Journal of Prime Research in Mathematics Vol. 15(2019), 21-48

MODIFIED BETA GENERALIZED LINEAR FAILURE RATE DISTRIBUTION: THEORY AND APPLICATIONS

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Abstract. In this paper we introduce a new comprehensive six-parameter distribution called the modified beta generalized linear failure rate distribution. One of the interest of this distribution is to generalize some well-known flexible distributions discussed in the literature, such as (i) the beta linear failure rate distribution, (ii) the generalized linear failure rate distribution, (iii) the beta exponential distribution, (iv) the beta Rayleigh distribution and (v) the generalized exponential distribution, among others. We derive some of its statistical properties, including the moments, the moment generating function, the quantile function, the order statistics and the mean deviations. We propose the method of maximum likelihood for estimating the model parameters. A simulation study is performed in order to investigate the performance of the maximum likelihood estimators. A real data set is used to illustrate the importance and the flexibility of the new distribution.

Key words : modified beta distribution, generalized linear failure rate distribution, moments, moment generating function, maximum likelihood estimation.
AMS SUBJECT : Primary 60E05, 62E15, 62F10.

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1. Introduction and Motivation

The development of (probability) distributions that have the ability to extract essential informations from lifetime data remains an important challenge for the statisticians. Indeed, the classical lifetime distributions such as the exponential distribution, the Rayleigh distribution, the linear failure rate distribution or the generalized exponential distribution have some limitations. One of the common limitation of these distributions is that they can not have increasing, decreasing and bathtub shaped hazard functions. This point is an obstacle for the perfect analysis of a wide variety of lifetime data. A suitable alternative is proposed by the so-called generalized linear failure rate distribution introduced by [20]. It is a three-parameter distribution characterized by the cumulative distribution function (cdf) given by

$$G(x) = \left[1 - e^{-(\lambda x + \frac{\theta}{2}x^2)}\right]^{\alpha}, \quad x > 0, \tag{1}$$

where ($\lambda > 0$ and $\theta \geq 0$) or ($\lambda \geq 0$ and $\theta > 0$) and $\alpha > 0$. Here λ and θ denote the scale parameters and α denotes the shape parameter of the distribution. The corresponding probability density function (pdf) is given by

$$g(x) = \alpha(\lambda + \theta x)e^{-(\lambda x + \frac{\theta}{2}x^2)}\left[1 - e^{-(\lambda x + \frac{\theta}{2}x^2)}\right]^{\alpha - 1}. \tag{2}$$

The corresponding hazard rate function (hrf) is given by

$$h(x) = \frac{\alpha(\lambda + \theta x)e^{-(\lambda x + \frac{\theta}{2}x^2)}\left[1 - e^{-(\lambda x + \frac{\theta}{2}x^2)}\right]^{\alpha - 1}}{1 - \left[1 - e^{-(\lambda x + \frac{\theta}{2}x^2)}\right]^{\alpha}}. \tag{3}$$

Let us mention that the generalized linear failure rate distribution contains several known lifetime distributions. Indeed, when $\theta = 0$ and $\alpha = 1$, we obtain the exponential distribution with parameter λ, when $\lambda = 0$ and $\alpha = 1$, we obtain the Rayleigh distribution with parameter θ, when $\alpha = 1$, we have the linear failure rate distribution with parameters λ and θ and when $\theta = 0$, we obtain the generalized exponential distribution with parameters λ and α.

On the other side, the last two decades of research in the field have been marked by the development of new general methods for building meaningful distributions from a baseline of an existing distribution. The most common approach is to use a so-called generator of distributions derived to common distributions with well-known structural properties. We refer to the extensive review of [22], and the references therein. Using several kinds of generators, several extensions of the linear failure rate distribution have been proposed (some of them extending the generalized linear failure rate distribution by using a power parameter in their constructions). See, for instance, [15] used the Kumaraswamy generator, [4] used the McDonald generator, [5] used the odd generalized exponential (OGE) generator, [19] used the beta and the geometric generators, and [13] used the odd generator. In this study, we focus our attention on an extension of the generalized linear failure rate distribution obtained by the use of the modified beta generator developed by [18]. A feature of this generator is to perfectly combine the complementary flexible properties of the beta generator introduced by [6] and the Marshall-Olkin generator introduced by [14]. We then benefit of this great advantage to increase the flexibility of the generalized linear failure rate distribution. The resulting distribution is naturally called the modified beta generalized linear failure rate distribution (MBGLFR for short). To the best of our knowledge, it has never been explored in the literature. In this study, we extensively studied its statistical and practical properties, and motivate its use for the analysis of complex data sets via simulation studies and the consideration of a concrete application.

The remainder of the article is organized as follows. Section 2 defines the MBGLFR distribution. The main structural properties of the MBGLFR distribution are investigated in Section 3. Section 4 provides the necessary to the estimation of the unknown parameters with the maximum likelihood method. A simulation study is performed to illustrate the theoretical results. The applicability of the MBGLFR model is studied in Section 4, with comparison to other competing probability models. Concluding remarks are provided in Section 5.

2. Modified beta generalized linear failure rate distribution

This section is devoted to the presentation of the MBGLFR distribution. Firstly, we recall the basics of the modified beta generator, then we present the crucial functions of the MBGLFR distribution, with an analytical and graphical study of the shapes of the related pdf and hrf.

2.1. **The modified beta generator.** First of all, let us recall the construction of the modified beta generator introduced by [18]. Let $c > 0$, $G(x)$ be a cdf and $g(x)$ be a related pdf. Then the modified beta generator is characterized by the cdf given by

$$F(x) = I_{\frac{cG(x)}{1-(1-c)G(x)}}(a, b), \tag{4}$$

where $a, b > 0$, $B(a, b)$ denotes the beta function defined by $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ and $I_x(a, b)$ denotes the incomplete beta function ratio defined by $I_x(a, b) = (1/B(a, b)) \int_0^x t^{a-1}(1-t)^{b-1}dt$, $x \in [0, 1]$. The related pdf is given by

$$f(x) = \frac{c^a g(x) \left[G(x)\right]^{a-1} \left[1 - G(x)\right]^{b-1}}{B(a, b) \left[1 - (1 - c)G(x)\right]^{a+b}}, \qquad x \in \mathbb{R}. \tag{5}$$

The related hrf is given by

$$h(x) = \frac{c^a g(x)\,[G(x)]^{a-1}\,[1-G(x)]^{b-1}}{B(a,b)\,[1-(1-c)G(x)]^{a+b}\left(1-I_{\frac{cG(x)}{1-(1-c)G(x)}}(a,b)\right)}, \qquad x \in \mathbb{R}.$$

The related reversed hrf is given by

$$r(x) = \frac{c^a g(x)\,[G(x)]^{a-1}\,[1-G(x)]^{b-1}}{B(a,b)\,[1-(1-c)G(x)]^{a+b}\,I_{\frac{cG(x)}{1-(1-c)G(x)}}(a,b)}, \qquad x \in \mathbb{R}.$$

2.2. **The MBGLFR distribution.** Let us now present the MBGLFR distribution. Using the general formulas above with the cdf of the generalized linear failure rate distribution as baseline, i.e., having the cdf $G(x)$ given by (1) (and the pdf $g(x)$ given by (2)), the cdf given by (4) becomes

$$F(x) = I_{\frac{c\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha}}{1-(1-c)\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha}}}(a,b), \quad x > 0. \tag{6}$$

The related pdf given by (5) becomes

$$f(x) = \frac{c^a \alpha(\lambda+\theta x)e^{-(\lambda x+\frac{\theta}{2}x^2)}\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha a-1}\left[1-\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha}\right]^{b-1}}{B(a,b)\left[1-(1-c)\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha}\right]^{a+b}}, \quad x > 0. \tag{7}$$

The related hrf is given by

$$h(x) = \frac{c^a \alpha(\lambda+\theta x)e^{-(\lambda x+\frac{\theta}{2}x^2)}\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha a-1}\left[1-\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha}\right]^{b-1}}{B(a,b)\left[1-(1-c)\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha}\right]^{a+b}\left(1-I_{\frac{c\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha}}{1-(1-c)\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha}}}(a,b)\right)}, \quad x > 0. \tag{8}$$

The related reversed hrf is given by

$$r(x) = \frac{c^a \alpha(\lambda+\theta x)e^{-(\lambda x+\frac{\theta}{2}x^2)}\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha a-1}\left[1-\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha}\right]^{b-1}}{B(a,b)\left[1-(1-c)\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha}\right]^{a+b} I_{\frac{c\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha}}{1-(1-c)\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha}}}(a,b)},$$

$x > 0$.

2.3. **Special cases.** The MBGLFR distribution has the faculty to approach different lifetime distributions when its parameters are changed. A non-exhaustive list is given below.

(1) When $\alpha = 1$ and $c = 1/(1-p)$ with $p \in (0,1)$, we obtain the beta geometric generalized linear failure rate distribution introduced by [19].
(2) When $\alpha = 1$ and $c = 1$, we obtain the beta generalized linear failure rate distribution introduced by [10].
(3) When $\alpha = c = b = 1$, then we get the generalized linear failure rate distribution introduced by [20].
(4) When $\alpha = c = 1$ and $\lambda = 0$, we get the beta Rayleigh distribution proposed by [2].
(5) When $\alpha = c = 1$ and $\theta = 0$, we obtain the beta exponential distribution which is introduced by [17].
(6) For $\alpha = c = b = 1$ and $\theta = 0$, we get the generalized exponential distribution proposed by [8].
(7) When $a = b = \alpha = c = 1$, we get the linear failure rate distribution.

Potential new special distributions are listed below.

(1) When $\alpha = 1$, we get the modified beta linear failure rate distribution.
(2) When $\alpha = 1$ and $\lambda = 0$, we get the modified beta Rayleigh distribution.
(3) When $\alpha = 1$ and $\theta = 0$, we obtain the modified beta exponential distribution.
(4) When $\alpha = c = 1$, we get the beta linear failure rate distribution distribution.

Note: For the sake of simplicity in exposition, we suppose in the next that $\lambda > 0$ and $\theta > 0$, which does not exclude the case where they are very small. This is the most interesting cases in practice. The cases ($\lambda > 0$ and $\theta = 0$) and ($\lambda = 0$ and $\theta > 0$) can be investigated in a similar way.

2.4. **Asymptotes and shapes of the pdf and the hrf.** Let us now investigate the asymptotes for $f(x)$. We have

$$f(x) \sim \frac{c^a}{B(a,b)} \alpha \lambda^{a\alpha} x^{a\alpha-1}, \quad x \to 0.$$

So, for $a \in (0, 1/\alpha)$, we have $\lim_{x\to 0} f(x) = +\infty$, for $a = 1/\alpha$, we have $\lim_{x\to 0} f(x) = c^{1/\alpha}\lambda\alpha/B(1/\alpha, b)$ and for $a > 1/\alpha$, we have $\lim_{x\to 0} f(x) = 0$. We see that the parameter θ play no role. On the other hand, we have

$$f(x) \sim \frac{1}{c^b B(a,b)} \alpha^b \theta x e^{-b(\lambda x + \frac{\theta}{2}x^2)}, \quad x \to +\infty.$$

Therefore, we have $\lim_{x\to +\infty} f(x) = 0$ in all cases. The shapes of $f(x)$ can be described analytically; the critical points x_* of the pdf $f(x)$ satisfies the equation given by $\partial \log(f(x_*))/\partial x = 0$, which corresponds to

$$\begin{aligned}
&\frac{\theta}{\lambda + \theta x_*} - (\lambda + \theta x_*) + (a\alpha - 1)(\lambda + \theta x_*) \frac{e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}}{1 - e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}} \\
&- (b-1) \frac{\alpha(\lambda + \theta x_*) e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)} \left[1 - e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\right]^{\alpha-1}}{1 - \left[1 - e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\right]^{\alpha}} \\
&+ (a+b)(1-c) \frac{\alpha(\lambda + \theta x_*) e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)} \left[1 - e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\right]^{\alpha-1}}{1 - (1-c)\left[1 - e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\right]^{\alpha}} = 0.
\end{aligned}$$

As usual in the theory of extrema, a critical point x_* corresponds to a local maximum if $\partial^2 \log(f(x_*))/\partial x^2 < 0$, a local minimum if $\partial^2 \log(f(x_*))/\partial x^2 > 0$ and a point of inflection if $\partial^2 \log(f(x_*))/\partial x^2 = 0$.

Let us now focus on the hrf $h(x)$. Similarly to $f(x)$, we have

$$h(x) \sim \frac{c^a}{B(a,b)} \alpha \lambda^{a\alpha} x^{a\alpha-1}, \quad x \to 0.$$

So, for $a \in (0, 1/\alpha)$, we have $\lim_{x\to 0} h(x) = +\infty$, for $a = 1/\alpha$, we have $\lim_{x\to 0} h(x) = c^{1/\alpha}\lambda\alpha/B(1/\alpha, b)$ and for $a > 1/\alpha$, we have $\lim_{x\to 0} h(x) = 0$. We have

$$h(x) \sim b(\lambda + \theta x), \quad x \to +\infty.$$

Therefore, we have $\lim_{x\to +\infty} h(x) = +\infty$, except in the case $\theta \to 0$ where $\lim_{x\to +\infty} h(x) = b\lambda$. The critical points x_* of $h(x)$ satisfies the equation

$\partial \log(h(x_*))/\partial x = 0$, i.e.,

$$\frac{\theta}{\lambda+\theta x_*} - (\lambda+\theta x_*) + (a\alpha-1)(\lambda+\theta x_*)\frac{e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}}{1-e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}}$$
$$-(b-1)\frac{\alpha(\lambda+\theta x_*)e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\left[1-e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\right]^{\alpha-1}}{1-\left[1-e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\right]^{\alpha}}$$
$$+(a+b)(1-c)\frac{\alpha(\lambda+\theta x_*)e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\left[1-e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\right]^{\alpha-1}}{1-(1-c)\left[1-e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\right]^{\alpha}}$$
$$+\frac{c^a\alpha(\lambda+\theta x_*)e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\left[1-e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\right]^{\alpha a-1}\left[1-\left[1-e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\right]^{\alpha}\right]^{b-1}}{B(a,b)\left[1-(1-c)\left[1-e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\right]^{\alpha}\right]^{a+b}\left[1-I_{\frac{c\left[1-e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\right]^{\alpha}}{1-(1-c)\left[1-e^{-(\lambda x_* + \frac{\theta}{2}x_*^2)}\right]^{\alpha}}}(a,b)\right]}$$
$$=0.$$

Then a point x_* corresponds to a local maximum if $\partial^2 \log(h(x_*))/\partial x^2 < 0$, a local minimum if $\partial^2 \log(h(x_*))/\partial x^2 > 0$ and a point of inflection if $\partial^2 \log(h(x_*))/\partial x^2 = 0$.

The critical points and the shapes of $f(x)$ and $h(x)$ can be explored graphically. Figures 1 and 2 show the plots for $f(x)$ and $h(x)$ respectively, for selected parameter values. In particular, we see that the hrf can be increasing, decreasing or bathtub shaped, with more different curvatures in comparison to the former generalized linear failure rate distribution (see [20]).

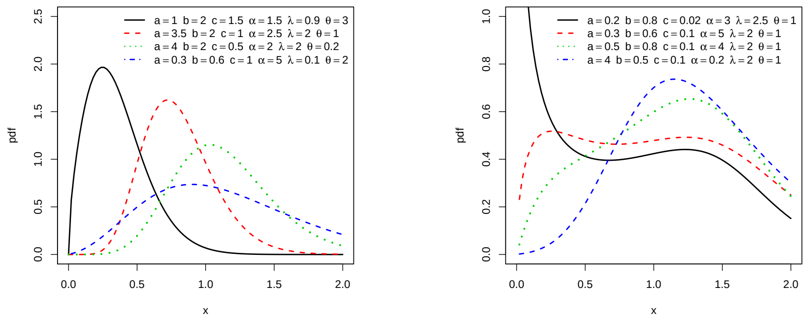

FIGURE 1. Some plots of the pdf $f(x)$ for selected parameter values.

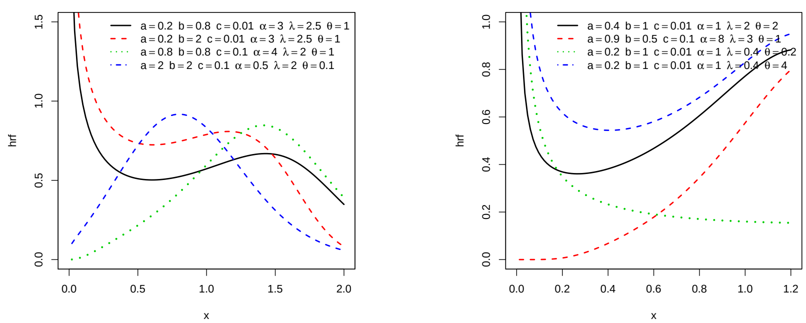

FIGURE 2. Some plots of the hrf $h(x)$ for selected parameter values.

3. Statistical properties

We now study some important statistical properties of the MBGLFR distribution, i.e., a linear representation of the cdf (and the pdf), the moments, the moment generating function, the quantile function, the order statistics and the mean deviations.

3.1. **Linear representation for the cdf.** Proposition 1 below shows that $F(x)$ can be expressed as a linear combination of survival functions of linear failure rate distributions.

Proposition 1. *For any integer k and $\gamma \in \mathbb{R}$, let us set* $\binom{\gamma}{k} = \frac{\gamma(\gamma-1)\dots(\gamma-k+1)}{k!}$. *The cdf $F(x)$ can be expressed as*

$$F(x) = \sum_{m=0}^{+\infty} v_m[c] S_m(x), \tag{9}$$

where

$$v_m[c] = \begin{cases} \frac{(-1)^m c^a}{B(a,b)} \sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} \binom{\alpha(\ell+a+k)}{m} \binom{-(a+k)}{\ell} \binom{b-1}{k} \frac{c^k (-1)^{k+\ell}(1-c)^\ell}{a+k} & \text{if } c \in (0,1], \\ \frac{(-1)^m}{B(a,b)} \sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} \sum_{q=0}^{\ell} \binom{-(a+k)}{\ell} \binom{\ell}{q} \binom{\alpha(q+a+k)}{m} \binom{b-1}{k} \frac{(-1)^{\ell+q+k}(c-1)^\ell c^{-\ell}}{a+k} & \text{if } c > 1 \end{cases} \tag{10}$$

and $S_m(x) = e^{-(m\lambda x + \frac{m\theta}{2} x^2)}$ is the survival function related to the linear failure rate distribution with parameters $m\lambda$ and $m\theta$.

Proof. The generalized binomial formula can be formulated as follows: for any z such that $|z| < 1$ and $\gamma \in \mathbb{R}$, we have $(1+z)^\gamma = \sum_{k=0}^{+\infty} \binom{\gamma}{k} z^k$. By this formula, we can express $F(x)$ as the following series:

$$\begin{aligned} F(x) = I_{\frac{cG(x)}{1-(1-c)G(x)}}(a,b) &= \frac{1}{B(a,b)} \sum_{k=0}^{+\infty} \binom{b-1}{k} (-1)^k \int_0^{\frac{cG(x)}{1-(1-c)G(x)}} t^{a+k-1} dt \\ &= \frac{1}{B(a,b)} \sum_{k=0}^{+\infty} \binom{b-1}{k} \frac{(-1)^k}{a+k} \left[\frac{cG(x)}{1-(1-c)G(x)} \right]^{a+k}. \end{aligned}$$

Let us suppose that $c \in (0,1]$. Using again the generalized binomial formula, we get

$$\left[\frac{cG(x)}{1-(1-c)G(x)}\right]^{a+k} = c^{a+k}\sum_{\ell=0}^{+\infty}\binom{-(a+k)}{\ell}(-1)^{\ell}(1-c)^{\ell}[G(x)]^{\ell+a+k} \quad (11)$$

and again

$$[G(x)]^{\ell+a+k} = \left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha(\ell+a+k)} = \sum_{m=0}^{+\infty}\binom{\alpha(\ell+a+k)}{m}(-1)^m S_m(x).$$

By putting the above equalities together, we prove the announced formula for $c \in (0,1]$ (with the related expression for $v_m[c]$).

Let us now investigate the case $c > 1$. We still have

$$F(x) = \frac{1}{B(a,b)}\sum_{k=0}^{+\infty}\binom{b-1}{k}\frac{(-1)^k}{a+k}\left[\frac{cG(x)}{1-(1-c)G(x)}\right]^{a+k}.$$

By noticing that $\frac{cG(x)}{1-(1-c)G(x)} = \frac{G(x)}{1-(1-\frac{1}{c})(1-G(x))}$, by applying the generalized binomial formula and the standard binomial formula, we obtain

$$\begin{aligned}\left[\frac{cG(x)}{1-(1-c)G(x)}\right]^{a+k} &= [G(x)]^{a+k}\sum_{\ell=0}^{+\infty}\binom{-(a+k)}{\ell}(-1)^{\ell}(c-1)^{\ell}c^{-\ell}[1-G(x)]^{\ell} \\ &= \sum_{\ell=0}^{+\infty}\sum_{q=0}^{\ell}\binom{-(a+k)}{\ell}\binom{\ell}{q}(-1)^{\ell+q}(c-1)^{\ell}c^{-\ell}[G(x)]^{q+a+k}.\end{aligned}$$

We also have

$$[G(x)]^{q+a+k} = \left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha(q+a+k)} = \sum_{m=0}^{+\infty}\binom{\alpha(q+a+k)}{m}(-1)^m S_m(x).$$

By combining these equalities, we prove the announced formula for $c > 1$ (with the related expression for $v_m[c]$). □

An important consequence of Proposition 1 is the following. Let $u_m(x)$ be a pdf related to the (standard) linear failure rate distribution with parameters $m\lambda$ and $m\theta$. By the differentiation of infinite series, we can express the pdf $f(x)$ as

$$f(x) = \sum_{m=0}^{+\infty} w_m[c]u_m(x), \quad (12)$$

where

$$w_m[c] = -v_m[c] = \begin{cases} \dfrac{(-1)^{m+1}c^a}{B(a,b)} \displaystyle\sum_{k=0}^{+\infty}\sum_{\ell=0}^{+\infty} \binom{\alpha(\ell+a+k)}{m}\binom{-(a+k)}{\ell}\binom{b-1}{k} \frac{c^k(-1)^{k+\ell}(1-c)^\ell}{a+k} & \text{if } c \in (0,1], \\ \dfrac{(-1)^{m+1}}{B(a,b)} \displaystyle\sum_{k=0}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{q=0}^{\ell} \binom{-(a+k)}{\ell}\binom{\ell}{q}\binom{\alpha(q+a+k)}{m}\binom{b-1}{k} \frac{(-1)^{\ell+q+k}(c-1)^\ell c^{-\ell}}{a+k} & \text{if } c > 1. \end{cases}$$

Note: Hereafter, we denote by X a random variable having the cdf $F(x)$ given by (6) (and the pdf $f(x)$ given by (7)) and by Y_m a random variable following the linear failure rate distribution with parameters $m\lambda$ and $m\theta$, i.e., having the survival function $S_m(x)$ (and the pdf $u_m(x)$, i.e., the pdf given by (2) with $m\lambda$ instead of λ and $m\theta$ instead of θ).

3.2. **Moments.** Here, moments of the MBGLFR distribution are presented. They are crucial since some most important features of the distribution can be defined through moments (dispersion, skewness, kurtosis ...). The result below presents a sum expression for the r-th moment of X.

Proposition 2. *Let r be a positive integer. Let $\Gamma(x)$ be the gamma function defined by $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$, $x > 0$. Then the r-th moment of X is given by*

$$\mu'_r = \frac{1}{\lambda^{r+2}} \sum_{m=0}^{+\infty}\sum_{k=0}^{+\infty} w_m[c](-1)^k \frac{\theta^k}{2^k k!\lambda^{2k}} \frac{1}{m^{k+r+1}} \left(m\lambda^2 + \theta(2k+r+1)\right)\Gamma(2k+r+1),$$

where $w_m[c]$ is defined as in Proposition 1.

Proof. Using the linear representation given by (12), the r-th ordinary moment of X is given by

$$\mu'_r = \mathbb{E}\left(X^r\right) = \int_{-\infty}^{+\infty} x^r f(x)dx = \sum_{m=0}^{+\infty} w_m[c] \int_{-\infty}^{+\infty} x^r u_m(x)dx$$
$$= \sum_{m=0}^{+\infty} w_m[c]\mathbb{E}(Y_m^r).$$

Using the exponential series expansion: $e^{-\frac{m\theta}{2}x^2} = \sum_{k=0}^{+\infty}(-1)^k \frac{m^k\theta^k}{2^k k!}x^{2k}$, $x \in \mathbb{R}$, we obtain

$$\mathbb{E}(Y_m^r) = m\int_0^{+\infty} x^r(\lambda+\theta x)e^{-(m\lambda x+\frac{m\theta}{2}x^2)}dx$$
$$= \sum_{k=0}^{+\infty}(-1)^k \frac{m^{k+1}\theta^k}{2^k k!}\int_0^{+\infty} x^{2k+r}(\lambda+\theta x)e^{-m\lambda x}dx. \quad (13)$$

By doing the change of variable $u = m\lambda x$ and by using the property $\Gamma(x+1) = x\Gamma(x)$, $x > 0$, we obtain

$$\int_0^{+\infty} x^{2k+r}(\lambda+\theta x)e^{-m\lambda x}dx$$
$$= \lambda\int_0^{+\infty} x^{2k+r}e^{-m\lambda x}dx + \theta\int_0^{+\infty} x^{2k+r+1}e^{-m\lambda x}dx$$
$$= \lambda\frac{1}{(m\lambda)^{2k+r+1}}\Gamma(2k+r+1) + \theta\frac{1}{(m\lambda)^{2k+r+2}}\Gamma(2k+r+2)$$
$$= \frac{1}{(m\lambda)^{2k+r+2}}\left(m\lambda^2 + \theta(2k+r+1)\right)\Gamma(2k+r+1). \quad (14)$$

By putting all these equalities together, we end the proof of Proposition 2. □

Proposition 2 can be used to determine all quantities depending on the moments. In particular, the mean of X is given by $\mathbb{E}(X) = \mu_1'$ and the variance of X is given by $\mathbb{V}(X) = \mu_2' - (\mu_1')^2$. The r-th central moment of X is given by

$$\mu_r = \mathbb{E}\left[(X-\mu_1')^r\right] = \sum_{k=0}^{r}\binom{r}{k}(-1)^k(\mu_1')^k\mu_{r-k}'.$$

The r-th cumulants of X can be obtained by the recursive formula: $\kappa_r = \mu_r' - \sum_{k=1}^{r-1}\binom{r-1}{k-1}\kappa_k\mu_{r-k}'$, with $\kappa_1 = \mu_1'$. The skewness of X is given by $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and the kurtosis of X is given by $\gamma_2 = \kappa_4/\kappa_2^2$. Finally, the moment generating function of X is given by

$$M_X(t) = \mathbb{E}(e^{tX}) = \sum_{k=0}^{+\infty}\frac{t^k}{k!}\mu_k', \quad t \le 0,$$

(an alternative expression, which really generate moments, will be given in Subsection 3.3).

Table 1 presents the numerical values of some moments (order 1, 2, 3 and 4), the skewness γ_1 and the kurtosis γ_2 of X for selected values of the parameters.

TABLE 1. Some moments, skewness and kurtosis of X for the following selected parameters values in order (a, b, c, α, β, θ); (i): $(1, 2, 1.5, 1.5, 0.9, 3)$, (ii): $(4, 2, 0.5, 2, 2, 0.2)$, (iii): $(5, 5, 0.1, 2, 2, 0.1)$ and (iv): $(5, 5, 0.4, 2, 2, 0.5)$.

	(i)	(ii)	(iii)	(iv)
$\mathbb{E}(X)$	0.1848323	1.436719	1.69751	1.13918
$\mathbb{E}(X^2)$	0.05675808	2.132057	2.912508	1.331062
$\mathbb{E}(X^3)$	0.02448038	3.262164	5.048277	1.592475
$\mathbb{E}(X^4)$	0.01358948	5.138691	8.835829	1.947968
$\mathbb{V}(X)$	0.0225951	0.06789533	0.03096792	0.03333187
γ_1	1.659701	0.2206476	-0.1597525	0.03524769
γ_2	4.096917	0.169661	0.186536	0.017433

3.3. **Moment generating function.** The result below presents a series expression for the moment generating function of X.

Proposition 3. *Let r be a positive integer. Then the moment generating function of X is given by*

$$M_X(t) = \sum_{m=0}^{+\infty}\sum_{k=0}^{+\infty} w_m[c](-1)^k \frac{m^{k+1}\theta^k}{2^k k!}\frac{1}{(m\lambda - t)^{2k+2}}\left[\lambda(m\lambda - t) + \theta(2k+1)\right]\Gamma(2k+1),$$
$$t \leq 0,$$

where $w_m[c]$ is defined as in Proposition 1.

Proof. Using the linear representation given by (12), the moment generating function of X is given by

$$M_X(t) = \int_{-\infty}^{+\infty} e^{tx} f(x)dx = \sum_{m=0}^{+\infty} w_m[c]\int_{-\infty}^{+\infty} e^{tx} u_m(x)dx = \sum_{m=0}^{+\infty} w_m[c] M_{Y_m}(t),$$

where $M_{Y_m}(t) = \mathbb{E}(e^{tY_m})$ (is the moment generating function of Y_m). Using the exponential series expansion: $e^{-\frac{m\theta}{2}x^2} = \sum_{k=0}^{+\infty}(-1)^k \frac{m^k\theta^k}{2^k k!}x^{2k}$, $x \in \mathbb{R}$, we obtain

$$\begin{aligned} M_{Y_m}(t) &= m\int_0^{+\infty} e^{tx}(\lambda+\theta x)e^{-(m\lambda x+\frac{m\theta}{2}x^2)}dx \\ &= m\int_0^{+\infty}(\lambda+\theta x)e^{-((m\lambda-t)x+\frac{m\theta}{2}x^2)}dx \\ &= \sum_{k=0}^{+\infty}(-1)^k\frac{m^{k+1}\theta^k}{2^k k!}\int_0^{+\infty} x^{2k}(\lambda+\theta x)e^{-(m\lambda-t)x}dx. \end{aligned}$$

By doing the change of variable $u = [m\lambda - t]x$ and by using the property $\Gamma(x+1) = x\Gamma(x)$, $x > 0$, we obtain

$$\begin{aligned} &\int_0^{+\infty} x^{2k}(\lambda+\theta x)e^{-(m\lambda-t)x}dx \\ &= \lambda\int_0^{+\infty} x^{2k}e^{-(m\lambda-t)x}dx + \theta\int_0^{+\infty} x^{2k+1}e^{-(m\lambda-t)x}dx \\ &= \lambda\frac{1}{(m\lambda-t)^{2k+1}}\Gamma(2k+1) + \theta\frac{1}{(m\lambda-t)^{2k+2}}\Gamma(2k+2) \\ &= \frac{1}{(m\lambda-t)^{2k+2}}\left[\lambda(m\lambda-t)+\theta(2k+1)\right]\Gamma(2k+1). \end{aligned}$$

By combining all these equalities, we complete the proof of Proposition 3. □

3.4. **Quantile function.** Let $I_u^{-1}(a,b)$ be the inverse of the incomplete beta function ratio $I_u(a,b)$. Then the quantile function of X is given by

$$Q(u) = \frac{1}{\theta}\left\{-\lambda + \sqrt{\lambda^2 - 2\theta\log\left(1-\left[\frac{I_u^{-1}(a,b)}{c+(1-c)I_u^{-1}(a,b)}\right]^{\frac{1}{\alpha}}\right)}\right\}, \quad u \in (0,1).$$

Since $I_u(a,b) \sim (auB(a,b))^{\frac{1}{a}}$ when $u \to 0$, we have

$$Q(u) \sim \frac{1}{\lambda c^{\frac{1}{\alpha}}}(auB(a,b))^{\frac{1}{a\alpha}}, \quad u \to 0.$$

The median of X is given by $M = Q(0.5)$. By definition of the quantile function, for a random variable U following the uniform distribution over $(0,1)$, the random variable $X = Q(U)$ follows the MBGLFR distribution. Alternatively, noticing that $I_U^{-1}(a,b)$ follows the beta distribution with parameters a and b, the following characterization holds.

Lemma 4. *Let V be a random variable following the beta distribution with parameters a and b. Then the random variable X given by*

$$X = \frac{1}{\theta}\left\{-\lambda + \sqrt{\lambda^2 - 2\theta \log\left(1 - \left[\frac{V}{c+(1-c)V}\right]^{\frac{1}{\alpha}}\right)}\right\},$$

follows the MBGLFR distribution. On the other side, let X be a random variable following the MBGLFR distribution. Then the random variable given by

$$V = \frac{c\left[1 - e^{-(\lambda X + \frac{\theta}{2}X^2)}\right]^{\alpha}}{1 + (c-1)\left[1 - e^{-(\lambda X + \frac{\theta}{2}X^2)}\right]^{\alpha}},$$

follows the beta distribution with parameters a and b.

We can also use $Q(u)$ to define other measures of skewness as, for instance, the Bowley skewness and the Moors kurtosis. They are respectively defined by

$$B = \frac{Q(0.75) + Q(0.25) - 2Q(0.5)}{Q(0.75) - Q(0.25)}$$

and

$$M_o = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(0.75) - Q(0.25)}.$$

In comparison to γ_1 and γ_2, the interest of B and M_o is to be less sensitive to outliers and to always exist (which is not the case for γ_1 and γ_2). Further details can be found in [11] and [16].

3.5. **Order statistics.** The order statistics play a determinant role in statistics. They naturally arise in reliability theory and life testing. Here, we present a result characterizing the pdf of the i-th order statistic of the MBGLFR distribution in terms of sum of pdfs of linear failure rate distributions.

Proposition 5. *Let $X_1, \ldots, X_n$ be the random sample from X and $X_{i:n}$ be the i-th order statistic. Then the pdf of $X_{i:n}$ can be expressed as a linear combination of pdfs of linear failure rate distributions.*

Proof. The pdf of the i-th order statistic denoted by $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x)[F(x)]^{i-1}[1 - F(x)]^{n-i}, \qquad x \in \mathbb{R}.$$

By using the binomial formula and the linear expansions (9) and (12), we get

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j f(x) [F(x)]^{j+i-1}$$
$$= \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \sum_{m=0}^{+\infty} w_m[c] u_m(x) \left[\sum_{k=0}^{+\infty} v_k[c] S_k(x)\right]^{j+i-1}.$$

A result from [7] on power series can be formulated as follows. As soon as the sums exist, for a positive integer s, a sequence of real numbers $(a_k)_{k\in\mathbb{N}}$ and $y \in \mathbb{R}$, we have

$$\left(\sum_{k=0}^{+\infty} a_k y^k\right)^s = \sum_{k=0}^{+\infty} d_{s,k} y^k,$$

where the coefficients $(d_{s,k})_{k\in\mathbb{N}}$ are determined by the following relations: $d_{s,0} = a_0^s$ and, for any $m \geq 1$,

$$d_{s,m} = (ma_0)^{-1} \sum_{\ell=1}^{m} (\ell(s+1) - m) a_\ell d_{s,m-\ell}.$$

Noticing that $S_k(x) = \left(e^{-(\lambda x + \frac{\theta}{2} x^2)}\right)^k$, the formula above applied with $s = j + i - 1$, $a_k = v_k[c]$ and $y = e^{-(\lambda x + \frac{\theta}{2} x^2)}$ yields

$$\left[\sum_{k=0}^{+\infty} v_k[c] S_k(x)\right]^{j+i-1} = \sum_{k=0}^{+\infty} d_{j+i-1,k}[c] S_k(x),$$

where $d_{j+i-1,0}[c] = (v_0[c])^{j+i-1}$ and, for any $m \geq 1$,

$$d_{j+i-1,m}[c] = \frac{1}{mv_0[c]} \sum_{k=1}^{m} (k(j+i) - m) v_k[c] d_{j+i-1,m-k}[c].$$

By combining the equalities above, we obtain

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-i} \sum_{m=0}^{+\infty} \sum_{k=0}^{+\infty} \binom{n-i}{j} (-1)^j w_m[c] d_{j+i-1,k}[c] u_m(x) S_k(x).$$

We end the proof by observing that

$$u_m(x) S_k(x) = m(\lambda + \theta x) e^{-(m+k)(\lambda x + \frac{\theta}{2} x^2)} = \frac{m}{m+k} u_{m+k}(x),$$

where $u_{m+k}(x)$ denotes the pdf of the linear failure rate distribution with parameters $(m+k)\lambda$ and $(m+k)\theta$. □

It follows from the proof of Proposition 5 that $f_{i:n}(x)$ can be written as

$$f_{i:n}(x) = \sum_{m=0}^{+\infty} \sum_{k=0}^{+\infty} z_{m,k}[c] u_{m+k}(x),$$

where

$$z_{m,k}[c] = \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \, w_m[c] d_{j+i-1,k}[c] \frac{m}{m+k}.$$

Therefore the r-th ordinary moment of $X_{i:n}$ can be expressed as

$$\mathbb{E}(X_{i:n}^r) = \int_{-\infty}^{+\infty} x^r f_{i:n}(x) dx = \sum_{m=0}^{+\infty} \sum_{k=0}^{+\infty} z_{m,k}[c] \int_{-\infty}^{+\infty} x^r u_{m+k}(x) dx.$$

By proceeding as in (13) and (14) with $m+k$ instead of m, we can show that

$$\int_{-\infty}^{+\infty} x^r u_{m+k}(x) dx =$$
$$\sum_{\ell=0}^{+\infty} \frac{(-1)^\ell \theta^\ell}{(k+m)^{\ell+r+1} \lambda^{2\ell+r+2} 2^\ell \ell!} \left[(m+k)\lambda^2 + \theta(2\ell+r+1) \right] \Gamma(2\ell+r+1).$$

By combining the equalities above, we obtain a sum expression for the r-th ordinary moment of $X_{i:n}$.

3.6. **Mean deviations.** The mean deviation of X about the mean μ_1' is given by

$$\delta_1 = \mathbb{E}(|X - \mu_1'|) = \int_{-\infty}^{+\infty} |x - \mu_1'| f(x) dx.$$

The mean deviation of X about the median M is given by

$$\delta_2 = \mathbb{E}(|X - M|) = \int_{-\infty}^{+\infty} |x - M| f(x) dx.$$

They are two measures of spread in a population. The following result shows expressions for these two quantities.

Proposition 6. *Let $\gamma(d, x)$ be the lower incomplete gamma function defined by $\gamma(d, x) = \int_0^x t^{d-1} e^{-t} dt$, $d > 0$, $x \geq 0$. Then we have*

$$\delta_1 = 2\mu_1' F(\mu_1') - 2m_*(\mu_1'), \quad \delta_2 = \mu_1' - 2m_*(M),$$

where, for any $t \geq 0$,

$$m_*(t) = \int_{-\infty}^{t} xf(x)dx$$
$$= \sum_{m=0}^{+\infty}\sum_{k=0}^{+\infty} w_m[c](-1)^k \frac{\theta^k}{2^k k! m^{k+2}\lambda^{2k+3}} \left[m\lambda^2\gamma(2k+2, m\lambda t) + \theta\gamma(2k+3, m\lambda t)\right]. \tag{15}$$

Proof. We have

$$\begin{aligned}\delta_1 &= \int_{-\infty}^{\mu_1'}(\mu_1' - x)f(x)dx + \int_{\mu_1'}^{+\infty}(x - \mu_1')f(x)dx \\ &= \int_{-\infty}^{\mu_1'}(\mu_1' - x)f(x)dx - \int_{-\infty}^{\mu_1'}(x - \mu_1')f(x)dx \\ &= 2\mu_1' F(\mu_1') - 2m_*(\mu_1').\end{aligned}$$

It follows from (12) that

$$m_*(t) = \sum_{m=0}^{+\infty} w_m[c] \int_{-\infty}^{t} x u_m(x)dx.$$

Using similar mathematical arguments to (13) and (14) but with the integration over $(0, t)$ instead of $(0, +\infty)$ and with $r = 1$, we get

$$\int_{-\infty}^{t} x u_m(x)dx = \sum_{k=0}^{+\infty}(-1)^k \frac{m^{k+1}\theta^k}{2^k k!} \int_0^t x^{2k+1}(\lambda + \theta x)e^{-m\lambda x}dx,$$

with

$$\int_0^t x^{2k+1}(\lambda + \theta x)e^{-m\lambda x}dx = \lambda \int_0^t x^{2k+1}e^{-m\lambda x}dx + \theta \int_0^t x^{2k+2}e^{-m\lambda x}dx$$
$$= \lambda \frac{1}{(m\lambda)^{2k+2}}\gamma(2k+2, m\lambda t) + \theta \frac{1}{(m\lambda)^{2k+3}}\gamma(2k+3, m\lambda t)$$
$$= \frac{1}{(m\lambda)^{2k+3}}\left[m\lambda^2\gamma(2k+2, m\lambda t) + \theta\gamma(2k+3, m\lambda t)\right].$$

We end the proof of the first equality by putting these equalities together. For δ_2, using $F(M) = 0.5$, we obtain

$$\begin{aligned}\delta_2 &= \int_{-\infty}^{M}(M - x)f(x)dx + \int_{M}^{+\infty}(x - M)f(x)dx \\ &= \int_{-\infty}^{M}(M - x)f(x)dx + \mu_1' - M - \int_{-\infty}^{M}(x - M)f(x)dx \\ &= \mu_1' - 2m_*(M).\end{aligned}$$

The second equality in the proof follows. □

One can also use (15) to determine the Bonferroni curve and the Lorenz curve which are very useful in many applied areas (economics, reliability, demography, medicine...). They are respectively given by

$$B(p) = \frac{1}{p\mu_1'} m_*(Q(p)), \qquad L(p) = \frac{1}{\mu_1'} m_*(Q(p)), \qquad p \in (0,1).$$

4. Statistical inference

Here, the estimations of the parameters λ, θ, α, a, b and c of the MBGLFR distribution are performed with the maximum likelihood method.

4.1. **Maximum likelihood estimation.** The most common method of parametric estimation is the maximum likelihood method. The resulting estimators, called the maximum likelihood estimators, enjoy remarkable properties. Among others, it can be used when constructing confidence intervals and regions in test statistics. Approximation for the maximum likelihood estimators in distribution theory is easily handled either analytically or numerically. Further details can be found in [12]. Here, we determine the maximum likelihood estimators of the parameters of the MBGLFR distribution from complete samples only. Let $x_1, \ldots, x_n$ be a random sample of size n from the MBGLFR distribution. Let $\phi = (a, b, c, \alpha, \lambda, \theta)^T$ be the 6×1 vector of parameters. The total log-likelihood function for ϕ is given by

$$\begin{aligned} L_n = L_n(\phi) &= na \log c + n \log \alpha - n \log B(a,b) + \sum_{i=1}^{n} \log(\lambda + \theta x_i) \\ &- \lambda \sum_{i=1}^{n} x_i - \frac{\theta}{2} \sum_{i=1}^{n} x_i^2 + (\alpha a - 1) \sum_{i=1}^{n} \log\left[1 - e^{-(\lambda x_i + \frac{\theta}{2} x_i^2)}\right] \\ &+ (b-1) \sum_{i=1}^{n} \log\left[1 - \left[1 - e^{-(\lambda x_i + \frac{\theta}{2} x_i^2)}\right]^{\alpha}\right] \\ &- (a+b) \sum_{i=1}^{n} \log\left[1 - (1-c)\left[1 - e^{-(\lambda x_i + \frac{\theta}{2} x_i^2)}\right]^{\alpha}\right]. \end{aligned}$$

The associated components of the score function is igven by

$$U_n(\phi) = \left[\frac{\partial L_n}{\partial a}, \frac{\partial L_n}{\partial b}, \frac{\partial L_n}{\partial c}, \frac{\partial L_n}{\partial \alpha}, \frac{\partial L_n}{\partial \lambda}, \frac{\partial L_n}{\partial \theta}\right]^T.$$

Let us introduce the digamma function $\psi(x)$ defined by $\psi(x) = \Gamma'(x)/\Gamma(x)$. Then, since we can express $B(a,b)$ as $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, we have

$\partial \log B(a,b)/\partial a = \psi(a) - \psi(a+b)$. Therefore

$$\frac{\partial L_n}{\partial a} = n\log c - n\psi(a) + n\psi(a+b) + \alpha\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right] - \sum_{i=1}^{n}\log\left[1-(1-c)\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]^{\alpha}\right],$$

$$\frac{\partial L_n}{\partial b} = -n\psi(b) + n\psi(a+b) + \sum_{i=1}^{n}\log\left[1-\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]^{\alpha}\right] - \sum_{i=1}^{n}\log\left[1-(1-c)\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]^{\alpha}\right],$$

$$\frac{\partial L_n}{\partial c} = \frac{na}{c} - (a+b)\sum_{i=1}^{n}\frac{\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]^{\alpha}}{1-(1-c)\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]^{\alpha}},$$

$$\begin{aligned}\frac{\partial L_n}{\partial \alpha} = \frac{n}{\alpha} &+ a\sum_{i=1}^{n}\log\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right] \\ &- (b-1)\sum_{i=1}^{n}\frac{\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]^{\alpha}\log\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]}{1-\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]^{\alpha}} \\ &+ (a+b)\sum_{i=1}^{n}\frac{(1-c)\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]^{\alpha}\log\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]}{1-(1-c)\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]^{\alpha}},\end{aligned}$$

$$\begin{aligned}\frac{\partial L_n}{\partial \lambda} &= \sum_{i=1}^{n}\frac{1}{\lambda+\theta x_i} - \sum_{i=1}^{n}x_i + (\alpha a-1)\sum_{i=1}^{n}\frac{x_i e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}}{1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}} \\ &- \alpha(b-1)\sum_{i=1}^{n}\frac{x_i e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]^{\alpha-1}}{1-\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]^{\alpha}} \\ &+ \alpha(a+b)\sum_{i=1}^{n}\frac{(1-c)x_i e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]^{\alpha-1}}{1-(1-c)\left[1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right]^{\alpha}}\end{aligned}$$

and

$$\frac{\partial L_n}{\partial \theta} = \sum_{i=1}^{n} \frac{x_i}{\lambda + \theta x_i} - \frac{1}{2}\sum_{i=1}^{n} x_i^2 + (\alpha a - 1)\sum_{i=1}^{n} \frac{\frac{1}{2}x_i^2 e^{-(\lambda x_i + \frac{\theta}{2}x_i^2)}}{1 - e^{-(\lambda x_i + \frac{\theta}{2}x_i^2)}}$$
$$- \alpha(b-1)\sum_{i=1}^{n} \frac{\frac{1}{2}x_i^2 e^{-(\lambda x_i + \frac{\theta}{2}x_i^2)}\left[1 - e^{-(\lambda x_i + \frac{\theta}{2}x_i^2)}\right]^{\alpha-1}}{1 - \left[1 - e^{-(\lambda x_i + \frac{\theta}{2}x_i^2)}\right]^{\alpha}}$$
$$+ \alpha(a+b)\sum_{i=1}^{n} \frac{(1-c)\frac{1}{2}x_i^2 e^{-(\lambda x_i + \frac{\theta}{2}x_i^2)}\left[1 - e^{-(\lambda x_i + \frac{\theta}{2}x_i^2)}\right]^{\alpha-1}}{1 - (1-c)\left[1 - e^{-(\lambda x_i + \frac{\theta}{2}x_i^2)}\right]^{\alpha}}.$$

The maximum likelihood estimator of the vector of parameters ϕ, say $\widehat{\phi} = (\widehat{a}, \widehat{b}, \widehat{c}, \widehat{\alpha}, \widehat{\lambda}, \widehat{\theta})$, is obtained by solving the nonlinear system $U_n(\phi) = 0$. These equations cannot be solved analytically, and statistical software (R, SAS...) can be used to solve them numerically via iterative methods. For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The 6×6 observed information matrix is given by $I_n(\phi) = \{-I_{uv}\}_{(u,v)\in\{a,b,c,\alpha,\lambda,\theta\}^2}$, where

$$I_{uv} = \frac{\partial^2 L_n}{\partial u \partial v}.$$

The general expressions of I_{uv} can be found in [18, Section 3] with $G(x)$ defined by (6). Applying the usual large sample approximation, maximum likelihood estimators of ϕ, i.e $\widehat{\phi}$ can be treated as being approximately $N_6(\phi, J_n(\phi)^{-1})$, where $J_n(\phi) = \mathbb{E}\left[I_n(\phi)\right]$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\phi} - \phi)$ is $\mathcal{N}_6(0, J(\phi)^{-1})$, where $J(\phi) = \lim_{n\to\infty} n^{-1} I_n(\phi)$ is the unit information matrix. This asymptotic behavior remains valid if $J(\phi)$ is replaced by the average sample information matrix evaluated at $\widehat{\phi}$, say $n^{-1}I_n(\widehat{\phi})$. The estimated asymptotic multivariate normal $N_6(\phi, I_n(\widehat{\phi})^{-1})$ distribution of $\widehat{\phi}$ can be used to construct approximate confidence intervals for the parameters. For any $\gamma \in (0,1)$, a $100(1-\gamma)\%$ asymptotic confidence interval (ACI) for each parameter ϕ_r is given by

$$ACI_r = \left(\widehat{\phi}_r - z_{\frac{\gamma}{2}}\sqrt{\widehat{I_{rr}}}, \widehat{\phi}_r + z_{\frac{\gamma}{2}}\sqrt{\widehat{I_{rr}}}\right),$$

where $\widehat{\phi}_r$ is the MLE of ϕ_r, $\widehat{I_{rr}}$ is the corresponding estimation of I_{rr} and z_γ is the upper 100γ-th percentile of the standard normal distribution.

4.2. **Simulation.** Due to obvious difficulties to compare the theoretical performances of the different maximum likelihood estimates (MLEs) for the

MBGLFR distribution. Therefore, simulation is needed to compare the performances of the MLE mainly with respect to their mean square errors (MSEs) for different sample sizes. A numerical study is performed using Mathematica 9 software. Different sample sizes are considered through the experiments at size $n = 50$, 100 and 300. The experiment will be repeated 1000 times. In each experiment, the estimates of the parameters will be obtained by maximum likelihood method of estimation. The means and MSEs for the different estimates will be reported from these experiments.

TABLE 2. The MLEs and MSEs of parameters of the MBGLFR distribution.

n	Parameters	Initial	MLE	MSE	Initial	MLE	MSE
50	a	1.20	1.0150	0.0511	1.00	1.0195	0.015400
	b	0.80	1.0570	0.1216	0.80	1.0368	0.111000
	c	0.50	0.4986	0.0013	0.50	0.5068	0.001400
	α	2.00	3.0801	1.5397	2.00	2.9507	1.144700
	λ	0.25	1.1084	0.8736	0.25	1.0530	0.739200
	θ	0.50	1.3793	3.8545	0.50	1.2210	1.537500
100	a	1.20	1.0110	0.0408	1.00	1.0060	0.009500
	b	0.80	1.0303	0.0778	0.80	1.0170	0.066500
	c	0.50	0.5018	0.0005	0.50	0.5020	0.000800
	α	2.00	2.9711	1.0646	2.00	2.9509	1.044300
	λ	1.20	1.0083	0.0390	0.25	1.0326	0.651100
	θ	0.50	1.1086	0.5313	0.50	1.0616	0.410300
300	a	0.80	0.9933	0.0447	1.00	0.9997	0.003000
	b	0.50	0.5033	0.0002	0.80	1.0086	0.050100
	c	2.00	2.9469	0.9487	0.50	0.5000	0.000300
	α	0.50	1.0017	0.2983	2.00	2.9246	0.900500
	λ	1.20	1.0150	0.0511	0.25	1.0159	0.599292
	θ	0.25	0.9943	0.5687	0.50	1.0248	0.302500
50	a	0.80	1.01190	0.081700	0.50	1.1495	0.689600
	b	0.80	1.07230	0.114200	0.80	1.0228	0.080300
	c	0.50	0.49640	0.001900	0.50	0.5119	0.004400
	α	2.00	2.97440	1.251500	2.00	2.6504	0.612700
	λ	0.25	1.11030	0.835100	0.25	1.0248	0.648200
	θ	0.50	1.20680	0.869400	0.50	1.0767	0.466000
100	a	0.80	0.99460	0.052000	0.50	1.0519	0.393300
	b	0.80	1.02100	0.063000	0.80	1.0381	0.073900
	c	0.50	0.49850	0.000800	0.50	0.4970	0.001900
	α	2.00	2.90730	0.936600	2.00	2.6946	0.608100
	λ	0.25	1.04250	0.654900	0.25	1.0435	0.658400
	θ	0.50	1.05010	0.353800	0.50	1.0747	0.379000
300	a	0.80	1.00350	0.047300	0.50	1.0286	0.303200
	b	0.80	1.01050	0.050200	0.80	1.0062	0.047000
	c	0.50	0.49900	0.000300	0.50	0.5019	0.000800
	α	2.00	2.83940	0.752100	2.00	2.6374	0.447600
	λ	0.25	1.01618	0.597699	0.25	1.0046	0.577609
	θ	0.50	1.02530	0.293600	0.50	1.0145	0.275000
50	a	1.20	1.02740	0.052500	0.80	1.03830	0.111600
	b	1.00	1.00890	0.020900	1.20	1.00880	0.050300
	c	0.50	0.50730	0.001600	0.50	0.50550	0.002000
	α	2.00	2.01030	0.056600	2.00	1.60190	0.181900
	λ	0.25	1.05160	0.750900	0.25	1.06560	0.757500
	θ	0.50	1.14560	1.022200	0.50	1.13860	0.964000
100	a	1.20	1.01510	0.042200	0.80	1.03730	0.078100
	b	1.00	1.00810	0.016700	1.20	0.99880	0.046900
	c	0.50	0.50460	0.000800	0.50	0.50930	0.001200
	α	2.00	2.00000	0.029500	2.00	1.57810	0.178000
	λ	0.25	1.02730	0.658200	0.25	1.00770	0.622500
	θ	0.50	1.08510	0.508800	0.50	1.03040	0.365200
300	a	1.20	0.99920	0.041500	0.80	1.01810	0.055300
	b	1.00	1.00290	0.003600	1.20	0.99820	0.042900
	c	0.50	0.49990	0.000200	0.50	0.50270	0.000500
	α	2.00	2.01070	0.010200	2.00	1.57920	0.151100
	λ	0.25	1.01628	0.601701	0.25	1.00005	0.576287
	θ	0.50	1.01830	0.291100	0.50	1.00570	0.271700

4.3. **Applications.** This section provides an application to show how the MBGLFR distribution can be applied in practice. We compare MBGLFR to Kumaraswamy Weibull-exponential (Kw-WE) by [9] and other well known distributions in litrature, Kumaraswamy-Weibull (Kw-W), beta Weibull(BW) and Weibull (W) models. The MLEs are computed using Quasi-Newton Code for Bound Constrained Optimization and the log-likelihood function evaluated. The goodness-of-fit measures, Anderson-Darling (A*), Cramer-von Mises (W*), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and log-likelihood ($\hat{\ell}$) values are computed. The lower values of these criteria, the better fit. The value for the Kolmogorov Smirnov (KS) statistic and its P-value are also provided.

The following data represent the survival times (in days) of 72 pigs infected with virulent tubercle bacilli, observed and reported by [3]. The data are as follows: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55. The required computations are carried out in the R software. Table 4 lists the maximum likelihood estimates (and the corresponding standard errors in parentheses) of the unknown parameters of the MBGLFR distribution. The values of log likelihood, AIC, BIC, *W**, *A**, *KS*, P-Value for all the models are listed in Table 5. The proposed MBGLFR model fits these data better than the other models according to the Table 5. The plots of the fitted pdfs, cdfs of some distributions are displayed for visual comparison in Figure 5. The MBGLFR model may be an interesting alternative to other models available in the literature for modeling positive real data.

The estimated variance-covariance matrix of the MLEs ($\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\theta}$) of the parameters of the MBGLFR distribution for the data set is given by

$$\begin{pmatrix} 0.11144487 & 0.10584027 & -0.1973001 & -0.7954907 & -0.01736836 & -0.05833310 \\ 0.10584027 & 0.14396049 & -0.4818798 & -0.8597145 & -0.02380887 & -0.07662741 \\ -0.19730006 & -0.48187977 & 90.0809792 & 1.9707454 & -0.50662976 & -1.64273388 \\ -0.79549068 & -0.85971454 & 1.9707454 & 7.1804502 & 0.28364724 & 0.45959755 \\ -0.01736836 & -0.02380887 & -0.5066298 & 0.2836472 & 0.0025787 & 0.01287525 \\ -0.05833310 & -0.07662741 & -1.6427339 & 0.4595976 & 0.01287525 & 0.05400238 \end{pmatrix}.$$

The confidence intervals (CIs) for the parameters of the MBGLFR distribution are given in Table 3.

The shape of the most appropriate pdf or hrf for modeling can be determined by a graphical analysis of the data set. In this context, we can use the total time on test plot (TTT) (see [1]) for the hrf and a basic kernel density estimator for the pdf (see [21]). Figure 3 shows that concave shape for the TTT plot, indicating that the data set has increasing hrf. Figure 4 shows that the pdf

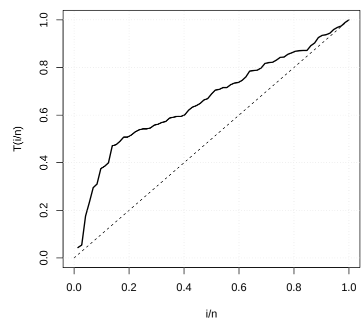

FIGURE 3. TTT plots for the considered data set.

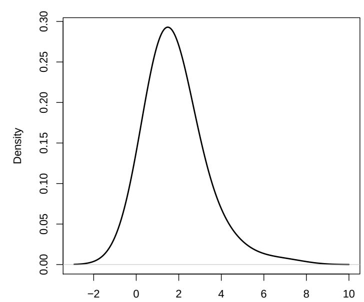

FIGURE 4. Kernel density plots for the considered data set.

is unimodal with right skewed. Hence the MBGLFR model is in principle a suitable model for fitting this kind of data set.

TABLE 3. Confidence intervals for the parameters of the MBGLFR distribution for the considered data set (the lower bounds have been remplaced put to 0 since they take negative values).

CI	a	b	c	α	λ	θ
95%	[0, 1.0752]	[0, 1.3390]	[0, 35.6590]	[0, 8.5570]	[0, 0.1778]	[0, 0.7055]
99%	[0, 1.283986]	[0, 1.577236]	[0, 41.543040]	[0, 10.230880]	[0, 0.209206]	[0, 0.851490]

Table 4. MLEs (standard errors in parentheses).

Distribution	Estimates					
MBGLFR(a, b, c, α, λ, θ)	0.4153	0.5860	17.0560	3.2654	0.0784	0.2439
	(0.3367)	(0.3842)	(09.4911)	(2.6998)	(0.0507)	(0.2355)
Kw-WE(λ, a, b, c, β)	3.63748	1.84467	2.14003	0.79822	0.03756	
	(2.58073)	(2.76562)	(1.12893)	(1.06864)	(0.11321)	
BW(a, b, c, β)	2.73456	0.90765	0.66618	0.32174		
	(1.59435)	(1.49643)	(0.24362)	(0.43221)		
Kw-W(a, b, c, β)	4.12327	2.94308	0.45855	0.21630		
	(5.83511)	(8.10936)	(0.51350)	(0.24837)		
W(c, β)	1.04782	0.10459				
	(0.06757)	(0.00933)				

Table 5. The $\hat{\ell}$, AIC, BIC, W^*, A^*, KS, P-Value values for the considered data set.

Distributon	$\hat{\ell}$	AIC	BIC	W^*	A^*	KS	P-Value
MBGLFR	100.1782	212.0563	218.0163	0.0465	0.3192	0.0745	0.8187
Kw-WE	102.9913	215.9826	227.3659	0.1141	0.7548	0.1045	0.4103
BW	102.7950	213.5901	222.6967	0.1097	0.7255	0.1010	0.4537
Kw-W	102.7097	213.4195	222.5261	0.1077	0.7107	0.1009	0.4557
W	104.0168	212.7336	219.5869	0.1602	0.9758	0.1134	0.3121

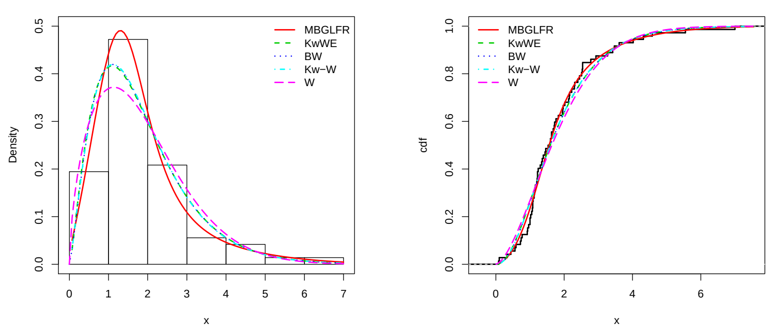

Figure 5. Plots of estimated pdfs and cdfs for given data set.

5. Concluding remarks

In this paper, we introduce a new distribution referred to as the modified beta generalized linear failure rate. It generalizes the well-known; beta linear failure rate distribution, the generalized linear failure rate distribution, beta geometric generalized linear failure rate distribution, the beta exponential distribution, the beta Rayleigh distribution, the generalized exponential distribution, and the linear failure rate distribution. Besides it contains some new sub-models. Some basic properties are derived. The maximum likelihood estimators of the parameters are obtained. Simulation studies as well as a real data application are described to show superior performance of the proposed model versus some other existing models.

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