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QUASI INVO-REGULAR RINGS

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Abstract. We define the class of *quasi invo-regular* rings and prove that they curiously coincide with the so-called invo-regular rings, recently introduced and explored by the present author in Ann. Univ. Mariae Curie-Sklodowska – Sect. Math. (2018).

Key words : unit-regular rings, invo-regular rings, quasi invo-regular rings.
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1. Introduction and Background

Throughout the text of the current short paper, all rings R are assumed to be associative, containing the identity element 1, which differs from the zero element 0 of R. Our standard terminology and notations are mainly in agreement with those from [6]. For instance, $U(R)$ denotes the set of all units in R, $Id(R)$ the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R and $J(R)$ the Jacobson radical of R. As for the specific notions, they will be recollected below explicitly.

It is classically well known that a ring R is said to be *unit-regular* if, for every $r \in R$, there exists $u \in U(R)$ such that $r = rur$ (see, e.g., [5]). Moreover, referring to [7], a ring R is said to be *clean* if, for each $r \in R$, there exist $u \in R$ and $e \in Id(R)$ such that $r = u + e$. It was shown in [1] that unit-regular rings are rather special sorts of clean rings. However, in the case when $u \in R$ is an arbitrary element depending on r (*not* necessarily a unit), these rings are called *regular* and they generally are *not* longer clean.

On the other hand, in [3] were investigated the so-called *invo-regular* rings that are rings which form a proper subclass of unit-regular rings, provided $u^2 = 1$, that is, u is an involution. These rings were completely characterized

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there as being a subdirect product of family of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$; thus they are surprisingly commutative.

So, we come to our key concept.

Definition 1. *A ring R is called quasi invo-regular if, for any $x \in R$, there exists $v \in R$ such that $x = xvx$ and v or $1 - v$ is an involution.*

Such an element v is usually termed *quasi-involution*. Thus $v^2 = 1$ or $1 - v = w$ whence $v = 1 - w$ for some $w^2 = 1$. This allows us to write that $x = xvx$ or that $x = x^2 - xwx$.

The leitmotif of this brief article is to visualize the most important properties of the newly defined ring class. Curiously, we shall prove in the sequel that these quasi invo-regular rings do coincide with the already known and characterized invo-regular rings.

2. Main Results

We begin with the following decomposition property.

Proposition 1. *For every quasi invo-regular ring $6 = 0$ and $R \cong R_1 \times R_2$, where either $R_1 = \{0\}$ or R_1 is a quasi invo-regular ring of characteristic 2, and either $R_2 = \{0\}$ or R_2 is a quasi invo-regular ring of characteristic 3.*

Proof. Given $x = 2$, one writes that $2 = 4v$ or $2 = 4 - 4w$ for some involutions $v, w \in R$. In both cases we, however, have that $2 = 4v$ or $2 = 4w$ which after squaring leads to $12 = 0$. Now, with the Chinese Remainder Theorem at hand, one decomposes $R \cong R_1 \times R_2$ for some two quasi invo-regular rings R_1, R_2, where $4 = 0$ in R_1 and hence, in view of the above, $2 = 0$ in R_1, and where $3 = 0$ in R_2, as stated. □

We now arrive at our rather surprising result.

Theorem 2. *The next three points are equivalent:*

(i) R is quasi invo-regular;

(ii) R is invo-regular;

(iii) R is a subdirect product of family of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

Proof. The equivalence (ii) $\iff$ (iii) was proved in [3].

The implication (ii) $\Rightarrow$ (i) being elementary, we shall be concentrated on the reverse one (i) $\Rightarrow$ (ii). To that goal, we first appeal to Proposition 1 to decompose R as $R = R_1 \times R_2$, where either R_1, R_2 are zero rings (*not* necessarily simultaneously), or R_1 and R_2 are quasi invo-regular rings of characteristic 2 and 3, respectively.

Furthermore, we deal with two possible cases, namely:

Case 1: Consider R_1. Clearly, $J(R_1) = \{0\}$ and, for any element $x \in R_1$, it must be that $x = xvx$ for some $v \in R_1$ such that $v^2 = 1$ or $v^2 = 0$. Therefore, each unit u in R_1 has to be an involution, that is, $u^2 = 1$ and hence $(u-1)^2 = 0$ implying that $u \in 1 + Nil(R_1)$, i.e., $U(R_1) = 1 + Nil(R_1)$. Since R_1 is (unit-)regular, we furthermore may apply [2] or [4] concluding that R_1 is necessarily a boolean ring.

Case 2: Consider R_2. For any $x \in R_2$, one writes that $x = xvx$ with $v^2 = 1$ or $(1 - v)^2 = 1$. The latter gives that $v^2 = 2v = -v$ because $3 = 0$ here. Thus $(-v)^2 = -v$ shows that $-v$ is an idempotent. But we may also equivalently write that $-x = (-x)(-v)(-x)$ and so, as $(-v)^2 = 1$ whenever $v^2 = 1$, we may without loss of generality assume that $x = xvx$ and that v is an involution or an idempotent by replacing $x \to -x$. We claim that every unit u of R_2 must be an involution, that is, $u^2 = 1$. In fact, substituting x by u^{-1}, it readily follows that $u^2 = 1$ or $u^2 = u$ which means in the second situation that $u = 1$ is an involution too. Furthermore, for each nilpotent $q \in R_2$, $1 + q$ being a unit has to be an involution as well. Thus $(1 + q)^2 = 1$ yields that $q^2 = -2q = q$ which riches us that $q(q - 1) = 0$ whence $q = 0$ as $q - 1$ inverts in R_2. This deduces that R_2 is of necessity abelian, i.e., every idempotent is central. Since $x = xvx$ assures that xv is an idempotent, we derive that $x = xvx = x^2v$. If v is an involution, we have nothing to do, so let us assume that v is just an idempotent. Hence $xv = x^2v = x$. Consequently, $x = xvx = x.x = x^2 = x.1.x$. This immediately ensures that R_2 is invo-regular of characteristic 3.

These two cases obviously imply, in turn, that $R \cong R_1 \times R_2$ is invo-regular, too, as pursued. □

The next comments and subsequent discussion might be somewhat useful.

Remark 1. *Seemingly, the quasi-involution property gives nothing new in the stated above definition of quasi invo-regular rings. However, it could be essential in the general situation when we consider the generalized unit-regularity by replacing "unit" with "quasi-unit".*

We shall say that a ring R is *quasi unit-regular* if, for any $x \in R$, there is $u \in R$ such that $x = xux$ and u or $1 - u$ is a unit. This element u will be called *quasi-unit*.

We end our work with the following intriguing question:

Problem 1. *Are all quasi unit-regular rings clean?*

If yes, this will considerably extend the well-known fundamental result due to Camillo-Khurana from [1] which states that unit-regular rings are always clean. If not, what can be said in this direction provided that u or $1 - u$ is a torsion unit. We shall normally call this element u *torsion quasi-unit*.

It is also principally known that all artinian rings R with zero $J(R)$ are always unit-regular (cf. [5]). Even something a little more – the artinian rings are themselves both noetherian and clean.

So, we close with the following challenging query:

Conjecture. *A ring is artinian if, and only if, it is both noetherian and clean with nil Jacobson radical.*

It is worthwhile noticing that in noetherian rings any nil-ideal (especially, the nil-radical) is necessarily nilpotent, so that "nil" in this case is definitely equivalent to "nilpotent".

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