{ce}(P_m^*) = \frac{d(v_o)}{e(v_o)} + \frac{d(v_{\nu/2})}{e(v_{\nu/2})} + \frac{d(v'_{\nu/2})}{e(v'_{\nu/2})} + \sum_{i=1}^{[\nu/2]-1} \frac{d(v_i)}{e(v_i)} + \sum_{i=1}^{[\nu/2]-1} \frac{d(v'_i)}{e(v'_i)}$$

$$+ \sum_{i=1}^{\nu/2}\sum_{j=1}^{q-1}\sum_{t=1}^{P_i^q} \frac{d(v^t)_{ij}}{e(v^t)_{ij}} + \sum_{i=1}^{\nu/2}\sum_{j=1}^{q-1}\sum_{t=1}^{P_i'^q} \frac{d(v'^t)_{ij}}{e(v'^t)_{ij}} + \sum_{i=1}^{\nu/2}\sum_{t=1}^{P_i^q} \frac{d(v^t)_{iq}}{e(v^t)_{iq}} + \sum_{i=1}^{\nu/2}\sum_{t=1}^{P_i'^q} \frac{d(v'^t)_{iq}}{e(v'^t)_{iq}}$$

$$= \frac{2+P_o^q}{q+[\nu-1]/2} + \frac{1+P_{\nu/2}^q}{i-1+q+[\nu+1]/2} + \frac{1+P_{\nu/2}'^q}{i-1+q+[\nu+1]/2}$$

$$+ \sum_{i=1}^{[\nu/2]-1} \frac{2+P_i^q}{q-1+i+[\nu+1]/2} + \frac{2+P_i'^q}{q-1+i+[\nu+1]/2}$$

$$+ \sum_{i=1}^{\nu/2}\sum_{j=1}^{q-1}\sum_{t=1}^{P_i^q} \frac{2}{2j+i-1+[\nu+1]/2} + \sum_{i=1}^{\nu/2}\sum_{j=1}^{q-1}\sum_{t=1}^{P_i'^q} \frac{2}{2j+i-1+[\nu+1]/2}$$

$$+ \sum_{i=1}^{\nu/2}\sum_{t=1}^{P_i^q} \frac{1}{2q+i-1+[\nu+1]/2} + \sum_{i=1}^{\nu/2}\sum_{t=1}^{P_i'^q} \frac{1}{2q+i-1+[\nu+1]/2}$$

$$\xi^{ce}(P_m^*) = \frac{2+P_o^q}{q+[\nu-1]/2} + \frac{2+P_{\nu/2}^q + P_{\nu/2}'^q}{i-1+q+[\nu+1]/2} + \sum_{i=1}^{[\nu/2]-1} \frac{4+P_i^q + P_i'^q}{q-1+i+[\nu+1]/2}$$

$$+ \sum_{i=1}^{\nu/2}\sum_{j=1}^{q-1} \frac{2P_i^q + 2P_i'^q}{2j+i-1+[\nu+1]/2} + \sum_{i=1}^{\nu/2} \frac{P_i^q + P_i'}{2q+i-1+[\nu+1]/2}.$$

4. Conclusion

In this paper, we have computed the connective eccentricity index for the various families of the path-thorn graphs such as complete path-thorn, complete bipartite path-thorn, path path-thorn, cycle path-thorn and star path-thorn graphs. Moreover, the obtained results are shown in the suitable forms with the help of mathematical expressions consisting on various summations.

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