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SOME NEW INEQUALITIES RELATED TO (α, m)-CONVEX FUNCTIONS

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Abstract. In this paper, we establish some results for functions whose derivatives in absolute values at certain power is generalized (α, m)-convex function. We derive some new estimates to the right-hand side of Hermite-Hadamard inequality for functions whose absolute values of second derivatives raised to real powers are m-convex functions. Special cases are discussed as applications of our results. The technique of this paper may stimulate further research in this field.
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1. Introduction

The theory of convex analysis has been appeared as one of the most interesting and powerful technique to study a wide class of unrelated problems in a unified framework. Convexity theory has numerous applications in business, industry, art and medicine. In recent years, various inequalities for convex functions and their variant forms are being developed and studied using novel techniques, see [1, 2, 4, 5, 8, 10, 13].

Inequalities theory plays a very important role in developing many new results in the theory of convex functions. One of the most important inequality related to convex function is the Hermite-Hadamard inequality, see [11, 12]. This inequality is used to find the upper and lower bounds for the integrals. This inequality has been a subject of extensive research since its discovery. For the recent applications,, various refinements, and other aspects of Hermite-Hadamard inequalities, see [7, 10, 13, 27, 28] and the references therein. It is known that the minimum of a differentiable convex functions on the convex

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sets in a normed spaces can be characterized by a in equality, which is called variational inequality. For the applications, formulation, numerical methods and various aspects of variational inequalities, see [17, 18, 19, 20] and the references therein.

In recent years, convex functions have been generalized in various directions using interesting and innovative techniques. An important extension of convex functions is m-convex function, defined by the Toader, an intermediate form between the usual convexity and star shaped property [28]. The further generalization in m-convexity is (α, m)-convex functions, introduced by Mihesan [15]. For more details see [3, 6, 14, 26, 28].

Gordji et al. [9] introduced an important class of convex functions, which is called generalized (φ-convex) convex function. These generalized convex functions are nonconvex functions. For recent developments, see [16, 21, 22, 23, 24, 25, 26] and the references therein.

Motivated and inspired by the ongoing research activities, we derive some results for the generalized (α, m)-convex functions. We establish some new Hermite-Hadamard inequality for the functions whose derivatives in absolute values raise to some certain power is generalized (α, m)-convex function. We obtain some new estimates to the right-hand side of Hermite-Hadamard inequality for functions whose absolute values of second derivatives raised to real powers are generalized m-convex functions. Several special cases, which can be obtained from our main results, are also discussed.

2. Preliminaries

Let $I = [a, b]$ be an interval in real line $\mathbb{R}$. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a homogenous and differentiable function on the interior I^0 of I and let $f : I \to \mathbb{R}$ be continuous and $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous bifunction.

Definition 1. [9]. *Let I be an interval in real line $\mathbb{R}$. A function $I = [a, b] \to \mathbb{R}$ is said to be generalized convex, if there exists a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, such that*

$$f(ta + (1-t)b) \leq f(b) + t\eta(f(a), f(b)), \quad \forall a, b \in I, t \in [0, 1]. \tag{1}$$

If $\eta(f(a), f(b)) = f(a) - f(b)$, then we obtain the classical definition of a convex functions.

Definition 2. *A function $f : I = [0, b] \to \mathbb{R}$ is said to be generalized m-convex function, where $m \in (0,1]$ with respect to a bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f(tma + (1-t)b) \leq (1-t)[f(b)] + t[f(b) + \eta(mf(a), f(b))], \quad \forall a, b \in I, t \in [0, 1].$$

If $\eta(f(a), f(b)) = f(a) - f(b)$, then the definition 2 reduces to

Definition 3. [25] *A function $f : I = [0, b] \to \mathbb{R}$ is said to be m-convex, where $m \in$(0,1], if*

$$f(mta + (1 - t)b) \leq tmf(a) + (1 - t)f(b), \quad \forall a, b \in I, t \in [0, 1].$$

It is denoted by $K_m(b)$, the class of all m-convex function on $[0, b]$.

Definition 4. *A function $f : I = [a, b] \to \mathbb{R}$, $a < b$ is said to be generalized (α, m)-convex with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where $(\alpha, m) \in [0, 1]^2$, if*

$$f(mta + (1 - t)b) \leq (1 - t^{\alpha})[f(b)] + t^{\alpha}[f(b) + \eta(mf(a), f(b))], \quad \forall a, b \in I, t \in [0, 1].$$

Now we will discuss some special cases.

(1) If (α, m)=$(1, m)$, then we obtain generalized m-convex functions [15].
(2) If (α, m)=$(1, 1)$, then we have ordinary generalized convex functions [9].
(3) If (α, m)=$(1, 0)$, then we obtain star shaped generalized convex functions [6].

If $\eta(f(a), f(b)) = f(a) - f(b)$, then the Definition 4 reduces to

Definition 5. [15] *A function $f : I = [0, b] \to \mathbb{R}$, $b > 0$ is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if*

$$f(mta + (1 - t)b) \leq t^{\alpha}mf(a) + (1 - t^{\alpha})f(b), \quad \forall a, b \in I, t \in [0, 1].$$

It is denoted by $K_m^{\alpha}(b)$, the class of all (α, m)-convex function on $[a, b]$ for which $f(0) \leq 0$.

Every generalized m-convex function is generalized convex function, but some times the converse is not true.

Example 1. *Consider $f : [0, +\infty) \to \mathbb{R}$ as*

$$f(x) = cx$$

and define a bifunction $\eta(x, y) = x + y$ for all $x, y \in \mathbb{R}^+$. Then f is clearly generalized m-convex, but the converse is not true.

Definition 6. *The beta function, also called the Euler integral of the first kind, is a special function defined by*

$$\beta(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}, \quad x, y > 0$$

where $\Gamma(.)$ is a Gamma function.

We need the following results, which can be viewed as a significant refinement in the previously known results, see [3, 20].

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on the interior I^0 of I, such that $f' \in L[a, b]$, where a, $b \in I$ and $a < b$. If f is homogenous, then*

$$\frac{mf(a) + f(b)}{2} - \frac{1}{b - ma} \int_{ma}^{b} f(x)\mathrm{d}x = \frac{b - ma}{2} \int_{0}^{1} (1 - 2t)f'(tma + (1 - t)b)\mathrm{d}t.$$

Proof. Let

$$\begin{aligned} I &= \frac{b - ma}{2} \int_{0}^{1} (1 - 2t)f'(tma + (1 - t)b)\mathrm{d}t \\ &= \frac{b - ma}{2} \left[\frac{(1 - 2t)f(tma + (1 - t)b)}{ma - b} \Big|_0^1 - \frac{1}{ma - b} \int_{0}^{1} f(tma + (1 - t)b)(-2)\mathrm{d}t \right] \\ &= \frac{b - ma}{2} \left[\frac{-f(ma) - f(b)}{ma - b} + \frac{2}{ma - b} \int_{0}^{1} f(tma + (1 - t)b)\mathrm{d}t \right] \\ &= \left[\frac{f(ma) + f(b)}{2} - \frac{1}{b - ma} \int_{ma}^{b} f(x)\mathrm{d}x \right] \\ &= \left[\frac{mf(a) + f(b)}{2} - \frac{1}{b - ma} \int_{ma}^{b} f(x)\mathrm{d}x \right], \end{aligned}$$

where we have used the fact that$f(mx) = mf(x)$. □

In the similar way, one can prove the following result.

Lemma 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a homogenous and twice differentiable function on the interior I^0 of I , $a < b$ with $m \in (0, 1]$ and a, $b \in I$. If $f'' \in L[a, b]$, then*

$$\begin{aligned} &\frac{mf(a) + f(b)}{2} - \frac{1}{b - ma} \int_{ma}^{b} f(x)\mathrm{d}x \\ = &\frac{(b - ma)^2}{2} \left[\int_{0}^{1} (t - t^2)f''(tma + (1 - t)b)\mathrm{d}t \right]. \end{aligned}$$

3. Main results

In this section, we derive our main results.

Theorem 3. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a homogenous and differentiable function on the interior I^0 of I and $a < b$ such that $f' \in L[a, b]$, where a, $b \in I$. If $|f'|$ is generalized (α, m)-convex functions on $[a, b]$ for $\alpha \in [0, 1]$ and m*

$\in (0,1]$. *Then*

$$
\begin{aligned}
&\left|\frac{mf(a)+f(b)}{2}-\frac{1}{b-ma}\int_{ma}^{b} f(x)\mathrm{d}x\right| \\
\leq\;& \frac{b-ma}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\min\Bigg[\Bigg(\frac{1}{(\alpha+1)(\alpha+2)}\{\alpha+(\frac{1}{2})^{\alpha}\}|f'(b)|^{q} \\
&+\frac{1}{(\alpha+1)(\alpha+2)}\{\frac{\alpha^2+\alpha+2}{2}-(\frac{1}{2})^{\alpha}\}|f'(b)+\eta(mf'(a),f'(b)]|^{q}\Bigg)^{\frac{1}{q}} \\
&,\Bigg(\frac{1}{(\alpha+1)(\alpha+2)}\{\alpha+(\frac{1}{2})^{\alpha}\}|mf'(a)|^{q}+\frac{1}{(\alpha+1)(\alpha+2)}\{\frac{\alpha^2+\alpha+2}{2} \\
&-(\frac{1}{2})^{\alpha}\}|mf'(a)+\eta(f'(b),mf'(a)]|^{q}\Bigg)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

Proof. Here we consider two cases, namely $q=1$ and $q>1$. For $q=1$, using Lemma 1, we have

$$
\begin{aligned}
&\left|\frac{mf(a)+f(b)}{2}-\frac{1}{b-ma}\int_{ma}^{b} f(x)\mathrm{d}x\right| \\
\leq\;& \frac{b-ma}{2}\int_0^1 |1-2t||f'(mta+(1-t)b)|\mathrm{d}t \\
\leq\;& \frac{b-ma}{2}\int_0^1 |1-2t|\{(1-t^{\alpha})|f'(b)|+t^{\alpha}|(f'(b)+\eta(mf'(a),f'(b))|\}\mathrm{d}t \\
=\;& \frac{b-ma}{2}\Bigg\{\int_0^1 |1-2t|\{(1-t^{\alpha})|f'(b)|\}\mathrm{d}t \\
&+\int_0^1 |1-2t|\{t^{\alpha}|f'(b)+\eta(mf'(a),f'(b))|\}\mathrm{d}t \\
=\;& \frac{b-ma}{2}\Bigg\{\int_0^{\frac{1}{2}}(1-2t)(1-t^{\alpha})|f'(b)|\mathrm{d}t+\int_{\frac{1}{2}}^1(2t-1)(1-t^{\alpha})|f'(b)|\mathrm{d}t \\
&+\int_0^{\frac{1}{2}}(1-2t)t^{\alpha}|f'(b)+\eta(mf'(a),f'(b))|\mathrm{d}t \\
&+\int_{\frac{1}{2}}^1(2t-1)t^{\alpha}|f'(b)+\eta(mf'(a),f'(b))|\mathrm{d}t \\
=\;& \frac{b-ma}{2}\Bigg[\frac{1}{(\alpha+1)(\alpha+2)}\{\alpha+(\frac{1}{2})^{\alpha}\}|f'(b)| \\
&+\frac{1}{(\alpha+1)(\alpha+2)}\{\frac{\alpha^2+\alpha+2}{2}-(\frac{1}{2})^{\alpha}\}|f'(b)+\eta(mf'(a),f'(b)|\Bigg].
\end{aligned}
$$

Similarly we obtain

$$
\begin{aligned}
&|\frac{mf(a)+f(b)}{2}-\frac{1}{b-ma}\int_{ma}^{b}f(x)\mathrm{d}x|\\
\leq\ &\frac{b-ma}{2}\Big[\frac{1}{(\alpha+1)(\alpha+2)}\{\alpha+(\frac{1}{2})^{\alpha}\}|mf'(a)|\\
&+\frac{1}{(\alpha+1)(\alpha+2)}\{\frac{\alpha^2+\alpha+2}{2}-(\frac{1}{2})^{\alpha}\}|mf'(a)+\eta(f'(b),mf'(a))|\Big].
\end{aligned}
$$

Now for $q>1$, using the Holder's inequality $\frac{1}{p}+\frac{1}{q}=1$, for q and $p=\frac{q}{q-1}$, we obtain

$$
\begin{aligned}
&|\frac{mf(a)+f(b)}{2}-\frac{1}{b-ma}\int_{ma}^{b}f(x)\mathrm{d}x|\\
\leq\ &\frac{b-ma}{2}\int_0^1|1-2t||f'(mta+(1-t)b)|\mathrm{d}t\\
\leq\ &\frac{b-ma}{2}\Big(\int_0^1|1-2t|\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_0^1|1-2t||f'(mta+(1-t)b)|^q\mathrm{d}t\Big)^{\frac{1}{q}}\\
\leq\ &\frac{b-ma}{2}\Big[\Big(\int_0^1|1-2t|\mathrm{d}t\Big)^{1-\frac{1}{q}}\\
&\Big(\int_0^1|1-2t|[(1-t^{\alpha})|f'(b)|^q+t^{\alpha}|(f'(b)+\eta(mf'(a),f'(b)))|^q]\mathrm{d}t\Big)^{\frac{1}{q}}\\
=\ &\frac{b-ma}{2}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Big(\frac{1}{(\alpha+1)(\alpha+2)}\{\alpha+(\frac{1}{2})^{\alpha}\}|f'(b)|^q\\
&+\frac{1}{(\alpha+1)(\alpha+2)}\{\frac{\alpha^2+\alpha+2}{2}-(\frac{1}{2})^{\alpha}\}|f'(b)+\eta(mf'(a),f'(b)]|^q\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|\frac{mf(a)+f(b)}{2}-\frac{1}{b-ma}\int_{ma}^{b}f(x)\mathrm{d}x|\\
=\ &\frac{b-ma}{2}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Big(\frac{1}{(\alpha+1)(\alpha+2)}\{\alpha+(\frac{1}{2})^{\alpha}\}|mf'(a)|^q\\
&+\frac{1}{(\alpha+1)(\alpha+2)}\{\frac{\alpha^2+\alpha+2}{2}-(\frac{1}{2})^{\alpha}\}|mf'(a)+\eta(f'(b),mf'(a))|^q\Big)^{\frac{1}{q}},
\end{aligned}
$$

which competes the proof. □

Corollary 4. *If $\eta(f(b), f(a)) = f(b) - f(a)$, then under the assumption of Theorem 3, we have*

$$
\begin{aligned}
&|\frac{mf(a)+f(b)}{2} - \frac{1}{b-ma}\int_{ma}^{b} f(x)\mathrm{d}x| \\
\leq\ & \frac{b-ma}{4}\min\Bigg[\Bigg(\frac{1}{2}\Big[\frac{1}{(\alpha+1)(\alpha+2)}\{\alpha+(\frac{1}{2})^{\alpha}\}|f'(b)|^{q} \\
&+\frac{1}{(\alpha+1)(\alpha+2)}\{\frac{\alpha^2+\alpha+2}{2}-(\frac{1}{2})^{\alpha}\}|mf'(a)|^{q}\Big]\Bigg)^{\frac{1}{q}} \\
&,\Bigg(\frac{1}{2}\Big[\frac{1}{(\alpha+1)(\alpha+2)}\{\alpha+(\frac{1}{2})^{\alpha}\}|mf'(a)|^{q}+\frac{1}{(\alpha+1)(\alpha+2)}\{\frac{\alpha^2+\alpha+2}{2} \\
&-(\frac{1}{2})^{\alpha}\}|(f'(b)|^{q}\Big]\Bigg)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

Theorem 5. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a homogenous and differentiable function on the interior I^0 of I and $a < b$ such that $f' \in L[a,b]$, where a, b $\in I$. If $|f'|$ is generalized (α,m)-convex functions on $[a,b]$ for $\alpha \in [0,1]$, $m \in (0,1]$ and $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$. Then*

$$
\begin{aligned}
&|\frac{mf(a)+f(b)}{2} - \frac{1}{b-ma}\int_{ma}^{b} f(x)\mathrm{d}x| \\
\leq\ & \frac{b-ma}{2}\Bigg\{(\frac{1}{p+1})^{\frac{1}{p}}\{\frac{\alpha}{\alpha+1}|f'(b)|^{q}+\frac{1}{\alpha+1}|(f'(b)+\eta(mf'(a),f'(b))|^{q}\}^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

Proof. From Lemma 1 and using Holder's inequality, we have

$$
\begin{aligned}
&|\frac{f(a)+f(b)}{2}| - \frac{1}{b-ma}\int_{a}^{b} f(x)\mathrm{d}x| \\
\leq\ & \frac{b-ma}{2}\int_0^1 |1-2t||f'(ta+(1-t)b)|\mathrm{d}t \\
\leq\ & \frac{b-ma}{2}\int_0^1 \big(|1-2t|^{p}\mathrm{d}t\big)^{\frac{1}{p}}\Big(\int_0^1 |f'(mta+(1-t)b)|^{q}\mathrm{d}t\Big)^{\frac{1}{q}} \\
\leq\ & \frac{b-ma}{2}\Bigg(\int_0^1 |1-2t|^{p}\mathrm{d}t\Bigg)^{\frac{1}{p}}\Bigg(\int_0^1 (1-t^{\alpha})|f'(b)|^{q}+t^{\alpha}|(f'(b)+\eta(mf'(a),f'(b))|^{q}\Bigg)^{\frac{1}{q}} \\
=\ & \frac{b-ma}{2}\Bigg\{(\frac{1}{p+1})^{\frac{1}{p}}\{\frac{\alpha}{\alpha+1}|f'(b)|^{q}+\frac{1}{\alpha+1}|(f'(b)+\eta(mf'(a),f'(b))|^{q}\}^{\frac{1}{q}}\Bigg\},
\end{aligned}
$$

which is the required result. □

Corollary 6. *If $\eta(f(b), f(a)) = f(b) - f(a)$ and $\alpha = m = 1$, then under the assumption of Theorem 5, we have*

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2}\right| - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Big| \\ \leq \;& \frac{b-a}{2}\left\{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\frac{1}{2}[|f'(b)|^q + |f'(a)|^q]\right\}^{\frac{1}{q}}\right\}. \end{aligned}$$

Theorem 7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a homogenous and twice differentiable function on the interior I^0 of I and $a < b$ such that $f'' \in L[a, b]$, where a, b $\in I$. If $|f''|$ is generalized (α, m)-convex functions on $[a, b]$ for $\alpha \in [0, 1]$, $m \in (0, 1]$ and $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned} & \left|\frac{mf(a)+f(b)}{2}\right| - \frac{1}{b-ma}\int_{ma}^b f(x)\mathrm{d}x\Big| \\ \leq \;& \frac{(b-ma)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\Bigg[\left(\frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)}\right)|f''(b)|^q \\ & + \frac{1}{(\alpha+2)(\alpha+3)}|f''(b) + \eta(mf''(a), f''(b)]^q|\Bigg]^{\frac{1}{q}}. \end{aligned}$$

Proof. Here we consider two cases, $q = 1$ and $q > 1$.
For $q = 1$, using Lemma 2 and Holder's inequality, we have

$$\begin{aligned} & \left|\frac{mf(a)+f(b)}{2} - \frac{1}{b-ma}\int_{ma}^b f(x)\mathrm{d}x\right| \\ \leq \;& \frac{(b-ma)^2}{2}\int_0^1 (t-t^2)|f''(mta + (1-t)b)|\mathrm{d}t. \\ \leq \;& \frac{(b-ma)^2}{2}\int_0^1 (t-t^2)[(1-t^\alpha)|f''(b)| + t^\alpha|f''(b) + \eta(mf''(a), f''(b))|]\mathrm{d}t \\ = \;& \frac{b-ma}{2}\Bigg\{\int_0^1 (t-t^2)\{(1-t^\alpha)|f''(b)|\}\mathrm{d}t \\ & + \int_0^1 (t-t^2)\{t^\alpha|f''(b) + \eta(mf''(a), f''(b))|\}\mathrm{d}t \\ = \;& \frac{(b-ma)^2}{2}\Bigg[\frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)}|f''(b)| \\ & + \frac{1}{(\alpha+2)(\alpha+3)}|f''(b) + \eta(mf''(a), f''(b)|\Bigg]. \end{aligned}$$

Now for $q > 1$, using the Holder's inequality $\frac{1}{p} + \frac{1}{q} = 1$ for q and $p = \frac{q}{q-1}$, we obtain

$$
\begin{aligned}
&\left|\frac{mf(a)+f(b)}{2} - \frac{1}{b-ma}\int_{ma}^{b} f(x)\mathrm{d}x\right| \\
\leq\ & \frac{(b-ma)^2}{2}\int_0^1 (t-t^2)\left|f''(mta+(1-t)b)\right|\mathrm{d}t \\
\leq\ & \frac{(b-ma)^2}{2}\Big(\int_0^1 (t-t^2)\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_0^1 (t-t^2)\left|f''(mta+(1-t)b)\right|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\
\leq\ & \frac{(b-ma)^2}{2}\Big(\int_0^1 (t-t^2)\mathrm{d}t\Big)^{1-\frac{1}{q}} \\
&\Big(\int_0^1 (t-t^2)[(1-t^\alpha)\left|f''(b)\right|^q + t^\alpha\left|f''(b)+\eta(mf''(a),f''(b))\right|^q]\mathrm{d}t\Big)^{\frac{1}{q}} \\
=\ & \frac{(b-ma)^2}{2}\Big(\frac{1}{6}\Big)^{1-\frac{1}{q}}\Big[\Big(\frac{1}{6}-\frac{1}{(\alpha+2)(\alpha+3)}\Big)\left|f''(b)\right|^q \\
&+\frac{1}{(\alpha+2)(\alpha+3)}\left|f''(b)+\eta(mf''(a),f''(b))\right|^q\Big]^{\frac{1}{q}},
\end{aligned}
$$

which competes the proof. □

Corollary 8. *If $\eta(f(b), f(a)) = f(b) - f(a)$ and $\alpha = q = m = 1$, then under the assumption of Theorem 7, we have*

$$
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \tfrac{(b-a)^2}{24}\big(\left|f''(b)\right| + \left|f''(a)\right|\big).
$$

Theorem 9. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a homogenous and twice differentiable function on the interior I^0 of I and $a < b$ such that $f'' \in L[a, b]$, where a, $b \in I$. If $|f''|$ is generalized (α, m)-convex functions on $[a, b]$ for $\alpha \in [0, 1]$, $m \in (0, 1]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$
\begin{aligned}
&\left|\frac{mf(a)+f(b)}{2} - \frac{1}{b-ma}\int_{ma}^{b} f(x)\mathrm{d}x\right| \\
\leq\ & \frac{(b-ma)^2}{2}\Big[\frac{1}{q+1} - \beta(\alpha+1,q+1)\left|f''(b)\right|^q \\
&+\beta(\alpha+1,q+1)\left|f''(b)+\eta(mf''(a),f''(b))\right|^q\Big]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. From Lemma 2 and using Holder's inequality, we have

$$
\begin{aligned}
&|\frac{mf(a)+f(b)}{2} - \frac{1}{b-ma}\int_{ma}^{b} f(x)\mathrm{d}x| \\
\leq\ & \frac{(b-ma)^2}{2}\int_0^1 (t-t^2)|f''(mta+(1-t)b)|\mathrm{d}t \\
\leq\ & \frac{(b-ma)^2}{2}\Big(\int_0^1 (t)^p \mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 (1-t)^q|f''(mta+(1-t)b)|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\
\leq\ & \frac{(b-ma)^2}{2}\Big(\int_0^1 (t)^p \mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 (1-t)^q[(1-t^\alpha)|f''(b)|^q \\
& +t^\alpha|f''(b)+\eta(mf''(a),f''(b))|^q]\mathrm{d}t\Big)^{\frac{1}{q}} \\
=\ & \frac{(b-ma)^2}{2}\Big[\frac{1}{p+1}\Big]^{\frac{1}{p}}\Big[\frac{1}{q+1} - \beta(\alpha+1,q+1)|f''(b)|^q \\
& +\beta(\alpha+1,q+1)|f''(b)+\eta(mf''(a),f''(b))|^q\Big]^{\frac{1}{q}} \\
=\ & \frac{(b-ma)^2}{2}\Big[\frac{1}{q+1} - \beta(\alpha+1,q+1)|f''(b)|^q \\
& +\beta(\alpha+1,q+1)|f''(b)+\eta(mf''(a),f''(b))|^q\Big]^{\frac{1}{q}},
\end{aligned}
$$

where we have used the fact that $\frac{1}{2} < (\frac{1}{p+1})^{\frac{1}{p}} < 1$. □

Corollary 10. *If $\eta(f(b),f(a)) = f(b)-f(a)$ and $\alpha = m = 1$, then under the assumption of Theorem 9, we have*

$$
\begin{aligned}
&|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_{a}^{b} f(x)\mathrm{d}x| \\
\leq\ & \frac{(b-a)^2}{2}\Big[\frac{1}{q+1} - \beta(2,q+1)|f''(b)|^q + \beta(2,q+1)|f''(a)|^q\Big]^{\frac{1}{q}}.
\end{aligned}
$$

Theorem 11. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a homogenous and twice differentiable function on the interior I^0 of I and $a < b$ such that $f'' \in L[a,b]$, where a, b $\in I$. If $|f''|$ is generalized (α,m)-convex functions on $[a,b]$ for $\alpha \in[0,1]$, $m \in (0,1]$ and $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$. Then*

$$
\begin{aligned}
&|\frac{mf(a)+f(b)}{2}| - \frac{1}{b-ma}\int_{ma}^{b} f(x)\mathrm{d}x| \\
\leq\ & \frac{(b-ma)^2}{2}\big[\beta(p+1,p+1)\big]^{\frac{1}{p}}\Big[\frac{\alpha}{\alpha+1}|f''(b)|^q \\
& +\frac{1}{\alpha+1}|f''(b)+\eta(mf''(a),f''(b))|^q\Big]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. From Lemma 2 and using Holder's inequality, we have

$$\begin{aligned}
&\left|\frac{mf(a)+f(b)}{2}\right| - \frac{1}{b-ma}\int_{ma}^{b} f(x)\mathrm{d}x\Big| \\
\leq\ & \frac{(b-ma)^2}{2}\int_0^1 (t-t^2)|f''(mta+(1-t)b)|\mathrm{d}t \\
\leq\ & \frac{(b-ma)^2}{2}\Big(\int_0^1 (t-t^2)^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 |f''(mta+(1-t)b)|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\
\leq\ & \frac{(b-ma)^2}{2}\Big(\int_0^1 (t-t^2)^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 [(1-t^\alpha)|f''(b)|^q \\
& +t^\alpha|f''(b)+\eta(mf''(a),f''(b))|^q]\mathrm{d}t\Big)^{\frac{1}{q}} \\
=\ & \frac{(b-ma)^2}{2}\big[\beta(p+1,p+1)\big]^{\frac{1}{p}}\Big[\frac{\alpha}{\alpha+1}|f''(b)|^q \\
& +\frac{1}{\alpha+1}|f''(b)+\eta(mf''(a),f''(b))|^q\Big]^{\frac{1}{q}}
\end{aligned}$$

which is the required result. □

Corollary 12. *If $\eta(f(b),f(a)) = f(b)-f(a)$ and $\alpha = m = 1$, then under the assumption of Theorem 11, we have*

$$\begin{aligned}
&\Big|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Big| \\
\leq\ & \frac{(b-a)^2}{2}\big[\beta(p+1,p+1)\big]^{\frac{1}{p}}\Big[\frac{1}{2}\{|f''(b)|^q+|f''(a)|^q\}\Big]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 13. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a homogenous and twice differentiable function on the interior I^0 of I and $a < b$ such that $f'' \in L[a,b]$, where a, b $\in I$. If $|f''|$ is generalized (α, m)-convex functions on $[a,b]$ for $\alpha \in [0,1]$, $m \in (0,1]$ and $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$. Then*

$$\begin{aligned}
&\Big|\frac{mf(a)+f(b)}{2} - \frac{1}{b-ma}\int_{ma}^{b} f(x)\mathrm{d}x\Big| \\
\leq\ & \frac{(b-ma)^2}{2}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Big[\frac{1}{(q+1)(q+2)} - \beta(\alpha+2,q+1)|f''(b)|^q \\
& +\beta(\alpha+2,q+1)|f''(b)+\eta(mf''(a),f''(b))|^q\Big]^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 2 and using Holder's inequality, we have

$$
\begin{aligned}
& \left|\frac{mf(a)+f(b)}{2}-\frac{1}{b-ma}\int_{ma}^{b} f(x)\mathrm{d}x\right| \\
\leq\ & \frac{(b-ma)^2}{2}\int_0^1 (t-t^2)|f''(mta+(1-t)b)|\mathrm{d}t \\
\leq\ & \frac{(b-ma)^2}{2}\Big(\int_0^1 t\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t(1-t)^q|f''(mta+(1-t)b)|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\
\leq\ & \frac{(b-ma)^2}{2}\Big(\int_0^1 t\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t(1-t)^q[(1-t^\alpha)|f''(b)|^q \\
& +t^\alpha|f''(b)+\eta(mf''(a),f''(b))|^q]\mathrm{d}t\Big)^{\frac{1}{q}} \\
=\ & \frac{(b-ma)^2}{2}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Big[\frac{1}{(q+1)(q+2)}-\beta(\alpha+2,q+1)|f''(b)|^q \\
& +\beta(\alpha+2,q+1)|f''(b)+\eta(mf''(a),f''(b))|^q\Big]^{\frac{1}{q}},
\end{aligned}
$$

which is the required result. □

Corollary 14. *If $\eta(f(b),f(a))=f(b)-f(a)$ and $\alpha=m=1$, then under the assumption of Theorem 13, we have*

$$
\begin{aligned}
& \left|\frac{f(a)+f(b)}{2}\right|-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\
\leq\ & \frac{(b-a)^2}{2}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Big[\frac{1}{(q+1)(q+2)}-\beta(3,q+1)|f''(b)|^q \\
& +\beta(3,q+1)|f''(a)|^q\Big]^{\frac{1}{q}}.
\end{aligned}
$$

Conclusion

In this paper, we have introduced and studied a new class of generalized (α,m)-convex functions. Several new integral inequalities for these generalized functions have been derived, which have important applications in physics and material sciences. These estimates also useful in numerical analysis for finding the error bounds for the approximate solution. We have also discussed important several special cases, which can be obtained from our results.

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