Journal of Prime Research in Mathematics Vol. **16(1)***(2020), 46 - 51*

CHARACTERIZATIONS OF CHEVALLEY GROUPS USING ORDER OF THE FINITE GROUPS

ABID MAHBOOB[1,*], TASWER HUSSAIN[2], MISBAH AKRAM[3], SAJID MAHBOOB[4], NASIR ALI[4], ALI RAZA[4]

ABSTRACT. In this paper, we prove $\psi(A_1(4)) < \psi(G)$, $\forall$ groups which are not simple with order sixty, $A_1(4)$ is Chevalley group (Linear group) of order 60. Also we prove that $\psi(A_2(2)) < \psi(G)$ using higher order non-simple groups of order 168.

Key words: Chevalley groups, Finite groups, Simple groups, Element orders.
MSC: 20D05, 20B40.

1. INTRODUCTION

Consider the funtion

$$\psi(G) = \sum_{h \in G} o(h),$$

and we know that o(h) be order of h $\in G$. Now we explain the basic ideas which is already discussed in the papers papers [1, 2] which shows the minimum of ψ on the same order groups. In [1] Amiri et al. explained the function $\psi(G)$, which is the sum of element orders of a finite group G, and showed if G be a non-cyclic group with order n then $\psi(G) < \psi(C_n)$, where C_n denote the cyclic group of order n.

Recall that the function ψ is multiplicative, that is if G_1 and G_2 are two finite groups satisfying $\gcd(G_1 , G_2) = 1$, then $\psi(G_1 \times G_2) = \psi(G_1)\ \psi(G_2)$. We use the paper [1] which find the influence of ψ on the same order groups.

[1]Department of Mathematics, Division of Science and Technology, University of Education Lahore, Pakistan.
[2]GHSS Shergarh, Okara Pakistan.
[3]Department of Mathematics, GCU, Lahore, Pakistan.
[4]Department of Mathematics, The University of Education Lahore, Pakistan.
[*] Corresponding author: abid.mahboob@ue.edu.pk.

Notations and terminologies used in the entire paper are standard. Results related to group theory discussed briefly in [3, 4]. In 2011, Amiri and Amiri [2] turned their attention to the mini(ψ(G)). Moreover, the value of (ψ) on such a non-nilpotent group of order n is must less than the value of ψ on any nilpotent group of order n. In 2012, Amiri and Amiri [5] showed that $\psi(H) < \psi(A_n)$, for every proper subgroup H of S_n which is distinct from A_n. For more details about the subject matter of this paper, we refer [6-8].

In this paper, $A_1(4)$ represents Chevalley group of order 60 and $A_2(2)$ represents Chevalley group of order 168.

Theorem 1. *Suppose* $\mid C \mid = m$ *be a cyclic group. Then* $\psi(G) < \psi(C)$, $\forall$ *non-cyclic groups G of order m.*

These groups are investigated by orders and sums of elements order and the G is not determined by invariants $|G|$ and $\psi(G)$.

Example 1. *Let* G_0 *and* G_1 *be two non-isomorphic groups of order 27 such that* $\psi(G_1) = \psi(G_2)$.

Since ψ is multiplicative and G$_0$ and G$_1$ satisfying ($|G_0|, |G_1|$)=1, so $\psi(G_0 \times G_1) = \psi(G_0)\psi(G_1)$. In [2], we see

Theorem 2. *Let* $\mid G \mid = n$ *be a nilpotent group. Then* $\psi(G) \leq \psi(H)$ *for every nilpotent group H of order n, iff each Syl(G) is of prime exponent.*

Theorem 3. *Suppose* $n \in Z^+$ *s.t* $\exists$ *a non-nilpotent group of order n. Then K a non-nilpotent group of order n with* $\psi(K) < \psi(H)$ *for every nilpotent group H of order n.*

It is clear that G is distinct on the same order groups and conjectured in [2] and ψ(G) exist.

Conjecture 1. *Let G be a non-simple group and T be a simple group, if G has order equal to* $|T|$, *then* $\psi(T) < \psi(G)$.

Then there exist $|T|$ = n, so least of ψ, on all equal order groups, contained in T. Now we determine Conjecture 1 for A$_1$(4) and $A_2(2)$. Note that A$_1$(4) *and* $A_2(2)$ are observed by order and sum of element orders.

In this paper all notations are standard. Since prime is denoted by p, then number of $Syl_p(G)$ are $n_p = n_p(G)$ and Syl_p(G) be Sylow p-subgrups. For a positive integer n, $|C_n|$ = n be a cyclic group.

2. Mimimum of ψ on all groups of order 60

Since A$_1$(4) has elements of order 2, 3 and 5 and their elements are 15, 20 and 24 respectively. Therefore $\psi(A_1(4)) = 211$.

Theorem 4. *Let* $\mid G \mid = 60$. *Then* $\psi(G) \geq 211$ *and* $\psi(G)=211$ *iff* $G \cong A_1(4)$.

Proof. Since | G | = 60, then n_5 = 1 or 6 and n_3 = 1 or 10. If n_3 = 1 or n_5 = 1, then there is a cyclic subgroup in G has order fifteen. Thus G has minimum eight elements of order fifteen. So G has maximum forty five elements of order at least 2. Also G has minimum 4 elements of order five and minimum 2 elements of order three. Hence

$$\psi(G) \geq 8(15) + 4(5) + 2(3) + 45(2) + 1 = 236.$$

So we may assume that n_3 = 10 and n_5 = 6. Therefore G has 20 and 24 elements of order 3 and 5 respectively. If $T = \{y \in G \,|\, o(y) = 3 \text{ or } 5\}$, then $|T|$=44. If there exists an element h $\in G/T$ such that o(h) > 2, then $\psi(G) >$ 211. Let every element of G\T has order two.
$\Rightarrow |C_G(y)| = 4, \forall$ o(h) = 2. So intersection of two different Syl_2 of G are trivial and so n_2 = 5. Hence G ≅ S_5. It follows that G ≅ $A_1(4)$. As required. □

Corollary 5. *Let G be a non-simple group, if | G | = 60 then* $\psi(G) > \psi(A_1(4))$.

3. Groups of order 168 and mimimum of ψ

Since $A_2(2)$ has 21, 42, 56 and 48 elements of order 2, 4, 3 and 7 respectively and since $A_2(2)$ has 21 Sylow 2-subgroups isomorphic to D_8. n_3 = 28 and n_7 = 8. Therefore $\psi(A_2(2)) = 715$.

Lemma 6. *Let | G | = 168 and h ∈ G, if o(h) = 21 ≤ G, then* $\psi(G) > 715$.

Proof. Suppose that there are H_1, H_2 be 2 cyclic subgroups $H_1, H_2 \in$ G with orders 21, then G contain minimum elements 2Φ(21) is equal to twenty four has 21 order, thus maximum elements of G are 144 which have not 21 order. Hence

$$\psi(G) \geq 1 + 143(2) + 24(21) = 791 > 715.$$

Suppose G has only cyclic subgroup |R| = 21. Therefore n_7(G) = 1 and n_3(G) = 1. This shows

$$\frac{G}{C_G(P)} \hookrightarrow Aut(C_7)$$

where Q is $Syl_7(G)$. So $|C_G(Q)| = 2^3{\cdot}3{\cdot}7$ or $2^2 \cdot 3{\cdot}7$. As G contain a central subgroup Q. By [1] Corollary B, $\psi(G) = \psi(\frac{G}{Q})\psi(Q) > 715$. If $|C_G(Q)| = 2^2 \cdot 3 \cdot 7$, Q is central in $C_G(Q)$ and so $\psi(C_G(Q)) = 43 \cdot 24 > 715$, which is required. □

Lemma 7. *Let | G | = 168. If G contains no element order 21 and* $n_7 = 8$*, then either* $\psi(G) > 715$ *or* $G \cong A_2(2)$.

Proof. By supposition, E = N_G(Q) = MQ , where Q ∈ Syl_7(G), M ∈ Syl_3(G). Since n_3(E) = 7, n_3(G) ≥ 7 and so n_3 = 7 or 28.

If $E_1 \cap E_2$ of E is trivial, then $|E_1E_2| > 168$, which contradict the hypothesis. Therefore $|E_1 \cap E_2| = 3$ and since E has 8 conjugates, $n_3 = 28$, $\Rightarrow N_G(M) \cong S_3 \, or \, C_6$.

If $N_G(M) \cong C_6$, then number of elements of order six in G are 56 and each element is of order six, then

$$\psi(G) \geq 1 + 48(7) + 56(3) + 56(6) > 715.$$

If $N_G(M) \cong S_3$, then $C_G(M) = M$ and so the centralizer subgroup is a p-subgroup. If $n_2 \leq 7$, then G has maximum 49 2-elements without identity. Since order of G is 168, so $\exists$ r $\in$ G which contradicts as $\exists$ r $\in$ G is not a p-element. So $n_2 = 21$. If $P \cap P' = 1$ where P and P' be any 2 different $Syl_2(G)$, then the order of G is greater than 168. Hence there is y$\in I_1 \cap I_2$, where $I_i \in Syl_2$ (G) for i is equal to 1 or 2. If I_i is abelian then $|C_G(y)| > 8$, which is a contradiction. So I_i is not abelian. Following this $I_1 \cong D_8 \, or \, Q_8$. If o(y) = 4, then $C_G(y) = \langle y \rangle$ and y contain 42 conjugate elements contained in G. Since G contained 48, 56 and 42 elements and order of these elements are 7, 3 and 4 respectively. Hence G has 21 elements of order 2. Also G is simple as it has no minimal normal subgroup. Hence G $\cong A_2(2)$. Which is the required result. □

Lemma 8. *Let $|G| = 168$. If G have no element of order 21 and $n_7 = 1$, then $\psi(G) > 715$.*

Proof. Let Q be the only $Syl_7(G)$. Then $|C_G(Q)| = 2^3 \cdot 7$ or $2^2 \cdot 7$.

If $|C_G(Q)| = 2^3$. 7, then $C_G(Q) = Q \times I$, where $I \in Syl_2(G)$. So $n_2(G) = 1$, since $C_G(Q) \trianglelefteq G$. Then $\psi(C_G(Q)) \geq 15{\cdot}43 = 645$. As 3 is the minimum order of element in $G \backslash C_G(Q)$ and $|G \backslash C_G(Q)| = 112$, we have $\psi(G) \geq 645 + 112{\cdot}{+}645 > 715$.

Now $|C_G(Q)| = 2^2 \cdot 7$. Then $C_G(Q) = F \times Q$, where $|F| = 4$. Since $C_G(Q) \trianglelefteq G$ and F is characteristic in $C_G(Q)$, $F \trianglelefteq G$. Therefore F is the intersection of all Syl_2G.

Clearly $n_3 = 7$ or 28 and $n_2 = 7$ or 21. Let

$$B = \{y \in G \,|\, y \;\; be \quad 2\, element\}$$

and

$$C = \{y \in G \,|\, y \;\; be \quad 3\, element\}$$

Also these elements are non identity, so $|B|$ is thirty one or eighty seven and $|C|$ is fourteen or fifty six. Since y $\in G \backslash (B \cup C \cup Q)$, so either o(y) is six or o(y) greater than or equal to 12. If o(y) is equal to six, then G consist of $2n_3(G)$ elements of order six.

If $F \simeq C_4$, then $\psi(C_G(Q)) = 473$. Since $|G \backslash C_G(Q)| = 140$, $\psi(G) \geq 473 + 140(2) > 715$.

If F$\simeq C_2 \times C_2$, so G contain eighteen elements of order fourteen. Then we have:
1)- If $|B| = 31$ and $|C| = 14$, then

$$\psi(G) \geq 6(7) + 31(2) + 14(3) + |G\backslash(B \cup C \cup Q)|6 > 715.$$

2)- If $|B| = 87$ and $|C| = 14$, then

$$\psi(G) \geq 6(7) + 87(2) + 14(3) + |G\backslash(B \cup C \cup Q)|6 > 715.$$

3)- If $|B| = 31$ and $|C| = 56$, then

$$\psi(G) \geq 6(7) + 31(2) + 56(3) + |G\backslash(B \cup C \cup Q)|6 > 715.$$

4)- If $|B| = 87$ and $|C| = 56$ then G has 21 Syl_2(G). Let I $\in$ Syl$_2(G)$ and I $\cong C_2 \times C_2 \times C_2$. If y $\in$ F, then C$_g$(y) = IQ, since $|B \cup C \cup Q| = 150$ and G contain eighteen elements of order fourteen. So n$_2(C_G(y)) = 7$. A contradiction because C$_G$(y) contains 21 $Syl_2(G)$. Hence I $\neq$ (C$_2$)^3. Hence $Syl_2(G)$ has minimum one cyclic subgroup of order four. Since F $\cong$ C$_2 \times C_2$, the intersection of any two different $Syl_2(G)$ not have any four order element. Therefore G has 21 four order cyclic subgroups. Thus

$$\psi(G) \geq 42(4) + 87(2) + 56(3) + 18(14) > 715.$$

As required. □

Theorem 9. *Let* $\mid G \mid = 168$. *Then* $\psi(G) \geq 715$.

Corollary 10. *Let* $\mid G \mid = 168$ *and* $G \neq A_2(2)$. *Then* $\psi(G) > \psi(A_2(2))$.

Proof. If G satisfied the Lemmas, then it is true If G satisfied 3.2 , so $\psi(G) > 715$. □

References

[1] Amiri, H., Jafarian Amiri, S. M., & Isaacs, I. M. (2009). Sums of element orders in finite groups. *Communications in Algebra, 37*(9), 2978-2980.

[2] Amiri, H., & Jafarian Amiri, S. M. (2011). Sum of element orders on finite groups of the same order. *Journal of Algebra and its Applications, 10*(02), 187-190. 187-190.

[3] Mabuchi, T. (1990). Compactification of the Moduli Space of Einstein-Khler Orbifolds. *In Recent Topics in Differential and Analytic Geometry* (pp. 359-384). Academic Press.

[4] Isaacs, I. M. (2008). *Finite group theory* (Vol. 92). American Mathematical Soc.

[5] Amiri, H., & Jafarian Amiri, S. M. (2012). Sum of element orders of maximal subgroups of the symmetric group. *Communications in Algebra, 40*(2), 770-778.

[6] Tufail, M., & Qazi, R. (2019). A closer look at multiplication table of finite rings. *Open Journal of Mathematical Sciences., 3*(1), 285 288.

[7] Chelgham, M., & Kerada, M. (2018).On Torsion and finite extension of FC and τN_K groups in certain classes of finitely generated groups. *Open Journal of Mathematical Sciences., 2*(1), 351 360.

[8] Abdullateef, L. (2020). On algebraic properties of fundamental group of intuitionistic fuzzy topological spaces (IFTSs). *Open Journal of Mathematical Sciences., 4*(1), 34 47.