Journal of Prime Research in Mathematics Vol. 15*(2019),* 115-128

FORGOTTEN INDEX OF GENERALIZED F-SUM GRAPHS

H. M. AWAIS[1], M. JAVAID[1], M. JAMAL[2]

Abstract. Liu et al. [IEEE Access; 7(2019); 105479-105488] defined the concept of the generalized subdivided operations on graphs and obtained the generalized F-sum graphs. They also calculated the 1st and 2nd Zagreb indices of the generalized F-sum graphs. In the continuation of this work, we study the forgotten index (F-index) of the generalized F-sum graphs in terms of different topological indices (TI's) of their base graphs. In the end, the results of F-index on the generalized F-sum graphs acquired by the particular classes of alkane are also included.

Key words : Generalized operations; Zagreb indices; F-index.
AMS Classification: 05C12; 05C90; 05C35.

1. Introduction

A topological index (TI) is considered as a function $\phi : \sum \to R$ (set of real numbers) that links every element of the $\sum$ to a unique real number, where $\sum$ is collection of graphs. TI's forecast the bio-nature activities, chemical reactivities and physical attributes of the molecular graphs like heat of formation, heat of evaporation, viscosity, freezing point, boiling point, melting point, surface tension, stability, temperature, density, weight, polarizability, connectivity, and solubility. Several drugs, crystalline, and nano-materials that are utilized in various industries are analyzed with the help of TI's, see [14, 15, 10, 9, 8, 12, 22, 27]. These are also considered a useful tool to examine the quantitative structural properties relationships and quantitative structural activities relationships which connect molecular structures to their different molecular properties. For more detail, we refer to [12, 16, 17].

To study the paraffin's boiling point, Wiener first time used the distance-based TI [7]. Gutman and Trinajsti [19] calculated π-electrons total energy of a molecule through a TI called as the first Zagreb index. After that, many

[1]Department of Mathematics, School of Science, University of Management and Technology, Lahore, Pakistan; hmawais.umt@gmail.com, javaidmath@gmail.com.
[2]Department of Mathematics and Statistics, PMAS-Arid Agriculture University, Rawalpindi, Pakistan; jamal@uaar.edu.pk.

impressive TI's are introduced in chemical graph theory [20, 18] but the degree based TI's are well known than any other outcome, see [21].

In molecular graph theory, the different operations on a graph perform a fundamental role in the formation of different new classes of graphs, see [11]. Yan et al. [12] introduced the four operations S_1, R_1, Q_1, T_1 on graphs and obtained the Wiener index of these resultant graphs. For $F_1 \in \{S_1, R_1, Q_1, T_1\}$, Taeri and Eliasi [13] defined the F_1-sum graphs ($H_1 \; _{+F_1} \; H_2$) using the cartesian product on graphs $F(H_1)$ and H_2, where H_1 & H_2 are connected. They also studied the Wiener index of these F_1-sum graphs. Furthermore, Deng et al. [24], Shehnaz et al. [25] and Liu et al. [2] calculated the 1st & 2nd Zagreb indices, F-index and the 1st general Zagreb index of F_1-sum graph.

Recently, Liu et al. [1] introduced the concept of generalized operations on graphs and obtained the generalized F-sum (F_k-sum) graphs. They also calculated the 1st & 2nd Zagreb TI's of F_k-sum graphs. In the present article, we calculate the F-index for the generalized F-sum graphs (F_k-sum), where k presents a counting number. This article is arranged as; the section II contains some primary terminologies, and definitions. The section III covers the key outcomes, and finally in the section IV closing remarks are included with the application of the achieved outcomes.

2. Preliminaries

Let $H = (V(H), E(H))$ be a connected & simple graph having the node-set $V(H)$ as well as the edge-set $E(H) \subseteq V(H) \times V(H)$. Every node $u \in V(H)$ is named as atom in chemical graph theory and connection with in the two atoms is denoted by an edge. So the cardinalities of node-set and edge-set are known as order and size. The strength of edges which are incident on any node $v \in V(H)$ is known as its degree ($d_H(v)$). Here, we defined few topological indices that given:

Definition 2.1. For a simple graph H, the 1st Zagreb index ($M_1(H)$ and 2nd Zagreb index ($M_2(H)$) are $M_1(H) = \sum_{v\epsilon V(H)}[d_H(v)]^2 = \sum_{uv\epsilon E(H)}[d_H(u) + d_H(v)]$ and $M_2(H) = \sum_{uv\epsilon E(H)}[d_H(u) \times d_H(v)]$.

In 1972, Trinajsti and Gutman [19] explained the above TI's that are later utilized to calculate the structure based properties of (molecular) graphs like energy, connectivity, complexity, branching, chirality and hetero-system, see [16, 17]. However, there was another formula which was not studied by the mathematicians for many decades. In 2015, Gutman and Furtula [3] set a new name for this index as forgotten index and they have also got some meaningful outcomes in the same paper. For the graph H, the F-index is given as follows:

$$F(H) = \sum_{(v)\epsilon V(H)} [d_H(v)]^3 = \sum_{uv\epsilon E(H)} \left[[d_H(u)]^2 + [d_H(v)]^2\right].$$

The four generalized operations related to the subdivision of graphs defined in [1] are given as follows:
• The graph $S_k(H)$ is gained by the addition of k nodes in each edge of H, where $k \geq 1$ is an integral value.
• In $R_k(H)$, we join the vertices of $S_k(H)$ that were adjacent in H.
• $Q_k(H)$ is gained by joining the new added nodes of the edges having one vertex common in basic graph H.
• Finally, $T_k(H)$ is gained by both the operations R_k and Q_k on the graph $S_k(H)$. For more explanation, see Figure 1.

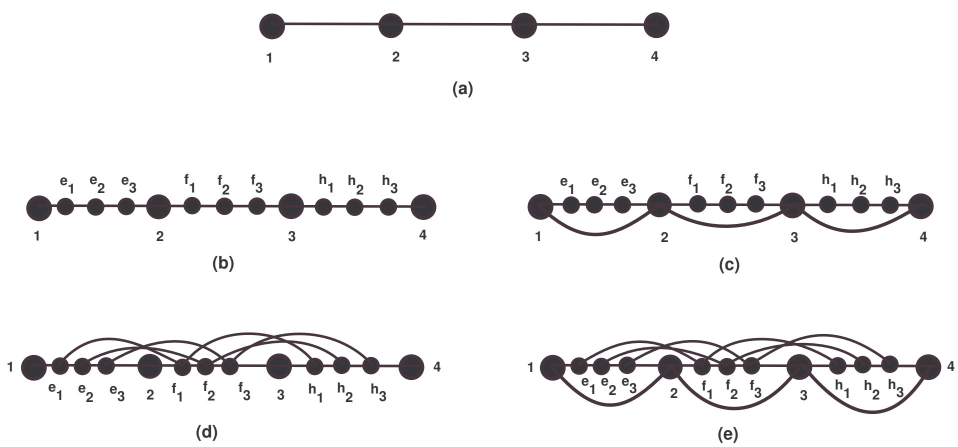

FIGURE 1. (a)$H \cong P_4$, (b)$S_3(H)$, (c)$R_3(H)$, (d)$Q_3(H)$ and (e)$T_3(H)$

Definition 2.2. Assume that H_1 & H_2 are two simple, undirected and connected graphs, $F_k \in \{S_k, R_k, Q_k, T_k\}$ is an operation and $F_k(H_1)$ is obtained after applying F_k on H_1 having edge-set $E(F_k(H_1))$ and node-set $V(F_k(H_1))$. The generalized F-sum graph $(H_1 \ {}_{+F_k} \ H_2)$ is a graph having node-set

$$V(H_1 \ {}_{+F_k} \ H_2) = V(F_k(H_1)) \times V(H_2) = (V(H_1) \cup E(H_1)) \times V(H_2)$$

such that two nodes (u_1, v_1) & (u_2, v_2) of $V(H_1 \ {}_{+F_k} \ H_2)$ are adjacent iff $[u_1 = u_2 \in V(H_1)$ & $(v_1, v_2) \in E(H_2)]$ or $[v_1 = v_2 \in V(H_2)$ & $(u_1, u_2) \in E(F_k(H_1))]$, where $k \geq 1$ is a positive number.

We noticed that the generalized F-sum graphs $(H_1 \;{}_{+F_k}\; H_2)$ contain $|V(H_2)|$ copies of graphs $F_k(H_1)$ that are labeled with the nodes of H_2. For more explanation FIGURE 2 and FIGURE 3 are given below.

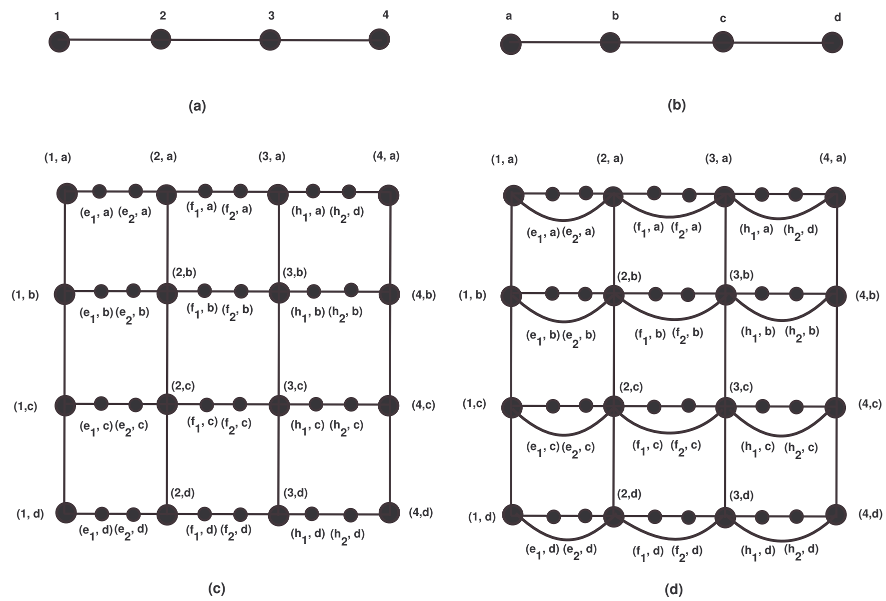

FIGURE 2. (a)$H_1 \cong P_4$, (b)$H_2 \cong P_4$, (c)$H_{1+S_2}H_2$, (d)$H_{1+R_2}H_2$

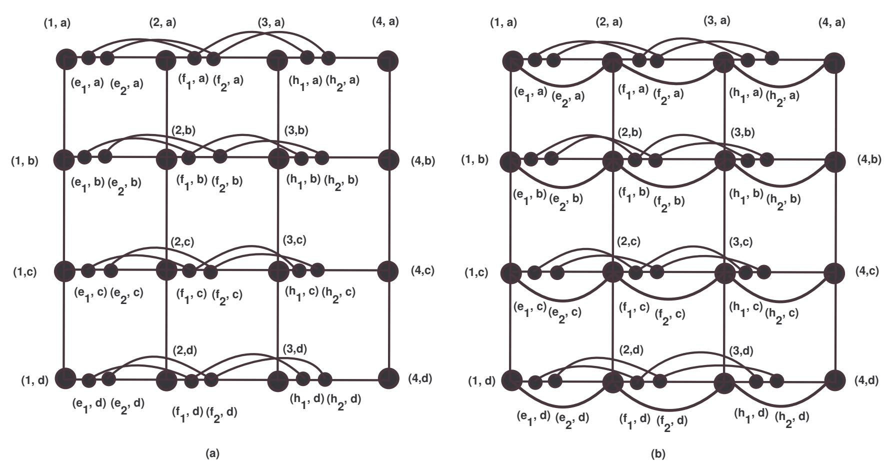

FIGURE 3. (a)$H_{1+Q_2}H_2$, (b)$H_{1+T_2}H_2$

3. Main Results

Now, we prove the key results of F-index for miscellaneous classes of graphs which are described in previous section. Before the main results, we assume that H_1 and H_2 be two connected & simple graphs with order $| V(H_1) |$ and $| V(H_2) |$ respectively.

Theorem 3.1.
For $| V(H_1) |, | V(H_2) | \geq 4$ and $k \geq 1$, the F-index of $F(H_1 +_{S_k} H_2)$ is

$| V(H_1) | F(H_2) + | V(H_2) | F(H_1) + 2 | E(H_2) | M_1(H_1) + 6 | E(H_1) | M_1(H_2)$
$+ 8 | E(H_2) || E(H_1) | + 8 | E(H_1) || V(H_2) | + 8 | V(H_2) || E(H_1) | (k-1)$.

Proof. Let $d(s,t) = d_{H_1 +_{S_k} H_2}(s,t)$ for $(s,t) \epsilon H_1 +_{S_k} H_2$.

$$F(H_1 +_{S_k} H_2) =$$

$$\sum_{(s,t)\epsilon V(H_1+_{S_k}H_2)} d(s,t)^3 = \sum_{(s_1,t_1)(s_2,t_2)\epsilon E(H_1+_{S_k}H_2)} [d(s_1,t_1)^2 + d(s_2,t_2)^2]$$

$$= \sum_{s\epsilon V(H_1)} \sum_{t_1t_2\epsilon E(H_2)} [d(s,t_1)^2 + d(s,t_2)^2] + \sum_{t\epsilon V(H_2)} \sum_{s_1s_2\epsilon E(S_k(H_1))} [d(s_1,t)^2 + d(s_2,t)^2]$$

$$= \sum_{s\epsilon V(H_1)} \sum_{t_1t_2\epsilon E(H_2)} [d(s,t_1)^2 + d(s,t_2)^2] + \sum_{t\epsilon V(H_2)} \sum_{\substack{s_1s_2\epsilon E(S_k(H_1)) \\ s_1\epsilon V(H_1),\ s_2\epsilon V(S_k(H_1)-H_1)}}$$

$$[d(s_1,t)^2 + d(s_2,t)^2] + \sum_{t\epsilon V(H_2)} \sum_{\substack{s_1s_2\epsilon E(S_k(H_1)) \\ s_1,\ s_2\epsilon V(S_k(H_1)-H_1)}} [d(s_1,t)^2 + d(s_2,t)^2]$$

$$= \sum 1 + \sum 2 + \sum 3.$$

Consider,

$$\sum 1 =$$

$$\sum_{s\epsilon V(H_1)} \sum_{t_1t_2\epsilon E(H_2)} [(d_{H_1}(s) + d_{H_2}(t_1))^2 + (d_{H_1}(s) + d_{H_2}(t_2))^2]$$

$$= \sum_{s\epsilon V(H_1)} \sum_{t_1t_2\epsilon E(H_2)} [2d_{H_1}(s)^2 + (d_{H_2}(t_1)^2 + d_{H_2}(t_2)^2) + 2d_{H_1}(s)(d_{H_2}(t_1) + d_{H_2}(t_2))]$$

$$= \sum_{s\epsilon V(H_1)} [| E(H_2) | \times 2d_{H_1}(s)^2 + F(H_2) + 2M_1(H_2)d_{H_1}(s)]$$

$$= 2 | E(H_2) | M_1(H_1) + | V(H_1) | F(H_2) + 4 | E(H_1) | M_1(H_2),$$

$$\sum 2 =$$

$$\sum_{t\epsilon V(H_2)} \sum_{\substack{s_1s_2\epsilon E(S_k(H_1))\\ s_1\epsilon V(H_1),\ s_2\epsilon V(S_k(H_1)-H_1)}} [(d_{S_k(H_1)}(s_1)+d_{H_2}(t))^2+d_{S_k(H_1)}(s_2)^2]$$

$$= \sum_{t\epsilon V(H_2)} \sum_{\substack{s_1s_2\epsilon E(S_k(H_1))\\ s_1\epsilon V(H_1),\ s_2\epsilon V(S_k(H_1)-H_1)}} [d_{S_k(H_1)}(s_1)^2+d_{H_2}(t)^2+2d_{S_k(H_1)}(s_1)d_{H_2}(t)$$

$$+d_{S_k(H_1)}(s_2)^2]$$

Since $s_1\epsilon V(H_1)$ and $s_2\epsilon V(S_k(H_1)-H_1)$, therefore

$$= \sum_{t\epsilon V(H_2)} [F(S_1(H_1))+2 \mid E(H_1) \mid d_{H_2}(t)^2+2\times 2 \mid E(H_1) \mid d_{H_2}(t)]$$

$$=\mid V(H_2) \mid F(S_1(H_1))+2 \mid E(H_1) \mid M_1(H_2)+8 \mid E(H_2) \mid\mid E(H_1) \mid,$$

and

$$\sum 3 =$$

$$(k-1) \sum_{t\epsilon V(H_2)} \sum_{\substack{s_1s_2\epsilon E(S_k(H_1))\\ s_1,\ s_2\epsilon V(S_k(H_1)-H_1)}} [4+4]$$

Since in this case $|E(S_k(H_1))| = |E(H_1)|(k-1)$, so

$$= |E(H_1)|(k-1) \sum_{t\epsilon V(H_2)} (8) = 8(k-1) \mid V(H_2) \mid\mid E(H_1).$$

Consequently,

$$F(H_1+_{S_k}H_2) =\mid V(H_1) \mid F(H_2)+ \mid V(H_2) \mid F(S_1(H_1))+2 \mid E(H_2) \mid M_1(H_1)$$
$$+6 \mid E(H_1) \mid M_1(H_2)+8 \mid E(H_2) \mid\mid E(H_1) \mid +8 \mid V(H_2) \mid\mid E(H_1) \mid (k-1)$$

Moreover, for $F(S_1(H_1)) = F(H_1) + 8 \mid E(H_1) \mid$, we have

$$F(H_1+_{S_k}H_2) =\mid V(H_1) \mid F(H_2)+ \mid V(H_2) \mid F(H_1)+2 \mid E(H_2) \mid M_1(H_1)+6$$
$$\mid E(H_1) \mid M_1(H_2)+8 \mid E(H_2) \mid\mid E(H_1) \mid +8 \mid E(H_1) \mid\mid V(H_2) \mid +8 \mid V(H_2) \mid$$
$$\mid E(H_1) \mid (k-1).$$

Theorem 3.2.
For $\mid V(H_1) \mid, \mid V(H_2) \mid\geq 4$ and $k \geq 1$, the F-index of $F(H_1 +_{R_k} H_2)$ is

$$8 \mid V(H_2) \mid F(H_1)+ \mid V(H_1) \mid F(H_2)+24 \mid E(H_2) \mid M_1(H_1)+12 \mid E(H_1) \mid$$
$$M_1(H_2)+8 \mid E(H_1) \mid\mid V(H_2) \mid +8(k-1) \mid V(H_2) \mid\mid E(H_1) \mid .$$

Proof.
Consider,

$$F(H_1+_{R_k}H_2) = \sum_{(s_1,t_1)(s_2,t_2)\epsilon E(H_1+_{R_k}H_2)} [d(s_1,t_1)^2+d(s_2,t_2)^2]$$

$$= \sum_{s\epsilon V(H_1)} \sum_{t_1t_2\epsilon E(H_2)} [d(s,t_1)^2+d(s,t_2)^2]+ \sum_{t\epsilon V(H_2)} \sum_{s_1s_2\epsilon E(R_k(H_1))} [d(s_1,t)^2+d(s_2,t)^2]$$

$$= \sum 1+\sum 2.$$

Consider,

$$\sum 1 =$$

$$\sum_{s\epsilon V(H_1)} \sum_{t_1t_2\epsilon E(H_2)} [(d_{R_k(H_1)}(s)+d_{H_2}(t_1))^2+(d_{R_k(H_1)}(s)+d_{H_2}(t_2))^2]$$

$$= \sum_{s\epsilon V(H_1)} \sum_{t_1t_2\epsilon E(H_2)} [2d_{R_k(H_1)}(s)^2+(d_{H_2}(t_1)^2+d_{H_2}(t_2)^2)+2d_{R_k(H_1)}(s)(d_{H_2}(t_1) +d_{H_2}(t_2))]$$

$$= \sum_{s\epsilon V(H_1)} \sum_{t_1t_2\epsilon E(H_2)} [8d_{(H_1)}(s)^2+(d_{H_2}(t_1)^2+d_{H_2}(t_2)^2)+4d_{(H_1)}(s)(d_{H_2}(t_1) +d_{H_2}(t_2))]$$

$$= \sum_{s\epsilon V(H_1)} [8 \mid E(H_2) \mid d_{H_1}(s)^2+F(H_2)+4M_1(H_2)d_{H_1}(s_1)]$$

$$= 8 \mid E(H_2) \mid M_1(H_1)+ \mid V(H_1) \mid F(H_2)+8 \mid E(H_1) \mid M_1(H_2),$$

and

$$\sum 2 =$$

$$\sum_{t\epsilon V(H_2)} \sum_{s_1s_2\epsilon E(R_k(H_1))} [d(s_1,t)^2+d(s_2,t)^2]$$

$$= \sum_{t\epsilon V(H_2)} \sum_{\substack{s_1s_2\epsilon E(R_k(H_1)) \\ s_1,s_2\epsilon V(H_1)}} [d(s_1,t)^2+d(s_2,t)^2]$$

$$+ \sum_{t\epsilon V(H_2)} \sum_{\substack{s_1s_2\epsilon E(R_k(H_1)) \\ s_1\epsilon V(H_1) \\ s_2\epsilon V(R_k(H_1))-V(H_1)}} [d(s_1,t)^2+d(s_2,t)^2]$$

$$+ \sum_{t\epsilon V(H_2)} \sum_{\substack{s_1s_2\epsilon E(R_k(H_1)) \\ s_1,s_2\epsilon V(R_k(H_1))-V(H_1)}} [d(s_1,t)^2+d(s_2,t)^2]$$

$$= \sum^{'} 2 + \sum^{''} 2 + \sum^{'''} 2.$$

Consider for $s_1s_2 \epsilon V(H_1)$, we have $s_1s_2 \epsilon E(R_k(H_1))$ if and only if $s_1s_2 \epsilon E(H_1)$; for $s_1 \epsilon V(H_1)$, we obtain $d_{R_k(H_1)}(s_1) = 2d_{H_1}(s_1)$ and for $s_2 \epsilon V(R_k(H_1)) - V(H_1)$, we have $d_{R_k(H_1)}(s_2) = 2$. Now, consider

$$\sum^{'} 2 =$$

$$\sum_{t \epsilon V(H_2)} \sum_{s_1s_2 \epsilon E(H_1)} [d(s_1,t)^2 + d(s_2,t)^2]$$

$$= \sum_{t \epsilon V(H_2)} \sum_{s_1s_2 \epsilon E(H_1)} [(d_{R_k(H_1)}(s_1) + d_{H_2}(t))^2 + (d_{R_k(H_1)}(s_2) + d_{H_2}(t))^2]$$

$$= \sum_{t \epsilon V(H_2)} \sum_{s_1s_2 \epsilon E(H_1)} [d_{R_k(H_1)}(s_1)^2 + 2d_{H_2}(t)^2 + d_{R_k(H_1)}(s_2)^2 + 2d_{H_2}(t)(d_{R_k(H_1)}(s_1)$$

$$+ d_{R_k(H_1)}(s_2))]$$

$$= \sum_{t \epsilon V(H_2)} \sum_{s_1s_2 \epsilon E(H_1)} [4d_{H_1}(s_1)^2 + 2d_{H_2}(t)^2 + 4d_{H_1}(s_2)^2 + 2d_{H_2}(t)(2d_{H_1}(s_1) + 2d_{H_1}(s_2))]$$

$$= \sum_{t \epsilon V(H_2)} [4F(H_1) + 2 \mid E(H_1) \mid d_{H_2}(t)^2 + 4M_1(H_1)d_{H_2}(t)]$$

$$= 4 \mid V(H_2) \mid F(H_1) + 2 \mid E(H_1) \mid M_1(H_2) + 8 \mid E(H_2) \mid M_1(H_1),$$

$$\sum^{''} 2 =$$

$$\sum_{t \epsilon V(H_2)} \sum_{\substack{s_1s_2 \epsilon E(R_k(H_1)) \\ s_1 \epsilon V(H_1) \\ s_2 \epsilon V(R_k(H_1)) - V(H_1)}} [(d_{R_k(H_1)}(s_1) + d_{H_2}(t))^2 + d_{R_k(H_1)}(s_2)^2]$$

$$= \sum_{t \epsilon V(H_2)} \sum_{\substack{s_1s_2 \epsilon E(R_k(H_1)) \\ s_1 \epsilon V(H_1) \\ s_2 \epsilon V(R_k(H_1)) - V(H_1)}} [d_{R_k(H_1)}(s_1)^2 + d_{H_2}(t)^2 + 2(d_{R_k(H_1)}(s_1)d_{H_2}(t))$$

$$+ d_{R_k(H_1)}(s_2)^2]$$

$$= \sum_{t \epsilon V(H_2)} \sum_{\substack{s_1s_2 \epsilon E(R_k(H_1)) \\ s_1 \epsilon V(H_1) \\ s_2 \epsilon V(R_k(H_1)) - V(H_1)}} [4d_{H_1}(s_1)^2 + d_{H_2}(t)^2 + 4(d_{H_1}(s_1)d_{H_2}(t)) + 4]$$

$$= \sum_{t \epsilon V(H_2)} [4F(H_1) + 2 \mid E(H_1) \mid d_{H_2}(t)^2 + 4M_1(H_1)d_{H_2}(t) + 4 \times 2 \mid E(H_1) \mid]$$

$= 4 \mid V(H_2) \mid F(H_1) + 2 \mid E(H_1) \mid M_1(H_2) + 8 \mid E(H_2) \mid M_1(H_1) + 8$
$\mid V(H_2) \mid\mid E(H_1) \mid,$

and

$$\sum\nolimits^{'''} 2 =$$

$$(k-1) \sum_{t \epsilon V(H_2)} \sum_{\substack{s_1 s_2 \epsilon E(R_k(H_1)) \\ s_1, s_2 \epsilon V(R_k(H_1)) - V(H_1)}} [d_{R_k}(H_1)(s_1)^2 + d_{R_k(H_1)}(s_2)^2]$$

$= 8(k-1) \mid V(H_2) \mid\mid E(H_1) \mid .$

Hence

$F(H_1 +_{R_k} H_2) = 8 \mid V(H_2) \mid F(H_1) + \mid V(H_1) \mid F(H_2) + 24 \mid E(H_2) \mid M_1(H_1)$
$+12 \mid E(H_1) \mid M_1(H_2) + 8 \mid E(H_1) \mid\mid V(H_2) \mid + 8(k-1) \mid V(H_2) \mid\mid E(H_1) \mid .$

Theorem 3.3.
For $\mid V(H_1) \mid, \mid V(H_2) \mid \geq 4$ and $k \geq 1$, the F-index of $F(H_1 +_{Q_k} H_2)$ is
$3 \mid V(H_2) \mid F(H_1) + \mid V(H_1) \mid F(H_2) + 6 \mid E(H_2) \mid M_1(H_1) + 6 \mid E(H_1) \mid$
$M_1(H_2) + 4 \mid V(H_2) \mid M_2(H_1) + (k) \mid V(H_2) \mid [M_4(H_1) - F(H_1) - 4M_2(H_1)$

$$+ \sum_{u \epsilon V(H_1)} \left(\sum_{v \epsilon N_{H_1}(u)} d_{H_1}(u)(d_{H_1}(v) - 1) \right) + \sum_{uv \epsilon E(H_1)} d_{H_1}(u) d_{H_1}(v)(d_{H_1}(u)$$

$+d_{H_1}(v))] + 2(k-1) \mid V(H_2) \mid F(H_1).$

Proof.
Consider,

$$F(H_1 +_{Q_k} H_2) = \sum_{(s_1,t_1)(s_2,t_2) \epsilon E(H_1 +_{Q_k} H_2)} [d(s_1,t_1)^2 + d(s_2,t_2)^2]$$

$$= \sum_{s \epsilon V(H_1)} \sum_{t_1 t_2 \epsilon E(H_2)} [d(s,t_1)^2 + d(s,t_2)^2] + \sum_{t \epsilon V(H_2)} \sum_{s_1 s_2 \epsilon E(Q_k(H_2))} [d(s_1,t)^2 + d(s_2,t)^2]$$

$$= \sum 1 + \sum 2.$$

Now

$$\sum 1 =$$

$$\sum_{s \epsilon V(H_1)} \sum_{t_1 t_2 \epsilon E(H_2)} [(d_{H_1}(s) + d_{H_2}(t_1))^2 + (d_{H_1}(s) + d_{H_2}(t_2))^2]$$

$$= \sum_{s \epsilon V(H_1)} \sum_{t_1 t_2 \epsilon E(H_2)} [2d_{H_1}(s)^2 + (d_{H_2}(t_1)^2 + d_{H_2}(t_2)^2) + 2d_{H_1}(s)(d_{H_2}(t_1) + d_{H_2}(t_2))]$$

$$= \sum_{s \epsilon V(H_1)} [\mid E(H_2) \mid \times 2 d_{H_1}(s)^2 + F(H_2) + 2M_1(H_2) d_{H_1}(s)]$$

$$= 2 \mid E(H_2) \mid M_1(H_1) + \mid V(H_1) \mid F(H_2) + 4 \mid E(H_1) \mid M_1(H_2),$$

and

$$\sum 2 =$$

$$\sum_{t \epsilon V(H_2)} \sum_{s_1 s_2 \epsilon E(Q_k(H_1))} [d(s_1,t)^2 + d(s_2,t)^2]$$

$$= \sum_{t \epsilon V(H_2)} \sum_{\substack{s_1 s_2 \epsilon E(Q_k(H_1)) \\ s_1 \epsilon V(H_1) \\ s_2 \epsilon V(Q_k(H_1)) - V(H_1)}} [d(s_1,t)^2 + d(s_2,t)^2]$$

$$+ \sum_{t \epsilon V(H_2)} \sum_{\substack{s_1 s_2 \epsilon E(Q_k(H_1)) \\ s_1 s_2 \epsilon V(Q_k(H_1)) - V(H_1)}} [d(s_1,t)^2 + d(s_2,t)^2]$$

$$= \sum{}' 2 + \sum{}'' 2.$$

Now we talk,

$$\sum{}' 2 =$$

$$\sum_{t \epsilon V(H_2)} \sum_{\substack{s_1 s_2 \epsilon E(Q_k(H_2)) \\ s_1 \epsilon V(H_1) \\ s_2 \epsilon V(Q_k(H_1)) - V(H_1)}} [(d_{Q_k(H_1)}(s_1) + d_{H_2}(t))^2 + d_{Q_k(H_1)}(s_2)^2]$$

$$= \sum_{t \epsilon V(H_2)} \sum_{\substack{s_1 s_2 \epsilon E(Q_k(H_2)) \\ s_1 \epsilon V(H_1) \\ s_2 \epsilon V(Q_k(H_1)) - V(H_1)}} [d_{Q_k(H_1)}(s_1)^2 + d_{H_2}(t)^2 + 2 d_{Q_k(H_1)}(s_1) d_{H_2}(t)$$

$$+ d_{Q_k(H_1)}(s_2)^2]$$

$$= \sum_{t \epsilon V(H_2)} \sum_{s_1 \epsilon V(H_1)} [d_{Q_k(H_1)}(s_1)^2 + d_{H_2}(t)^2 + 2 d_{Q_k(H_1)}(s_1) d_{H_2}(t)]$$

$$+ \sum_{t \epsilon V(H_2)} \sum_{\substack{s_1 s_2 \epsilon E(Q_k(H_2)) \\ s_1 \epsilon V(H_1) \\ s_2 \epsilon V(Q_k(H_1)) - V(H_1)}} [d_{Q_k(H_1)}(s_2)^2]$$

$$= \mid V(H_2) \mid F(H_1) + 2 \mid E(H_1) \mid M_1(H_2) + 4 \mid E(H_2) \mid M_1(H_1) + \sum_{t \epsilon V(H_2)}$$

$$\sum_{\substack{s_1s_2\epsilon E(Q_k(H_1))\\ s_1\epsilon V(H_1)\\ s_2\epsilon V(Q_k(H_1))-V(H_1)}} [d_{Q_k(H_1)}(s_2)^2]$$

For $s_2\epsilon V(Q_k(H_1))-V(H_1)$ that is introduced into the edges uv of (H_1), we have $d_{Q_k(H_1)}(s_2)=d_{H_1}(u)+d_{H_1}(v)$. This gives

$$\sum_{\substack{s_1s_2\epsilon E(Q_k(H_1))\\ s_1\epsilon V(H_1)\\ s_2\epsilon V(Q_k(H_1))-V(H_1)}} [d_{Q_k(H_1)}(s_2)^2] = 2\sum_{uv\epsilon E(H_1)} [d_{H_1}(u)+d_{H_1}(v)]^2$$

$$= 2\sum_{uv\epsilon E(H_1)} [d_{H_1}(u)^2+d_{H_1}(v)^2+2d_{H_1}(u)d_{H_1}(v)] = 2F(H_1)+4M_2(H_1)$$

and

$$= \sum_{t\epsilon V(H_2)} \sum_{\substack{s_1s_2\epsilon E(Q_k(H_1))\\ s_1\epsilon V(H_1)\\ s_2\epsilon V(Q_k(H_1))-V(H_1)}} [d_{Q_k(H_1)}(s_2)]^2$$

$$= 2\mid V(H_2)\mid F(H_1)+4\mid V(H_2)\mid M_2(H_1)$$

so,

$$\sum{}' 2 =$$

$$\mid V(H_2)\mid F(H_1)+2\mid E(H_1)\mid M_1(H_2)+4\mid E(H_2)\mid M_1(H_1)+2\mid V(H_2)\mid F(H_1)$$
$$+4\mid V(H_2)\mid M_2(H_1),$$
$$= 3\mid V(H_2)\mid F(H_1)+2\mid E(H_1)\mid M_1(H_2)+4\mid E(H_2)\mid M_1(H_1)+4\mid V(H_2)\mid M_2(H_1),$$

and

$$\sum{}'' 2 =$$

$$\sum_{t\epsilon V(H_2)} \sum_{\substack{s_1s_2\epsilon E(Q_k(H_1))\\ s_1,s_2\epsilon V(Q_k(H_1))-V(H_1)}} [d(s_1,t)^2+d(s_2,t)^2]$$

Now we divide $\sum'' 2$ into $\sum 3$ and $\sum 4$ for the nodes, s_1 and s_2, where $s_1s_2\epsilon V(Q_k(H_1))-V(H_1)$. So $\sum'' 2=\sum 3+\sum 4$, where $\sum 3$ include the edges of $Q_k(H_1)$ having the similar edges of H_1 and $\sum 4$ of $Q_k(H_1)$ in two dissimilar adjacent edges of H_1.

$$\sum 3 =$$

$$\sum_{t\epsilon V(H_2)} \sum_{\substack{s_1s_2\epsilon E(Q_k(H_1))\\ s_1s_2\epsilon V(Q_k(H_1))-V(H_1)}} [d_{Q_k(H_1)}(s_1)^2+d_{Q_k(H_1)}(s_2)^2]$$

$$= (k-1)2 \sum_{t\epsilon V(H_2)} \sum_{uv\epsilon E(H_1)} [d_{H_1}(u)^2+d_{H_1}(v)^2] = 2(k-1) \mid V(H_2) \mid F(H_1),$$

and

$$\sum 4 =$$

$$\sum_{t\epsilon V(H_2)} \sum_{\substack{s_1s_2\epsilon E(Q_k(H_1))\\ s_1,s_2\epsilon V(Q_k(H_1))-V(H_1)}} [d_{Q_k(H_1)}(s_1)^2+d_{Q_k(H_1)}(s_2)^2]$$

$$= (k) \sum_{t\epsilon V(H_2)} \sum_{\substack{uv\epsilon E(H_1)\\ vw\epsilon E(H_2)}} [(d_{H_1}(u)+d_{H_1}(v))^2+(d_{H_1}(v)+d_{H_1}(w)^2],$$

where s_1 and s_2 are the nodes that are introduced into the edge set of uv and vw of H_1.

$$= (k) \mid V(H_2) \mid [\sum_{u\epsilon V(H_1)} d_{H_1}(v)^4-d_{H_1}(v)^3+ \sum_{v\epsilon V(H_1)} (d_{H_1}(v)-1) \sum_{\substack{u\epsilon V(H_1)\\ uv\epsilon E(H_1)}} d_{H_1}(u)^2$$

$$+2 \sum_{v\epsilon V(H_1)} d_{H_1}(v)(d_{H_1}(v)-1) \sum_{\substack{u\epsilon V(H_1)\\ uv\epsilon E(H_1)}} d_{H_1}(u)]$$

$$= (k) \mid V(H_2) \mid [M_4(H_1)-F(H_1)-4M_2(H_1)+ \sum_{u\epsilon V(H_1)} (\sum_{v\epsilon N_{H_1}(u)} d_{H_1}(u)$$

$$(d_{H_1}(v)-1))+ \sum_{uv\epsilon E(H_1)} d_{H_1}(u)d_{H_1}(v)(d_{H_1}(u)+d_{H_1}(v))],$$

Consequently, we have

$$3 \mid V(H_2) \mid F(H_1)+ \mid V(H_1) \mid F(H_2)+6 \mid E(H_2) \mid M_1(H_1)+6 \mid E(H_1) \mid$$
$$M_1(H_2)+4 \mid V(H_2) \mid M_2(H_1)+(k) \mid V(H_2) \mid [M_4(H_1)-F(H_1)-4M_2(H_1)$$
$$+ \sum_{u\epsilon V(H_1)} \left(\sum_{v\epsilon N_{H_1}(u)} d_{H_1}(u)(d_{H_1}(v) - 1) \right) + \sum_{uv\epsilon E(H_1)} d_{H_1}(u)d_{H_1}(v)(d_{H_1}(u)$$
$$+d_{H_1}(v))]+2(k-1) \mid V(H_2) \mid F(H_1).$$

Theorem 3.4.
For $\mid V(H_1) \mid, \mid V(H_2) \mid \geq 4$ and $k \geq 1$, the F-index of $F(H_1 +_{T_k} H_2)$ is

$$9 \mid V(H_2) \mid F(H_1)+ \mid V(H_1) \mid F(H_2)+24 \mid E(H_2) \mid M_1(H_1)+12 \mid E(H_1) \mid$$

$M_1(H_2)+4 \mid V(H_2) \mid M_2(H_1)+(k) \mid V(H_2) \mid \big[M_4(H_1)-F(H_1)-4M_2(H_1)$

$+\sum_{u\epsilon V(H_1)}\left(\sum_{v\epsilon N_{H_1}(u)} d_{H_1}(u)(d_{H_1}(v)-1)\right)+\sum_{uv\epsilon E(H_1)} d_{H_1}(u)d_{H_1}(v)(d_{H_1}(u)$

$+d_{H_1}(v))\big]+2(k-1) \mid V(H_2) \mid F(H_1).$

Proof. Proof is same as of Theorem 4.2 and Theorem 4.3.

4. Conclusion

In this paper, we computed the F-index of the generalized F-sum graphs. For m_1, $m_2 \geq 4$ and $k = 4$, assume that $H_1 = P_{m_1}$ and $H_2 = P_{m_2}$ are particular alkane called as paths of orders m_1 and m_2 respectively. Then the following outcomes are the direct consequences of the achieved results.

- $F(P_{m_1+_{S_4}}P_{m_2}) = 88m_1m_2 - 90m_1 - 66m_2 + 56,$
- $F(P_{m_1+_{R_4}}P_{m_2}) = 248m_1m_2 - 336m_1 - 182m_2 + 216,$
- $F(P_{m_1+_{Q_4}}P_{m_2}) = 264m_1m_2 - 522m_1 - 74m_2 + 72,$
- $F(P_{m_1+_{T_4}}P_{m_2}) = 416m_1m_2 - 655m_1 - 182m_2 + 216.$

5. Acknowledgement

The authors would like to express their honest appreciation to the referees for their priceless opinion and insightful remarks, which led to a development in the previous version of this manuscript.

References

[1] J-B. Liu, M. Javaid and H.M. Awais, "Computing Zagreb indices of the subdivision-related generalized operations of graphs", IEEE Access, 7(2019); 105479-105488.
[2] J-B. Liu, S. Javed, M. Javaid and K. Shabbir, "Computing first general Zagreb index of operations on graphs", IEEE Access, vol. 07, pp. 47494-47499, 2019.
[3] B. Furtula, I. Gutman, A forgotten topological index, J. Math. Chem., Vol. 53, pp. 1184-1190, 2015
[4] P. Sarkar, N. De and A. Pal, "The Forgotten Topological Index of Double Corona of Graphs Related to the Different Subdivision Graphs" MAPAS, vol. 2, pp. 8-15, Mar 2019.
[5] V. Lokesha and K. Z. Yasmeen, "SK indices, forgotten toplogical indices and hyper Zagreb index od Q operator of carbon nanocone", TWMS J. App. and Eng. Math., vol. 9, pp. 675-680, 2019.
[6] F. Yan, Q. Shang, S. Xia, Q. Wang and P. Ma, "Application of topological index in predicting ionic liquids densities by the quantitative structure property relationship method", J. Chem. Eng., vol. 60, pp. 734-739, 2015.

[7] H.J. Wiener, "Structural determination of paraffin boiling points", J. Amer. Chem., vol. 69, pp. 17-20, 1947.
[8] G. Rucker and C. Rucker, "On topological indices, boiling points, and cycloalkanes", J. Chem. Inf. Comput. Sci., vol. 39, pp. 788-802, 1999.
[9] M. Randić, "On characterization of molecular branching", J. Am. Chem., vol. 97, pp. 6609-6615, 1975.
[10] A. R. Matamala and E. Estrada, "Generalised topological indices: Optimisation methodology and physico-chemical interpretation", Chemical Physics Letters, vol. 410, pp. 343-347, 2005.
[11] D.M. Cvetkocić, M. Doob and H. Sachs, Spectra of Graphs: Theory and Application, Academic Press, New York, 1980.
[12] W. Yan, B-Y. Yang and Y-N. Yeh, "The behavior of Wiener indices and polynomials of graphs under five graph decorations", Appl. Math. Lett., vol. 20, pp. 290-295, 2007.
[13] M. Eliasi and B. Taeri, Four new sums of graphs and their Wiener indices, Discrete Applied Mathematics, vol. 157(4), pp. 794-803, 2009.
[14] H. González-Díaz, S. Vilar, L. Santana and E. Uriarte, Medicinal Chemistry and Bioinformatics - Current Trends in Drugs Discovery with Networks Topological Indices, Current Topics in Medicinal Chemistry, vol. 7(10), pp. 1015-1029, 2007.
[15] L.H. Hall and L.B. Kier, Molecular Connectivity in Chemistry and Drug Research, Academic Press Boston, MA, USA, 1976.
[16] M.V. Diudea (Ed.), QSPR/QSAR Studies by Molecular Descriptors, NOVA, New York, 2001.
[17] J. Devillers and A.T. Balaban, Topological Indices and Related Descriptors in QSAR and QSPR, Gordon Breach, Amsterdam, 1999.
[18] R. Todeschini, V. Consonni, R. Mannhold, H. Kubinyi and H. Timmerman, Handbook of Molecular Descriptors, Wiley-VCH, Weinheim, Germany, 2002.
[19] I. Gutman and N. Trinajsti, Graph theory and molecular orbitals. III. Total π-electron energy of alternant hydrocarbons, Chem. Phys., vol. 17, pp. 535-538, 1972.
[20] I. Gutman and O. Polansky, Mathematical Concepts in Organic Chemistry, Springer-Verlag, Berlin, 1986.
[21] I. Gutman, Degree-based topological indices, Croat. Chem. Acta, vol. 86, pp. 351-361, 2013.
[22] H. González-Diaz, S. Vilar, L. Santana and E. Uriarte, Medicinal chemistry and bioinformatics-current trends in drugs discovery with networks topological indices, Current Topics Medical Chem., vol. 7, no. 10, pp. 1025-1039, May 2007.
[23] B. Furtula and I. Gutman, A forgotten topological index. Journal of Mathematical Chemistry, vol. 53(4), pp. 1184-1190, 2015.
[24] H. Deng, D. Sarala, S.K. Ayyaswamy and S. Balachandran, The Zagreb indices of four operations on graphs, Applied Mathematics and Computation, vol. 275 pp. 422-431, 2016.
[25] S. Akhter and M.Imran, Computing the forgotten topological index of four operations on graphs, AKCE International Journal of Graphs and Combinatorics, vol. 14(1), pp. 70-79, 2017.
[26] X. Li and J. Zheng, A unified approach to the extremal trees for different indices, MATCH Commun Math Comput Chem., vol. 54 pp. 195-208, 2005.
[27] R. Gozalbes, J.P. Doucet and F. Derouin, Application of topological descriptors in QSAR and drug design: history and new trends, Curr. Drug Targets Infect. Disord., vol. 02, pp. 93-102, 2002.