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SIMILARITY MEASURES FOR PICTURE HESITANT FUZZY SETS AND THEIR APPLICATIONS IN PATTERN RECOGNITION

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Abstract. The concept of picture hesitant fuzzy set (PHFS) is a generalization of picture fuzzy set (PFS) and hesitant fuzzy set (HFS). Such kind of idea can be very helpful in problems where opinions are of more than two types i.e. yes, no, abstinence and refusal. The goal of this manuscript is to introduce the concept of similarity measures for PHFSs as a generalization of the similarity measures for PFS. We studied the basic concepts of PFSs, HFSs and PHFSs. We proposed some similarity measures for PHFSs such as cosine similarity measure, set-theoretic similarity measure and grey similarity measure for PHFSs. Some weighted similarity measures are also proposed where weight of the attributes is considered. Then these similarity measures for PHFSs are applied to a building material recognition problem. Finally, a comparative study of similarity measures of PHFSs is established with similarity measures of PFSs, HFSs, IFSs and IHFSs and the advantages of new work are studied.

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1. Introduction

The theory of fuzzy set (FS) introduced by Zadeh [1] is a great achievement and has some applications in various fields involving impression and uncertainty. A FS is based on a membership function which have grade of membership for each element of the universal set X on the interval [0,1]. Atanassov [2, 3] extended the notion of FS to IFS by adding non-membership grade along with the membership grade and is therefore characterized by the degree

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of membership grade and the degree of non-membership grade for each element on the universal set X. An IFS reduces to FS by taking the non-membership value equal to zero. The concept of IFS was further generalized to introduce interval-valued intuitionistic fuzzy set (IvIFS) by Atanassov and Gargov [4]. For some other recent study on FSs and IFSs, one may refer to Atanassov [5], Asiain et al. [6], Kumar and Garg [7, 8], Mahmood et al. [9] and Li [10].

In 2013, Cuong introduced PFS [11] and described some basic operation and properties of PFSs. A PFS is the direct extension of FS and IFS having a neutral membership grade along with membership grade and non-membership grade. Moreover, the PFS also described the degree of refusal of each element on the universal set X by subtracting the sum of membership grade, non-membership grade and neutral value from 1. A PFS is reduced to IFS by considering the neutral value equal to zero and further reduced to FS by taking non-membership and neutral value equal to zero. Basically, PFS is a diverse concept as compared to FS and IFS and it can be applied in various fields. The human opinions involving more answers of types: yes, no, abstain and refusal can be modeled by PFS conveniently than FS or IFS. The vote casting is a good example that can be modeled using PFS only. Some other recent work on PFS can be founded in [12, 13, 14, 15].

In 2009, Torra [16] introduced the concept of HFS. HFS is a direct generalization of FS. The HFS is characterized in terms of a function that give us some finite set of values of [0,1] interval. Some other extension of HFS have been developed by some researchers, including interval-valued hesitant fuzzy set (IvHFS) by Chen and Cai [17], interval-valued intuitionistic hesitant fuzzy set (IvIHFS) has been proposed by Zhang [18], bipolar-valued HFS proposed by Ullah et al [19] and further developed by Mahmood et al [20, 21]. For some other quality work on HFS one may refer to [22, 23, 24]. Recently, in 2018 the concept of PHFS is introduced by Wang et al [25] and its application in multi- attribute decision making (MADM) are examined. In a PHFS, we have the environment of PFSs as well as HFSs therefore provides us more suitable environment to handle complex problems.

Similarity measures of FSs are important topic in fuzzy mathematics and have gained some serious attention from researchers due to its successful applications in real life. Some similarity measures of FSs have been proposed by Pappis and Karacapilidis [26], Chen [27] and by Lehmann [28] . In 2002, Dengfeng and Chuntian [29] introduced the concept of similarity measures of IFSs and applied them to pattern recognition problems. The similarity measures introduced in [30] for IFSs have some limitations. Therefore in 2005, Liu [31] introduced some new similarity measures for IFSs. Liu's similarity measures [31] identifies that Dengfeng and Chuntian's [29] methods have the same limitation as Chen's [27]. For some other quality work on recently developed similarity measures on FSs and IFSs, one may refer to Mishra et al.

[32], Hwang et al. [33], Garg and Kumar [34], Wei and Wei [35], Ye [36], Xu and Cai [37], Mahmood et al. [38] etc. Recently , Wei [39] introduced the concept of cosine similarity measures for PFSs and studied their applications in strategic decision making while the concept of grey and set-theoretic similarity measures have been proposed by Wei in [40]. Some dice and generalized dice similarity measures have been discussed by Joshi and Kumar [41] and Wei and Gao respectively [42]. Zhang et al [43] introduced the cosine similarity measures for HFSs while Sun et al. [44] developed the grey similarity measures for HFSs and applied them in pattern recognition problems. Basically, similarity measures are very useful in real life problems such as decision making, machine learning, pattern recognitions, medical diagnosis etc. for some other recently developed similarity measures and their applications, one is referred to Ullah et al [45], Palmeira et al. [46] and Zhang et al. [47] etc.

As described earlier, a PHFS is an advanced form of PFS as well as HFS and can cope with complex information having hesitancy. The aim of this article is to analyze several similarity measures developed by Ye [36], Xu and Cai [37], Wei [39, 40] and Ullah et al [45] and developed the concepts of cosine, grey and set-theoretic similarity measures for PHFSs. The generalizations of new similarity measures over the pre-existing concepts is proved.

This paper is organized as in section first the history of existing concepts is discussed in detail. In section two, we discussed some basic definitions of IFSs, PFSs, HFSs and PHFSs. In section three, some similarity measures and some weighted measures for PHFSs are proposed based on the concept of the similarity measures of IFSs and PFSs. In section four, the similarity measures for PHFSs are applied to building material recognition problem and results are studied. Section five and six are based on a comparative study of proposed work and its advantages respectively. In section seven we summarized the article along with some future directions.

2. Preliminaries

In this section, we studied some basic definitions and notion related to IFSs, PFSs, HFSs, PHFSs. In our study by X we mean the universal set and μ, η and υ denote the grade of membership, grade of neutral and grade of non-membership on $[0,1]$ interval.

2.1. **Definition** [2]**.** An IFS A on X is of the shape $A = \{\langle x, \mu_A(x), \upsilon_A(x)\rangle \,|x \in X\}$ provided that $0 \leq \mu_A(x) + \upsilon_A(x) \leq 1$. Further, (μ, υ) is an intuitionistic fuzzy number (IFN).

2.2. **Definition** [11]**.** An PFS A on X is of the shape $A = \{< x, \mu_A(x), \eta_A(x), \upsilon_A(x) > |x \in X\}$ provided that $0 \leq \mu_A(x) + \eta_A(x) + \upsilon_A(x) \leq 1$. Moreover, $\pi_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \upsilon_A(x))$ is the degree of hesitancy of $x \in X$ in A. Further, (μ, η, υ) is a picture fuzzy number (PFN).

2.3. **Definition** [16]. HFS H on X is of the shape $H = \{\langle x, h(x)\rangle | \forall x \in X\}$ where $h : X \to [0,1]$ is a finite set of values. Moreover, $h(x)$ is called hesitant fuzzy number (HFN).

2.4. **Definition** [25]. A PHFS A on X is of the shape $A = \{< x, \mu_A(x), \eta_A(x), \upsilon_A(x) > | x \in X\}$ provided that $0 \leq Sup(\mu_A(x_i)) + Sup(\eta_A(x_i)) + Sup(\upsilon_A(x_i)) \leq 1$. Further, $\pi_A(x_i) = 1 - (Sup(\mu_A(x_i)) + Sup(\eta_A(x_i)) + Sup(\upsilon_A(x_i)))$ is the degree of refusal of $x_i \in X$ in A and (μ, η, υ) is a picture hesitant fuzzy number (PHFN).

2.5. **Definition** [25]. For two PHFNs $A = (\mu_A, \eta_A, \upsilon_A)$ and $B = (\mu_B, \eta_B, \upsilon_B)$, we have

$$\mathbf{1} : A \cup B = \left\{ \left\langle \begin{array}{c} x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \\ \min(\upsilon_A(x), \upsilon_B(x)) \end{array} \right\rangle | x \in X \right\}$$

$$\mathbf{2} : A \cap B = \left\{ \left\langle \begin{array}{c} x, \min(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \\ \max(\upsilon_A(x), \upsilon_B(x)) \end{array} \right\rangle | x \in X \right\}$$

$$\mathbf{3} : A^C = \{\langle \upsilon_A(x), \eta_A(x), \mu_A(x)\rangle | x \in X\}$$

2.6. **Definition** [36]. For two IFNs $A = (\mu_A, \upsilon_A)$ and $B = (\mu_B, \upsilon_B)$ on X, a cosine similarity measure is dafined as:

$$C^1_{IFS}(A,B) = \frac{1}{n} \sum_{i=1}^{n} \frac{\mu_A(x_i)\mu_B(x_i) + \upsilon_A(x_i)\upsilon_B(x_i)}{\sqrt{\mu_A^2(x_i) + \upsilon_A^2(x_i)}\sqrt{\mu_B^2(x_i) + \upsilon_B^2(x_i)}} \tag{1}$$

2.7. **Definition** [37]. For two IFNs $A = (\mu_A, \upsilon_A)$ and $B = (\mu_B, \upsilon_B)$ on X, a set-theroetic similarity measure is dafined as:

$$C^2_{IFS}(A,B) = \frac{1}{n} \sum_{i=1}^{n} \frac{\mu_A(x_i)\mu_B(x_i) + \upsilon_A(x_i)\upsilon_B(x_i)}{\max\left(\mu_A^2(x_i) + \upsilon_A^2(x_i), \mu_B^2(x_i) + \upsilon_B^2(x_i)\right)} \tag{2}$$

2.8. **Definition** [37]. For two IFNs $A = (\mu_A, \upsilon_A)$ and $B = (\mu_B, \upsilon_B)$ on X, the grey similarity measure is dafined as:

$$C^3_{IFS}(A,B) = \frac{1}{3n} \sum_{i=1}^{n} \left(\frac{\Delta\mu_{\min} + \Delta\mu_{\max}}{\Delta\mu_i + \Delta\mu_{\max}} + \frac{\Delta\upsilon_{\min} + \Delta\upsilon_{\max}}{\Delta\upsilon_i + \Delta\upsilon_{\max}} \right) \tag{3}$$

Where $\Delta\mu_i = |\mu_A(x_i) - \mu_B(x_i)|$, $\Delta\mu_{\min} = \min\{|\mu_A(x_i) - \mu_B(x_i)|\}$, $\Delta\mu_{\max} = \max\{|\mu_A(x_i) - \mu_B(x_i)|\}$, $\Delta\upsilon_i = |\upsilon_A(x_i) - \upsilon_B(x_i)|$, $\Delta\upsilon_{\min} = \min\{|\upsilon_A(x_i) - \upsilon_B(x_i)|\}$, $\Delta\upsilon_{\max} = \max\{|\upsilon_A(x_i) - \upsilon_B(x_i)|\}$.

2.9. **Definition** [40]**.** For two PFNs $A = (\mu_A, \eta_A, \upsilon_A)$ and $B = (\mu_B, \eta_B, \upsilon_B)$ on X, a cosine similarity measure is dafined as:

$$C^1_{PFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_A(x_i)\mu_B(x_i) + \eta_A(x_i)\eta_B(x_i) + \upsilon_A(x_i)\upsilon_B(x_i)}{\sqrt{\mu_A^2(x_i) + \eta_A^2(x_i) + \upsilon_A^2(x_i)}\sqrt{\mu_B^2(x_i) + \eta_B^2(x_i) + \upsilon_B^2(x_i)}} \quad (4)$$

2.10. **Defiition** [40]**.** For two PFNs $A = (\mu_A, \eta_A, \upsilon_A)$ and $B = (\mu_B, \eta_B, \upsilon_B)$ on X, a set-theoretic similarity measure is dafined as:

$$C^2_{PFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_A(x_i)\mu_B(x_i) + \eta_A(x_i)\eta_B(x_i) + \upsilon_A(x_i)\upsilon_B(x_i)}{\max\left(\mu_A^2(x_i) + \eta_A^2(x_i) + \upsilon_A^2(x_i), \mu_B^2(x_i) + \eta_B^2(x_i) + \upsilon_B^2(x_i)\right)} \quad (5)$$

2.11. **Definition** [40]**.** For two IFNs $A = (\mu_A, \eta_A, \upsilon_A)$ and $B = (\mu_B, \eta_B, \upsilon_B)$ on X, the grey similarity measure is dafined as:

$$C^3_{PFS}(A,B) = \frac{1}{3n}\sum_{i=1}^{n}\left(\frac{\Delta\mu_{\min} + \Delta\mu_{\max}}{\Delta\mu_i + \Delta\mu_{\max}} + \frac{\Delta\eta_{\min} + \Delta\eta_{\max}}{\Delta\eta_i + \Delta\eta_{\max}} + \frac{\Delta\upsilon_{\min} + \Delta\upsilon_{\max}}{\Delta\upsilon_i + \Delta\upsilon_{\max}}\right) \quad (6)$$

Where $\Delta\mu_i = |\mu_A(x_i) - \mu_B(x_i)|$, $\Delta\mu_{\min} = \min\{|\mu_A(x_i) - \mu_B(x_i)|\}$, $\Delta\mu_{\max} = \max\{|\mu_A(x_i) - \mu_B(x_i)|\}$, $\Delta\eta_i = |\eta_A(x_i) - \eta_B(x_i)|$, $\Delta\eta_{\min} = \min\{|\eta_A(x_i) - \eta_B(x_i)|\}$, $\Delta\eta_{\max} = \max\{|\eta_A(x_i) - \eta_B(x_i)|\}$, $\Delta\upsilon_i = |\upsilon_A(x_i) - \upsilon_B(x_i)|$, $\Delta\upsilon_{\min} = \min\{|\upsilon_A(x_i) - \upsilon_B(x_i)|\}$, $\Delta\upsilon_{\max} = \max\{|\upsilon_A(x_i) - \upsilon_B(x_i)|\}$.

2.12. **Definition** [43]**.** For a HFS on X. $S(h(x_i)) = \frac{1}{lh(x_i)}\sum_{\gamma\in h(x_i)}\gamma$ is called the score function of $h(x_i)$, where $lh(x_i)$ is the length of the $h(x_i)$.

Now we define the inclusion of two PHFSs.

2.13. **Defintion** [25]**.** Let A and B be the PHFSs. Then $A \subseteq B$ iff

a: $S(\mu_A(x)) \leq S(\mu_B(x)) \Longrightarrow \mu_A(x) \leq \mu_B(x)$
b: $S(\eta_A(x)) \leq S(\eta_B(x)) \Longrightarrow \eta_A(x) \leq \eta_B(x)$
c: $S(\upsilon_A(x)) \geq S(\upsilon_B(x)) \Longrightarrow \upsilon_A(x) \geq \upsilon_B(x)$

3. Similarity Measures

The aim of this section is to develop some similarity measures for PHFSs as generalization of similarity measures of IFSs and PFSs. With the help of some remarks we developed the similarity measures for intuitionistic hesitant fuzzy set (IHFS) also we defined concepts are demonstrated by examples. In our next study we denote the set of all PHFNs on X by PHFS(X). By $l\mu_A$ we mean the length of μ_A and similar. Further μ^*, η^* and υ^* denote the length and defined as: $\mu^* = \max(l\mu_A, l\mu_B)$, $\eta^* = \max(l\eta_A, l\eta_B)$ and $\upsilon^* = \max(l\upsilon_A, l\upsilon_B)$ respectively.

3.1. **Cosine Similarity Measure.** The work developed in this section is a generalization of the work of IFS [36, 37] and PFS [40].

3.1.1. *Definition.* For $A, B \in PHFS(X)$, we define the cosine similarity measure as:

$$C^1_{PHFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\mu_A^j(x_i)\,\mu_B^j(x_i) + \frac{1}{\eta^*}\sum_{k=1}^{\eta^*}\eta_A^k(x_i)\,\eta_B^k(x_i) + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\upsilon_A^m(x_i)\,\upsilon_B^m(x_i)}{\sqrt{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\left(\mu_A^j(x_i)\right)^2 + \frac{1}{\eta^*}\sum_{k=1}^{\eta^*}\left(\eta_A^k(x_i)\right)^2 + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\left(\upsilon_A^m(x_i)\right)^2}\cdot\sqrt{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\left(\mu_B^j(x_i)\right)^2 + \frac{1}{\eta^*}\sum_{k=1}^{\eta^*}\left(\eta_B^k(x_i)\right)^2 + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\left(\upsilon_B^m(x_i)\right)^2}} \right) \tag{7}$$

The cosine similarity measure for PHFSs A, B and C satisfy following conditions:

$1 : 0 \leq C^1_{PHFS}(A,B) \leq 1$
$2 : C^1_{PHFS}(A,B) = C^1_{PHFS}(B,A)$
$3 : C^1_{PHFS}(A,B) = 1$ iff $A = B$
$4 : A \subseteq B \subseteq C$, then $C^1_{PHFS}(A,C) \leq C^1_{PHFS}(A,B)$ and $C^1_{PHFS}(A,C) \leq C^1_{PHFS}(B,C)$.

Proof. The proof of first three parts is obvious. For condition no. (4), let A, B and $C \in PHFS(X)$. We know that if

a : $S(\mu_A(x)) \leq S(\mu_B(x)) \leq S(\mu_C(x)) \Longrightarrow \mu_A(x) \leq \mu_B(x) \leq \mu_C(x)$
b : $S(\eta_A(x)) \leq S(\eta_B(x)) \leq S(\eta_C(x)) \Longrightarrow \eta_A(x) \leq \eta_B(x) \leq \eta_C(x)$
c : $S(\upsilon_A(x)) \geq S(\upsilon_B(x)) \geq S(\upsilon_C(x)) \Longrightarrow \upsilon_A(x) \geq \upsilon_B(x) \geq \upsilon_C(x)$ for each $x \in X$.

$\Longrightarrow A \subseteq B \subseteq C$. Hence $C^1_{PHFS}(A,C) \leq C^1_{PHFS}(A,B)$ and $C^1_{PHFS}(A,C) \leq C^1_{PHFS}(B,C)$ □

3.1.2. *Defintion.* For $A, B \in PHFS(X)$, we define the weighted cosine similarity measure as:

$$W^1_{PHFS}(A,B) = \sum_{i=1}^{n} w_i \left(\frac{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*} \mu_A^j(x_i)\,\mu_B^j(x_i) + \frac{1}{\eta^*}\sum_{k=1}^{\eta^*} \eta_A^k(x_i)\,\eta_B^k(x_i) + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*} \upsilon_A^m(x_i)\,\upsilon_B^m(x_i)}{\sqrt{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\left(\mu_A^j(x_i)\right)^2 + \frac{1}{\eta^*}\sum_{k=1}^{\eta^*}\left(\eta_A^k(x_i)\right)^2 + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\left(\upsilon_A^m(x_i)\right)^2} \cdot \sqrt{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\left(\mu_B^j(x_i)\right)^2 + \frac{1}{\eta^*}\sum_{k=1}^{\eta^*}\left(\eta_B^k(x_i)\right)^2 + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\left(\upsilon_B^m(x_i)\right)^2}} \right) \tag{8}$$

Where $W = (w_1, w_2, w_3, ..., w_n)^T$ is the weighted vector of $x_i\,(i = 1, 2, 3, ..., n)^T$, with $\sum_{i=1}^{n} w_i = 1$. In particular, if we take $W = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$. Then the weighted cosine similarity reduces to cosine similarity measure. The weighted cosine similarity measure for PHFSs A, B and C satisfy following conditions:

1 : $0 \leq W^1_{PHFS}(A,B) \leq 1$
2 : $W^1_{PHFS}(A,B) = W^1_{PHFS}(B,A)$
3 : $W^1_{PHFS}(A,B) = 1$ iff $A = B$, if $i = 1, 2, 3, ..., n$.

Proof. Proof is straight forward □

3.1.3. *Remark.* The defintion 3.1.1 reduces to cosine similarity measure of IHFS, if we assume that $\eta_A = \eta_B = \{0\}$ and we write it as:

$$C^1_{IHFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\mu_A^j(x_i)\,\mu_B^j(x_i) + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\upsilon_A^m(x_i)\,\upsilon_B^m(x_i)}{\sqrt{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\left(\mu_A^j(x_i)\right)^2 + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\left(\upsilon_A^m(x_i)\right)^2}\cdot\sqrt{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\left(\mu_B^j(x_i)\right)^2 + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\left(\upsilon_B^m(x_i)\right)^2}}\right) \tag{9}$$

3.1.4. *Example.* Let $A, B \in PHFS(X)$. $A = \{(x_1, \{0.2\}, \{0.1, 0.3\}, \{0.4\}), (x_2, \{0.0, 0.1\}, \{0.0, 0.2\}, \{0.2, 0.4\})\}$ and $B = \{(x_1, \{0.1, 0.2\}, \{0.5\}, \{0.0, 0.1\}), (x_2, \{0.3\}, \{0.1, 0.2\}\{0.0, 0.3\})\}$. Then by using Eq. (7), we get

$$C^1_{PHFS}(A,B) = 0.8885$$

3.2. **Set-Theoretic Similarity Meassures.** In this portion, we shall propose another kind of similarity measures and weighted similarity measures which is the generalization of IFS and PFS [36, 37, 40].

3.2.1. *Definition .* For $A, B \in PHFS(X)$, we define the set-theoretic similarity measure as:

$$C^2_{PHFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\mu_A^j(x_i)\,\mu_B^j(x_i) + \frac{1}{\eta^*}\sum_{k=1}^{\eta^*}\eta_A^k(x_i)\,\eta_B^k(x_i) + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\upsilon_A^m(x_i)\,\upsilon_B^m(x_i)}{\max\left\{\begin{array}{c}\left(\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\left(\mu_A^j(x_i)\right)^2 + \frac{1}{\eta^*}\sum_{k=1}^{\eta^*}\left(\eta_A^k(x_i)\right)^2 + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\left(\upsilon_A^m(x_i)\right)^2\right), \\ \left(\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\left(\mu_B^j(x_i)\right)^2 + \frac{1}{\eta^*}\sum_{k=1}^{\eta^*}\left(\eta_B^k(x_i)\right)^2 + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\left(\upsilon_B^m(x_i)\right)^2\right)\end{array}\right\}}\right) \quad (10)$$

The set-theroetic similarity measure for PHFSs A, B and C satisfy following results:

1 : $0 \leq C^2_{PHFS}(A,B) \leq 1$

2 : $C^2_{PHFS}(A,B) = C^2_{PHFS}(B,A)$

3 : $C^2_{PHFS}(A,B) = 1$ iff $A = B$

4 : $A \subseteq B \subseteq C$, then $C^2_{PHFS}(A,C) \leq C^2_{PHFS}(A,B)$ and $C^2_{PHFS}(A,C) \leq C^2_{PHFS}(B,C)$.

Proof. Similar □

3.2.2. *Defintion.* For $A, B \in PHFS(X)$, we define the weighted set-theoretic similarity measure as:

$$W^2_{PHFS}(A,B) = \sum_{i=1}^{n} w_i \left(\frac{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*} \mu_A^j(x_i)\,\mu_B^j(x_i) + \frac{1}{\eta^*}\sum_{k=1}^{\eta^*} \eta_A^k(x_i)\,\eta_B^k(x_i) + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*} \upsilon_A^m(x_i)\,\upsilon_B^m(x_i)}{\max\left\{ \left(\frac{1}{\mu^*}\sum_{j=1}^{\mu^*} \left(\mu_A^j(x_i)\right)^2 + \frac{1}{\eta^*}\sum_{k=1}^{\eta^*} \left(\eta_A^k(x_i)\right)^2 + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*} \left(\upsilon_A^m(x_i)\right)^2 \right), \left(\frac{1}{\mu^*}\sum_{j=1}^{\mu^*} \left(\mu_B^j(x_i)\right)^2 + \frac{1}{\eta^*}\sum_{k=1}^{\eta^*} \left(\eta_B^k(x_i)\right)^2 + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*} \left(\upsilon_B^m(x_i)\right)^2 \right) \right\}} \right) \quad (11)$$

By taking $w_i = \frac{1}{n}$ equation (11) reduces to equation (10). The weighted set-theoretic similarity measure for PHFSs A, B and C satisfy following results:

1 : $0 \leq W^2_{PHFS}(A,B) \leq 1$
2 : $W^2_{PHFS}(A,B) = W^2_{PHFS}(B,A)$
3 : $W^2_{PHFS}(A,B) = 1$ iff $A = B$, if $i = 1, 2, 3, ..., n$.

Proof. Proof is straight forward □

3.2.3. *Remark.* The definition 3.2.1 reduces to the- set theoretic similarity measure of IHFS, if we assume that $\eta_A = \eta_B = \{0\}$ and we write it as:

$$C^2_{IHFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\mu_A^j(x_i)\,\mu_B^j(x_i) + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\upsilon_A^m(x_i)\,\upsilon_B^m(x_i)}{\max\left\{\begin{array}{c}\left(\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\left(\mu_A^j(x_i)\right)^2 + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\left(\upsilon_A^m(x_i)\right)^2\right), \\ \left(\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\left(\mu_B^j(x_i)\right)^2 + \frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\left(\upsilon_B^m(x_i)\right)^2\right)\end{array}\right\}}\right) \quad (12)$$

3.2.4. *Example.* Let $A, B \in PHFS(X)$. $A = \{(x_1, \{0.1\}, \{0.1, 0.3\}, \{0.2, 0.4\})$ $, (x_2, \{0.2, 0.4\}, \{0.0\}, \{0.3, 0.5\}\}$ and $B = \{(x_1, \{0.0, 0.1\}, \{0.0, 0.3\}, \{0.5\}), (x_2$ $, \{0.3\}, \{0.3, 0.4\}, \{0.1, 0.2\})\}$. Then by using Eq. (10), we get

$$C^2_{PHFS}(A,B) = 0.6083$$

3.3. **Grey Similarity Measures.** Following, we introduced the generalization of IFS and PFS which proposed in References [36, 37, 40].

3.3.1. *Definition .* For $A, B \in PHFS(X)$, we define the grey similarity measure as:

$$C^3_{PHFS}(A,B) = \frac{1}{3n}\sum_{i=1}^{n}\left(\frac{\Delta\mu_{\min}+\Delta\mu_{\max}}{\Delta\mu_i+\Delta\mu_{\max}} + \frac{\Delta\eta_{\min}+\Delta\eta_{\max}}{\Delta\eta_i+\Delta\eta_{\max}} + \frac{\Delta\upsilon_{\min}+\Delta\upsilon_{\max}}{\Delta\upsilon_i+\Delta\upsilon_{\max}}\right) \quad (13)$$

Where $\Delta\mu_i = \frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\left|\mu_A(x_i) - \mu_B(x_i)\right|, \Delta\mu_{\min} = \min$ $\left\{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\left|\mu_A(x_i) - \mu_B(x_i)\right|\right\}, \Delta\mu_{\max} = \max$ $\left\{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}\left|\mu_A(x_i) - \mu_B(x_i)\right|\right\}, \Delta\eta_i = \frac{1}{\eta^*}$ $\sum_{k=1}^{\eta^*}\left|\eta_A(x_i) - \eta_B(x_i)\right|, \Delta\eta_{\min} = \min$

$\left\{\frac{1}{\eta^*}\sum_{k=1}^{\eta^*}|\eta_A(x_i)-\eta_B(x_i)|\right\}, \Delta\eta_{\max} = \max$
$\left\{\frac{1}{\eta^*}\sum_{k=1}^{\eta^*}|\eta_A(x_i)-\eta_B(x_i)|\right\}, \Delta\upsilon_i = \frac{1}{\upsilon^*}$
$\sum_{m=1}^{\upsilon^*}|\upsilon_A(x_i)-\upsilon_B(x_i)|, \Delta\upsilon_{\min} = \min$
$\left\{\frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}|\upsilon_A(x_i)-\upsilon_B(x_i)|\right\}, \Delta\upsilon_{\max} = \max$
$\left\{\frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}|\upsilon_A(x_i)-\upsilon_B(x_i)|\right\}.$

Obviously, the grey similarity measures satisfy the following properties:
$1 : 0 \leq C^3_{PHFS}(A,B) \leq 1$
$2 : C^3_{PHFS}(A,B) = C^3_{PHFS}(B,A)$
$3 : C^3_{PHFS}(A,B) = 1$ iff $A = B$
$4 : A \subseteq B \subseteq C$, then $C^3_{PHFS}(A,C) \leq C^3_{PHFS}(A,B)$ and $C^3_{PHFS}(A,C) \leq C^3_{PHFS}(B,C)$.

Proof. Similar □

3.3.2. *Definition.* For $A, B \in PHFS(X)$, we define the weighted grey similarity measure as:

$$\begin{aligned} & W^3_{PHFS}(A,B) \\ = \ & \frac{1}{3}\sum_{i=1}^{n} w_i\left(\frac{\Delta\mu_{\min}+\Delta\mu_{\max}}{\Delta\mu_i+\Delta\mu_{\max}} + \frac{\Delta\eta_{\min}+\Delta\eta_{\max}}{\Delta\eta_i+\Delta\eta_{\max}} + \frac{\Delta\upsilon_{\min}+\Delta\upsilon_{\max}}{\Delta\upsilon_i+\Delta\upsilon_{\max}}\right) \end{aligned} \quad (14)$$

Where $\Delta\mu_i = \frac{1}{\mu^*}\sum_{j=1}^{\mu^*}|\mu_A(x_i)-\mu_B(x_i)|, \Delta\mu_{\min} = \min$
$\left\{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}|\mu_A(x_i)-\mu_B(x_i)|\right\}, \Delta\mu_{\max} = \max$
$\left\{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}|\mu_A(x_i)-\mu_B(x_i)|\right\}, \Delta\eta_i = \frac{1}{\eta^*}$
$\sum_{k=1}^{\eta^*}|\eta_A(x_i)-\eta_B(x_i)|, \Delta\eta_{\min} = \min$

$\left\{\frac{1}{\eta^*}\sum_{k=1}^{\eta^*}|\eta_A(x_i)-\eta_B(x_i)|\right\}, \Delta\eta_{\max} = \max$
$\left\{\frac{1}{\eta^*}\sum_{k=1}^{\eta^*}|\eta_A(x_i)-\eta_B(x_i)|\right\}, \Delta\upsilon_i = \frac{1}{\upsilon^*}$
$\sum_{m=1}^{\upsilon^*}|\upsilon_A(x_i)-\upsilon_B(x_i)|, \Delta\upsilon_{\min} = \min$
$\left\{\frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}|\upsilon_A(x_i)-\upsilon_B(x_i)|\right\}, \Delta\upsilon_{\max} = \max$
$\left\{\frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}|\upsilon_A(x_i)-\upsilon_B(x_i)|\right\}.$

By taking $w_i = \frac{1}{n}$ the equation (14) reduces to equation (13). The weighted set-theoretic similarity measure for PHFSs A, B and C satisfy following results:
$1 : 0 \leq W^3_{PHFS}(A,B) \leq 1$
$2 : W^3_{PHFS}(A,B) = W^3_{PHFS}(B,A)$
$3 : W^3_{PHFS}(A,B) = 1$ iff $A = B$, if $i = 1, 2, 3, ..., n$.

Proof. Proof is straight forward □

3.3.3. *Remark.* The definition 3.3.1 reduces to grey similarity measure of IHFS, if we assume that $\eta_A = \eta_B = \{0\}$ and we write it as:

$$C^3_{IHFS}(A,B) = \frac{1}{3n}\sum_{i=1}^{n}\left(\frac{\Delta\mu_{\min}+\Delta\mu_{\max}}{\Delta\mu_i+\Delta\mu_{\max}} + \frac{\Delta\upsilon_{\min}+\Delta\upsilon_{\max}}{\Delta\upsilon_i+\Delta\upsilon_{\max}}\right) \tag{15}$$

Where $\Delta\mu_i = \frac{1}{\mu^*}\sum_{j=1}^{\mu^*}|\mu_A(x_i)-\mu_B(x_i)|, \Delta\mu_{\min} = \min$
$\left\{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}|\mu_A(x_i)-\mu_B(x_i)|\right\}, \Delta\mu_{\max} = \max$
$\left\{\frac{1}{\mu^*}\sum_{j=1}^{\mu^*}|\mu_A(x_i)-\mu_B(x_i)|\right\}, \Delta\upsilon_i = \frac{1}{\upsilon^*}$
$\sum_{m=1}^{\upsilon^*}|\upsilon_A(x_i)-\upsilon_B(x_i)|, \Delta\upsilon_{\min} = \min$
$\left\{\frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}|\upsilon_A(x_i)-\upsilon_B(x_i)|\right\}, \Delta\upsilon_{\max} = \max$

$$\left\{\frac{1}{\upsilon^*}\sum_{m=1}^{\upsilon^*}\left|\upsilon_A\left(x_i\right)-\upsilon_B\left(x_i\right)\right|\right\}.$$

4. Applications

In this section, we apply the similarity measures developed in our manuscript in a building material is adopt from [40]

4.1. Building Material Recognition. In this phenomenon, we determined the class of an unknown building material using the approach of similarity measures of PHFSs. In such process information about some known building material is obtained from expects also on the known building material. Then the similarity measure of all known building material calculated with that of unknown material. The unknown building material is then placed into a class of that material with whom the similarity index is greater. The detail algorithm is described as:

Algorithm:

1. Obtain information about known and unknown building material in form of PHFNs.
2. Compute similarity measure of each known material with unknown material.
3. Rank the similarity measure of all known material with unknown material.
4. Classify the unknown material based on ranking.

4.2. Example . Let $A_i\,(1 \leq i \leq 4)$ denote four building material named as brick, stone, muddy and steal. Let $X = \{x_1, x_2, x_3, x_4\}$ be the space of attribute have weight $W = (0.27, 0.33, 0.11, 0.29)^T$. The information about the unknown and known building material is provided that in table 1. We classified that unknown material as follows.

Step 1: Information of unknown and known building material.

$$\begin{array}{ccccc} & x_1 & x_2 & x_3 & x_4 \\ A_1 & \begin{pmatrix} \{.0,.1\}, \\ \{.0,.3\}, \\ \{.0,.5\} \end{pmatrix} & \begin{pmatrix} \{.1\}, \\ \{.2,.4\}, \\ \{.3\} \end{pmatrix} & \begin{pmatrix} \{.2,.4\}, \\ \{.0,.5\}, \\ \{.0,.01\} \end{pmatrix} & \begin{pmatrix} \{.2,.3\}, \\ \{.0,.2\}, \\ \{.0,.3\} \end{pmatrix} \\ A_2 & \begin{pmatrix} \{.0,.2\}, \\ \{.1\}, \\ \{.0,.4\} \end{pmatrix} & \begin{pmatrix} \{.0,.1\}, \\ \{.3\}, \\ \{.0\} \end{pmatrix} & \begin{pmatrix} \{.0,.1\}, \\ \{.2,.4\}, \\ \{.0,.2\} \end{pmatrix} & \begin{pmatrix} \{.2,.5\}, \\ \{.0,.2\}, \\ \{.0,.1\} \end{pmatrix} \\ A_3 & \begin{pmatrix} \{.22\}, \\ \{.23,.27\}, \\ \{.0\} \end{pmatrix} & \begin{pmatrix} \{.1\}, \\ \{.2,.11\}, \\ \{.3\} \end{pmatrix} & \begin{pmatrix} \{.0,.5\}, \\ \{.1,.3\}, \\ \{.0,.2\} \end{pmatrix} & \begin{pmatrix} \{.0,.17\}, \\ \{.54,.63\}, \\ \{.1,.2\} \end{pmatrix} \\ A_4 & \begin{pmatrix} \{.42,.47\}, \\ \{.0,.53\}, \\ \{.0\} \end{pmatrix} & \begin{pmatrix} \{.0,.15\}, \\ \{.0,.71\}, \\ \{.14\} \end{pmatrix} & \begin{pmatrix} \{.1\}, \\ \{.0,.3\}, \\ \{.4,.5\} \end{pmatrix} & \begin{pmatrix} \{.0,.1\}, \\ \{.02,.6\}, \\ \{.3,.35\} \end{pmatrix} \\ A & \begin{pmatrix} \{.1,.2\}, \\ \{.2,.3\}, \\ \{.3,.4\} \end{pmatrix} & \begin{pmatrix} \{.1,.2\}, \\ \{.0,.1\}, \\ \{.0,.4\} \end{pmatrix} & \begin{pmatrix} \{.0,.2\}, \\ \{.4\}, \\ \{.3\} \end{pmatrix} & \begin{pmatrix} \{1.00\}, \\ \{0.00\}, \\ \{0.00\} \end{pmatrix} \end{array}$$

Step 2: Comparison of similarity measures.

Similarity Measures	(A, A_1)	(A, A_2)	(A, A_3)	(A, A_4)
$W^1_{PHFS}(A, A_i)$	0.8423	0.8147	0.6418	0.3978
$W^2_{PHFS}(A, A_i)$	0.6178	0.5563	0.5022	0.2880
$W^3_{PHFS}(A, A_i)$	0.8739	0.9101	0.8728	0.7489

Step 3: This step involves the ranking of similarity measures.

Similarity Measures	Ranking of (A, A_i)
$W^1_{PHFS}(A, A_i)$	$(A, A_1) \succ (A, A_2) \succ (A, A_3) \succ (A, A_4)$
$W^2_{PHFS}(A, A_i)$	$(A, A_1) \succ (A, A_2) \succ (A, A_3) \succ (A, A_4)$
$W^3_{PHFS}(A, A_i)$	$(A, A_2) \succ (A, A_1) \succ (A, A_3) \succ (A, A_4)$

Step 4: Upon ranking we get that the similarity measures of (A, A_1) is greater among all other similarity measures using W^1 and W^2. But using W^3, we get that A_2 has a greater value of similarity measures. These results show that the results of similarity measures using different approaches are different.

5. Comparative Study

The similarity measures introduce in this article are generalization of similarity measures for PFSs, IHFSs and HFSs. The following remarks identify that the similarity measures defined in Eq. (7) to (15) are generalization of similarity measures for PFSs, IHFSs and HFSs.

5.1. **Remark.** If we take all the PHFNs as singleton sets. Then equations (7) and (8) reduce to similarity measures of PFSs. If membership and non-membership grade are any two sets and neutral membership grade is empty set. Then equations (7) and (8) reduce to similarity measures of IHFSs. If we take membership grade as non-empty set while non-membership and neutral grades as empty sets. Then equations (7) and (8) reduce to similarity measures of HFSs. This show that similarity measures proposed in equations (7) and (8) are generalizations of similarity measures of PFSs, IHFSs and HFSs.

5.2. **Remark.** If we take all the PHFNs as singleton sets. Then equations (10) and (11) reduce to similarity measures of PFSs. If membership and non-membership grade are any two sets and neutral membership grade is empty set. Then equations (10) and (11) reduce to similarity measures of IHFSs. If we take membership grade as non-empty set while non-membership and neutral grades as empty sets. Then equations (10) and (11) reduce to similarity measures of HFSs. This show that similarity measures proposed in equations (10) and (11) are generalizations of similarity measures of PFSs, IHFSs and HFSs.

5.3. **Remark.** If we take all the PHFNs as singleton sets. Then equations (13) and (14) reduce to similarity measures of PFSs. If membership and non-membership grade are any two sets and neutral membership grade is empty set. Then equations (13) and (14) reduce to similarity measures of IHFSs. If we take membership grade as non-empty set while non-membership and neutral grades as empty sets. Then equations (13) and (14) reduce to similarity measures of HFSs. This show that similarity measures proposed in equations (13) and (14) are generalizations of similarity measures of PFSs, IHFSs and HFSs.

6. Advantages

The advantage of proposed new work lies in a fact that these proposed similarity measures can solve the problem lies in the environment of PHFSs, IHFSs, IFSs as well as HFSs. On the other hand, the existing concepts could not hold the data provided in the environment of PHFSs. If we look at Example 4.2, the data is purely in the form of PHFNs which cannot be processed by existing concepts.

Consider an example in the space of PFSs from [40]

6.1. **Example.** The data about unknown materials is provided in Table 4.

	A_1	A_2	A_3	A_4	A
x_1	$(.17, .53, .13)$	$(.51, .24, .21)$	$(.31, .39, .25)$	$(1, 0, 0)$	$(.91, .03, .05)$
x_2	$(.1, .81, .05)$	$(.62, .12, .07)$	$(.60, .26, .11)$	$(1, 0, 0)$	$(.78, .12, .07)$
x_3	$(.53, .33, .09)$	$(1, 0, 0)$	$(.91, .03, .02)$	$(.85, .09, .05)$	$(.90, .05, .02)$
x_4	$(.89, .08, .03)$	$(.13, .64, .21)$	$(.07, .09, .07)$	$(.74, .16, .10)$	$(.68, .08, .21)$
x_5	$(.42, .35, .18)$	$(.03, .82, .13)$	$(.04, .85, .10)$	$(.02, .89, .05)$	$(.05, .87, .06)$
x_6	$(.08, .89, .02)$	$(.73, .15, .08)$	$(.68, .26, .06)$	$(.08, .84, .06)$	$(.13, .75, .09)$
x_7	$(.33, .51, .12)$	$(.52, .31, .16)$	$(.15, .76, .07)$	$(.16, .71, .05)$	$(.15, .73, .08)$

The data provided in Table 4 can be easily converted to the environment of PHFSs given in Table 5 which is then solved using proposed new similarity measures and the results are displayed in Table 6.

	A_1	A_2	A_3	A_4	A
x_1	$\begin{pmatrix}\{.17\},\\ \{.53\},\\ \{.13\}\end{pmatrix}$	$\begin{pmatrix}\{.51\},\\ \{.24\},\\ \{.21\}\end{pmatrix}$	$\begin{pmatrix}\{.31\},\\ \{.39\},\\ \{.25\}\end{pmatrix}$	$\begin{pmatrix}\{1\},\\ \{0\},\\ \{0\}\end{pmatrix}$	$\begin{pmatrix}\{.91\},\\ \{.03\},\\ \{0.05\}\end{pmatrix}$
x_2	$\begin{pmatrix}\{.10\},\\ \{.81\},\\ \{.05\}\end{pmatrix}$	$\begin{pmatrix}\{.62\},\\ \{.12\},\\ \{.07\}\end{pmatrix}$	$\begin{pmatrix}\{.60\},\\ \{.26\},\\ \{.11\}\end{pmatrix}$	$\begin{pmatrix}\{1\},\\ \{0\},\\ \{0\}\end{pmatrix}$	$\begin{pmatrix}\{.78\},\\ \{.12\},\\ \{.07\}\end{pmatrix}$
x_3	$\begin{pmatrix}\{.53\},\\ \{.33\},\\ \{.09\}\end{pmatrix}$	$\begin{pmatrix}\{1\},\\ \{0\},\\ \{0\}\end{pmatrix}$	$\begin{pmatrix}\{.91\},\\ \{.03\},\\ \{.02\}\end{pmatrix}$	$\begin{pmatrix}\{.85\},\\ \{.09\},\\ \{.05\}\end{pmatrix}$	$\begin{pmatrix}\{.90\},\\ \{.05\},\\ \{.02\}\end{pmatrix}$
x_4	$\begin{pmatrix}\{.89\},\\ \{.08\},\\ \{.03\}\end{pmatrix}$	$\begin{pmatrix}\{.13\},\\ \{.64\},\\ \{.21\}\end{pmatrix}$	$\begin{pmatrix}\{.07\},\\ \{.09\},\\ \{.07\}\end{pmatrix}$	$\begin{pmatrix}\{.74\},\\ \{.16\},\\ \{.10\}\end{pmatrix}$	$\begin{pmatrix}\{.68\},\\ \{.08\},\\ \{.21\}\end{pmatrix}$
x_5	$\begin{pmatrix}\{.42\},\\ \{.35\},\\ \{.18\}\end{pmatrix}$	$\begin{pmatrix}\{.03\},\\ \{.82\},\\ \{.13\}\end{pmatrix}$	$\begin{pmatrix}\{.04\},\\ \{.85\},\\ \{.10\}\end{pmatrix}$	$\begin{pmatrix}\{.02\},\\ \{.89\},\\ \{.05\}\end{pmatrix}$	$\begin{pmatrix}\{.05\},\\ \{.87\},\\ \{.06\}\end{pmatrix}$
x_6	$\begin{pmatrix}\{.08\},\\ \{.89\},\\ \{.02\}\end{pmatrix}$	$\begin{pmatrix}\{.73\},\\ \{.15\},\\ \{.08\}\end{pmatrix}$	$\begin{pmatrix}\{.68\},\\ \{.26\},\\ \{.06\}\end{pmatrix}$	$\begin{pmatrix}\{.08\},\\ \{.84\},\\ \{.06\}\end{pmatrix}$	$\begin{pmatrix}\{.13\},\\ \{.75\},\\ \{.09\}\end{pmatrix}$
x_7	$\begin{pmatrix}\{.33\},\\ \{.51\},\\ \{.12\}\end{pmatrix}$	$\begin{pmatrix}\{.52\},\\ \{.31\},\\ \{.16\}\end{pmatrix}$	$\begin{pmatrix}\{.15\},\\ \{.76\},\\ \{.07\}\end{pmatrix}$	$\begin{pmatrix}\{.16\},\\ \{.71\},\\ \{.05\}\end{pmatrix}$	$\begin{pmatrix}\{.15\},\\ \{.73\},\\ \{.08\}\end{pmatrix}$

Similarity Measures	(A, A_1)	(A, A_2)	(A, A_3)	(A, A_4)
$W^1_{PHFS}(A, A_i)$	0.716	0.763	0.858	0.994
$W^2_{PHFS}(A, A_i)$	0.556	0.657	0.693	0.920
$W^3_{PHFS}(A, A_i)$	0.660	0.762	0.830	0.901

Now using the operators, we can compute the similarity measures of known building material with unknown material as follows from Table 5, which are exactly same as in [40]

Similarly, if we have information in the form of IHFS, such information could be converted into the environment of PHFS by considering the abstinence grade as empty. Then using proposed new similarity measures, such information could be processed. All this proves our claim.

7. Conclusion

In this paper, the basic concepts of IFSs, PFSs, HFSs and PHFSs are analyzed along with their similarity measures. It is observed that the existing similarity measures cannot be applied to the problems of PHFSs. Therefore, some similarity measures have been developed for PHFSs. These similarity measures consisting grey similarity measures, set-theoretic similarity measures as well as cosine similarity measures. The characteristics of these similarity measures have been investigated and some results are proved. Some examples are also solved in support of new work. To signify the proposed new work, a problem based on building material recognition has been discussed. The newly developed results are compared with existing results and the conditions under which the new results reduce to existing results have been enlightened. The advantages of proposed new work over the existing work have also been studied. In future the proposed new results can be utilized in decision making problem and can be extended to the environment of interval valued Picture fuzzy set.

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