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SOME PROPERTIES OF THE MAXIMAL GRAPH RELATED TO CO-IDEAL OF A COMMUTATIVE SEMIRING

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ABSTRACT. For a commutative semiring R with non-zero identity, the maximal graph of R, denoted by $MG(R)$, is the graph whose vertices are all elements of $UM(R)$ with two distinct vertices joined by an edge when there is a maximal co-ideal that contains both of them. In this paper, we study some properties of maximal graph such as planarity, radius, splitting and domination number.

Key words : Commutative semirings, maximal co-ideal, maximal graph.
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1. INTRODUCTION

Throughout this paper, every semiring R is assumed to be commutative with non-zero identity. For a semiring R, we denote by $Co-Max(R)$, $UM(R)$ and $IM(R)$, the set of maximal co-ideals, the union of all the maximal co-ideals and the intersection of all the maximal co-ideals of R, respectively. Also, if R is a ring, then R has no proper co-ideals, thus in this paper we consider the semiring which is not a ring.

The idea to associate a graph to a commutative ring, was first introduced by Beck [3], where he was mainly interested in coloring. In [2], Anderson and Livingston take the non-zero zero-divisors for the vertices of the graph and two distinct vertices x and y are adjacent if and only if $xy = 0$. They called this graph, *zero-divisor graph* and denoted by $\Gamma(R)$. In [9], Sharma and Bhatwadekar defined another graph on a ring R with vertices as elements of R and there is an edge between two distinct vertices x and y in R if and

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only if $Rx + Ry = R$. Further, in [8], Maimani et al. studied the graph that intoduced by Sharma and Bhatwadekar and called it *comaximal graph*. Some other investigations into properties of comaximal graph over a commutative ring may be found in [1, 7]. Gaur and Sharma in [5], introduced the concept of the *maximal graph* for a commutative ring R, denoted by $G(R)$, with vertices as elements of R, where two distinct vertices x and y are adjacent if and only if $x, y \in m$ for some maximal ideal m of R. They showed a ring R is finite if and only if clique number of the graph $G(R)$ is finite. Also, they showed that $\chi(R) = \omega(R)$ for a semilocal ring R.

For an arbitrary commutative semiring R, the maximal graph of R, denoted by $MG(R)$, was studied in [10]. The vertex-set of $MG(R)$ is $UM(R)$ and two distinct vertices x and y are adjacent if and only if $x, y \in m$ for some $m \in Co - Max(R)$. In [10], the authors considered the subgraphs $MG_1(R)$ and $MG_2(R)$ of $MG(R)$ with vertex-set $IM(R)$ and $UM(R) \setminus IM(R)$ and we investigated some properties of these graphs such as diameter, girth, clique number, chromatic number and connectivity. In this paper, we continue our study of maximal graph of commutative semirings and investigate some graph-theoretic properties of $MG(R)$ and $MG_2(R)$ such as planarity, radius, splitting and domination number.

First, we recall some definitions and notations of graphs which will be used in this paper. For a graph G, by $V(G)$ and $E(G)$, we denote the set of all vertices and all edges, respectively. We say that G is connected if there is a path between any two distinct vertices of G. The *components* of a graph G are its maximal connected subgraphs. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is $diam(G) = sup\{d(x, y)$: x and y are distinct vertices of G $\}$. A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. We denote the complete graph on n vertices by K_n. A *clique* in a graph G is a set of pairwise adjacent vertices. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a largest clique of G. Also, an *independent set* in a graph G is a set of pairwise non-adjacent vertices. We say that two subgraphs G_1 and G_2 of G are *disjoint* if G_1 and G_2 have no common vertices and no vertex of G_1 (respectively, G_2) is adjacent (in G) to any vertex not in G_1 (respectively, G_2). We write $G \setminus \{x\}$ or $G \setminus S$ for the subgraph of G obtained by deleting a vertex x or set of vertices S. An *induced subgraph* is a subgraph obtained by deleting a set of vertices. When $S \subseteq V(G)$, the induced subgraph $G[S]$ consists of the vertex-set S and all edges whose endpoints are contained in S.

Now, we recall some various notions about semiring. According to [4, 6, 11], we have the following definitions.

A *semiring* R is an algebraic system $(R, +, \cdot)$ such that $(R, +)$ is a commutative monoid with identity element 0 and $(R, \cdot)$ is a semigroup. In addition, operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates R (i.e. $x0 = 0x = 0$ for each $x \in R$). A semiring R is said to be *commutative* if $(R, \cdot)$ is a commutative semigroup and R is said to have an *identity* if there exists $1 \in R$ such that $1x = x1 = x$.

A non-empty subset I of R is called a *co-ideal* of R if and only if it is closed under multiplication and satisfies the condition that $a + r \in I$ for all $a \in I$ and $r \in R$. According to this definition, $0 \in I$ if and only if $I = R$. It is trivial that R has no proper co-ideal when R is ring. We say that I is a maximal co-ideal of R, if $I \neq R$ and there is no co-ideal J such that $I \subset J \subset R$. If R is a semiring which is not a ring then it must have a maximal proper co-ideal. Also, every maximal co-ideal contains 1.

An element x of a semiring R is called a *zero-sum* of R if there exists an element $y \in R$ such that $x + y = 0$. We will denote the set of all zero-sums of R by $ZS(R)$. It is easy to see that $ZS(R)$ is an ideal of R.

For a non-empty subset A of a semiring R, the set $F(A)$ of all elements of R of the form $a_1a_2...a_n + r$, where $a_i \in A$ for all $1 \leq i \leq n$ and $r \in R$, is a co-ideal of R containing A. So we can consider the co-ideal generated by the element $a \in R$ as follows: $F(a) = \{a^n + r : \ r \in R \text{ and } n \in N\}$.

For a commutative semiring R, in [10], proved that $x \in \sqrt{ZS(R)}$ if and only if $F(x) = R$ and from this we conclude that $R \setminus \sqrt{ZS(R)} = UM(R)$. It is shown that $x \in IM(R)$ if and only if x is adjacent to any vertex of $MG(R)$. According to this result, it is interesting that we investigate the characteristics of $MG_2(R)$.

2. Planarity and radius of maximal graph

In this section, we give a necessary and sufficient condition for the planarity of $MG(R)$ and $MG_2(R)$.

We recall that a graph G is said to be *planar*, if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. The following well-known result is due to Kuratowski (see [12]):

Theorem 1. [12] *A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.*

Proposition 2. *Let R be a semiring. If R has a maximal co-ideal m such that $|m| \geq 5$, then $MG(R)$ is not planar.*

Proof. Let m be a maximal co-ideal of R and $a_i \in m$ for $i = 1, ..., 5$. Since $a_1, ..., a_5$ form a complete subgraph of $MG(R)$ which is isomorphic to K_5, so by Theorem 1, $MG(R)$ is not planar. □

Lemma 3. *Let R be a c-semilocal semiring with $|Co - Max(R)| = n$. If $n \geq 3$, then $\omega(MG(R)) \geq n + 1$.*

Proof. Suppose that $Co-Max(R) = \{m_1, ..., m_n\}$. Thus by [11, Lemma 2.14], $|m| \geq n+1$ for each maximal co-ideal m of R because $n \geq 3$. Given that each m is clique, so $\omega(MG(R)) \geq n+1$. □

Theorem 4. *Let R be a c-semilocal semiring with $|Co - Max(R)| = n$. If $MG(R)$ is planar, then $n \leq 3$ and $|m| \leq 4$ for each maximal co-ideal m of R.*

Proof. Let $|Co-Max(R)| = n$ and $MG(R)$ be a planar graph. Thus $MG(R)$ does not contain a subdivision of K_5 or $K_{3,3}$ by Theorem 1. If $n \geq 4$, by Lemma 3, $MG(R)$ contains K_5 as a subgraph and so by Theorem 1, $MG(R)$ can not be planar. Hence we must have $|Co - Max(R)| \leq 3$. Now, if there is a maximal co-ideal m of R such that $|m| \geq 5$, then by Proposition 2, $MG(R)$ is not planar. Thus $|m| \leq 4$ for each maximal co-ideal m of R. □

Lemma 5. *Let R be a c-semilocal semiring with $|Co - Max(R)| = n$. If $n \geq 4$, then $\omega(MG_2(R)) \geq n+1$.*

Proof. Let $\{m_1, ..., m_n\}$ be the set of maximal co-ideals of R. Since $n \geq 4$, by [11, Lemma 2.16], we have $|m \setminus IM(R)| \geq n+1$ for any $m \in Co - Max(R)$. Now, since each maximal co-ideal m forms a clique, so $\omega(MG_2(R)) \geq n+1$. □

Theorem 6. *Let R be a c-semilocal semiring with $|Co - Max(R)| = n$. If $MG_2(R)$ is planar graph, then $n \leq 3$ and $|m \setminus IM(R)| \leq 4$ for every $m \in Co - Max(R)$.*

Proof. Suppose that $MG_2(R)$ is a planar graph. Thus by Theorem 1 and Lemma 5, we must have $|Co - Max(R)| \leq 3$. Also, $|m \setminus IM(R)| \leq 4$ for each maximal

co-ideal m of R , otherwise $MG_2(R)$ contains K_5 as a subgraph by Proposition 2. □

Next, we give an example of maximal graph over semiring R that is not planar.

Example 1. *Let $X = \{a, b, c, d\}$ and $R = (P(X), \cup, \cap)$ be a semiring, where $P(X)$ is power set of X and $1_R = X$. This semiring has four maximal co-ideals as follows:*

$$m_1 = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\},$$
$$m_2 = \{\{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\},$$
$$m_3 = \{\{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\},$$
$$m_4 = \{\{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}.$$

Observe that $|m| = 8$ and $|m \setminus IM(R)| = 7$ for any maximal co-ideal m of R. On the other hand, any maximal co-ideal is a clique of $MG(R)$, hence in this example, $MG(R)$ and $MG_2(R)$ contain K_5 as a subgraph. This implies $MG(R)$ and $MG_2(R)$ are not planar.

For a graph G, the *eccentricity* of a vertex x in G is $e(x) = Max\{d(y, x); y \in V(G)\}$. A *center* of G is a vertex a with smallest eccentricity. The eccentricity $e(a)$ is called the *radius* of G and is denoted by $rad(G)$. Thus, if $|V(G)| = 1$, $rad(G) = 0$ and, if G is complete graph with at least two vertices, then each vertex is center and $rad(G) = 1$. In a disconnected graph, the radius (and every eccentricity) is infinite, for distance between vertices in different components is infinite.

The following results give information about radius and center of the graphs $MG(R)$ and $MG_2(R)$.

Proposition 7. *Let R be a semiring with at least two maximal co-ideals. Then for the graph $MG(R)$ we have:*

$$e(x) = \begin{cases} 1 & \text{if } x \in IM(R) \\ 2 & \text{otherwise} \end{cases}$$

Proof. First, assume that $x \in IM(R)$. As for each $y \in MG(R)$, x is adjacent to y, thus $e(x) = 1$. Now, let $x \notin IM(R)$. Since $x \in MG(R)$, so $x \in m$ for some $m \in Co - Max(R)$. For each $y \in MG(R)$, if x and y are not adjacent, then $x - 1 - y$ is a path of length two in $MG(R)$. Hence $e(x) \leq 2$. By hypothesis, since R has at least two maximal co-ideals, there exists $m' \in Co - Max(R)$ in which $m' \neq m$. On the other hand, there exists $z \in m'$ such that $z \notin m$. This implies x and z are not adjacent and so $e(x) = 2$. □

Corollary 8. *Let R be a semiring with at least two maximal co-ideals. Then the elements of $IM(R)$ are centers of $MG(R)$ and $rad(MG(R)) = 1$.*

Proof. Since $|Co-Max(R)| \geq 2$, $MG(R)$ has at least two distinct vertices and thus $rad(MG(R)) \neq 0$. Also, in the Proposition 7, we showed that $e(x) = 1$ for any $x \in IM(R)$. Hence the elements of $IM(R)$ are centers of $MG(R)$ and $rad(MG(R)) = 1$. □

Proposition 9. *Let R be a semiring with $|Co-Max(R)| = 2$. Then $rad(MG_2(R)) = \infty$.*

Proof. We know that $MG_2(R)$ is a disconnected graph when R has two maximal co-ideals by [11, Theorem 3.7]. Hence for each $x \in MG_2(R)$ we have $e(x) = \infty$ and so $rad(MG_2(R)) = \infty$. □

Theorem 10. *Let R be a c-semilocal semiring with $|Co-Max(R)| \geq 3$. Then all elements of $MG_2(R)$ are center and $rad(MG_2(R)) = 2$.*

Proof. Suppose that $Co - Max(R) = \{m_1, ..., m_n\}$ with $n \geq 3$. By [11, Theorem 3.12], $MG_2(R)$ is a connected graph and $diam(MG_2(R)) \leq 2$, thus for each $x \in MG_2(R)$, $e(x) \leq 2$. Since $x \in MG_2(R)$, so $x \in m_i$ and $x \notin m_j$ for some $m_i, m_j \in Co - Max(R)$. Also, by [11, Remark 2.13], there exists

$y \in m_j \setminus \bigcup_{\substack{k=1 \\ k \neq j}}^{n} m_k$. Hence x and y are not adjacent and we have $e(x) = 2$. This implies all elements of $MG_2(R)$ are center and $rad(MG_2(R)) = 2$. $\square$

3. Splitting

A graph G is said to be a *split* graph, if its vertices can be partitioned into a clique K and an independent set S.

In this section we characterize the semirings that their maximal graph is split.

Remark 1. *Note that, if R is a c-local semiring with maximal co-ideal m, then $MG(R)$ is a split graph with $K = m \setminus \{x\}$ and $S = \{x\}$ for each $x \in m$. Now, suppose that R has at least two maximal co-ideals. Since any maximal co-ideal m of R forms a clique, we must have $|m \cap S| \leq 1$. If m and m' are distinct maximal co-ideals of R, we show that $m \cap S \neq m' \cap S$. Assume contrary that $m \cap S = m' \cap S$. On the other hand, there exist $a \in m \setminus m'$ and $b \in m' \setminus m$ such that are not adjacent in $MG(R)$, so both a and b can not belong to K. Without loss of generality, we assume that $b \in S$. Therefore, $S \cap m' = \{b\}$ and this implies $S \cap m = S \cap m' = \{b\}$. Hence $b \in m$, which is not possible. Also, $1 \in K$, because, if $1 \in S$, then $m \cap S = \{1\}$ for all maximal co-ideals of R, a contradiction. We may further assume that K is a maximal clique.*

Now, we give the main result of this section.

Theorem 11. *Let R be a semiring with $|Co\!-\!Max(R)| \geq 2$. Then $MG(R)$ is a split graph if and only if R contains exactly two maximal co-ideals m_1 and m_2 such that $|m_i \setminus m_j| = 1$ for some $1 \leq i \neq j \leq 2$ or R contains exactly three maximal co-ideals such that $|m_i \setminus \bigcup_{\substack{j=1 \\ i \neq j}}^{3} m_j| = 1$ for each $1 \leq i \leq 3$.*

Proof. We consider different two cases:

$Case(1)$: There exists a maximal co-ideal such as m contained in K.
Let $m' \in Co\!-\!Max(R)$ such that $m \neq m'$. Also, let $a \in m \setminus m'$ and $b \in m' \setminus m$. It is obvious that a and b are not adjacent in $MG(R)$, so we have $a \in K$ and $b \in S$. We show that $m' \setminus m = \{b\}$. If there exists $b \neq c \in m' \setminus m$, then a is not adjacent to c and so $c \in S$. This implies $b, c \in m' \cap S$ and so $b = c$ since $|m' \cap S| \leq 1$ by Remark 1. Now, suppose that R has a maximal co-ideal p that $p \neq m, m'$. Since $p \setminus (m \cup m') \neq \emptyset$ and $(p \cap m') \setminus m \neq \emptyset$ by [11, Remark 2.13], hence $p \setminus m$ has at least two distinct elements, which is impossible. Therefore, R has exactly two maximal co-ideals, because R is not c-local.

$Case(2)$: No maximal co-ideal is contained in K.
In this case, we show that R has exactly three maximal co-ideals. Let $m_1, ..., m_4$ be maximal co-ideals of R. There are distinct vertices $a \in (m_1 \cap m_2) \setminus (m_3 \cup m_4)$

and $b \in (m_3 \cap m_4) \setminus (m_1 \cup m_2)$ which are not adjacent. If $a \in S$, then $S \cap m_1 = S \cap m_2 = \{a\}$, which is a contradiction by Remark 1 and hence $a \in K$. Similarly, $b \in K$. So a and b are adjacent, that is impossible. Therefore, R has at most three maximal co-ideals.

Now, suppose that R has only two maximal co-ideals m and m'. As by assumption no maximal co-ideal is contained in K, so $m \cap S$ and $m' \cap S$ are not empty. Thus there exist $a, b \in MG(R)$ such that $m \cap S = \{a\}$ and $m' \cap S = \{b\}$. Now, since we assume that K is maximal clique and $a \notin K$, then $K \cup \{a\}$ is not a clique. Hence there exists $c \in K$ such that a is not adjacent to c. Therefore, $c \in (m' \setminus m) \cap K$. Similarly, $K \cup \{b\}$ is not a clique. Thus there exists $d \in K$ such that b is not adjacent to d. So $d \in (m \setminus m') \cap K$. This implies c is not adjacent to d provided $c, d \in K$, that is a contradiction. Thus R has exactly three maximal co-ideals.

Let m_1, m_2 and m_3 be distinct maximal co-ideals of R and x_1, x_2, x_3 be distinct elements of m_i such that $S \cap m_i = \{x_i\}$ for $1 \leq i \leq 3$. By Remark 1, since $m_i \neq m_j$ for $1 \leq i, j \leq 3$, $i \neq j$, then $x_i \in m_i \setminus \bigcup_{\substack{j=1 \\ i \neq j}}^{3} m_j$. Now, it is enough to show that $m_i \setminus \bigcup_{\substack{j=1 \\ i \neq j}}^{3} m_j = \{x_i\}$. If $x_i \neq y \in m_i \setminus \bigcup_{\substack{j=1 \\ i \neq j}}^{3} m_j$, then $y \in K$, since $S \cap m_i = \{x_i\}$. But, y is not adjacent to the elements of $\bigcap_{\substack{j=1 \\ i \neq j}}^{3} m_j \setminus m_i$, which can not be true. Hence $m_i \setminus \bigcup_{\substack{j=1 \\ i \neq j}}^{3} m_j = \{x_i\}$.

Conversely, suppose that $Co-Max(R) = \{m_1, m_2\}$ and $m_2 \setminus m_1 = \{x\}$. We can set $K = m_1$ and $S = \{x\}$. Now, suppose that R contains three maximal co-ideals m_1, m_2 and m_3 such that $m_i \setminus \bigcup_{\substack{j=1 \\ i \neq j}}^{3} m_j = \{x_i\}$ for each $i = 1, 2, 3$. In this case, we set $K = \bigcup_{i=1}^{3} m_i \setminus \{x_1, x_2, x_3\}$ and $S = \{x_1, x_2, x_3\}$. Thus $MG(R)$ is a split graph. $\square$

For a semiring R with maximal co-ideals $\{m_i\}_{i \in I}$, the condition $|m_i \setminus \bigcup_{i \neq j} m_j| = 1$ is not sufficient for a maximal graph can be split and we should have $|Co - Max(R)| = 2$ or $|Co - Max(R)| = 3$. To see this, consider the following example:

Example 2. (1) *Let $X = \{a, b, c\}$ and $R = (P(X), \cup, \cap)$ be a semiring, where $P(X)$ is power set of X. Clearly, $1_R = X$ and $0_R = \emptyset$. In this case, the maximal co-ideals of semiring R are as follows:*

$$m_1 = \{\{a\}, \{a, b\}, \{a, c\}, X\},$$
$$m_2 = \{\{b\}, \{a, b\}, \{b, c\}, X\},$$
$$m_3 = \{\{c\}, \{a, c\}, \{b, c\}, X\}.$$

In this example, $\{\{a, b\}, \{a, c\}, \{b, c\}, X\}$ is a clique and $\{\{a\},\{b\},\{c\}\}$ is an independent set. So $MG(R)$ is a split graph.

(2) *Let $X = \{a, b, c, d\}$ and $R = (P(X), \cup, \cap)$ be a semiring as defined in Example 1. As was observed the semiring R has four maximal co-ideals and*

$|m_i \setminus \bigcup_{\substack{j=1 \\ i\neq j}}^{4} m_j| = 1$ for each $1 \leq i \leq 4$. But $MG(R)$ is not a split graph because its vertices can not be partitioned into a clique and an independent set.

Remark 2. *Note that if R is not a c-local semiring, then $IM(R) \subseteq K$. Because, if $x \in S \cap IM(R)$, then by Remak 1, $S \cap m = \{x\}$ for any $m \in Co-Max(R)$ and this implies R has only one maximal co-ideal, a contradiction. Therefore, we can conclude the graph $MG_2(R)$ satisfies the condition of Theorem 11.*

Theorem 12. *Let R be a semiring. If there is a subset $\{x_1, x_2, x_3\}$ of the vertex-set $MG(R)$ such that x_i and x_j are not adjacent for $1 \leq i, j \leq 3$, $i \neq j$ and $MG(R)\setminus\{x_1, x_2, x_3\}$ is clique, then R has exactly three maximal co-ideals.*

Proof. By our assumption $MG(R)$ is a split graph with $K = MG(R)\setminus\{x_1, x_2, x_3\}$ and $S = \{x_1, x_2, x_3\}$. Since $x_i \in MG(R)$ for each i, then x_i is contained in maximal co-ideal such as m_i. We claim that there exists exactly one maximal co-ideal m_i that $x_i \in m_i$. If m_k is another maximal co-ideal such that $x_i \in m_k$ and $m_i \neq m_k$, then by Remark 1 we have $S \cap m_i = S \cap m_k = \{x_i\}$. This implies $m_i = m_k$. Also, since x_i and x_j are not adjacent, then $m_i \neq m_j$ for $1 \leq i, j \leq 3$, $i \neq j$. Thus R has at least three maximal co-ideals.

Now, we show that R can not contains more than three maximal co-ideals. Assume that R contains maximal co-ideal m_4 that $m_4 \neq m_i$ (for $i = 1, 2, 3$). We know that $|m_4 \cap S| \leq 1$. If $m_4 \cap S = \emptyset$, then $m_4 \subseteq K$ and by case (1) of Theorem 11, R contains at most two maximal co-ideals, a contradiction. Hence, we may assume that $|m_4 \cap S| = 1$. This implies $m_4 \cap S = \{x_i\}$ and therefore $m_4 = m_i$ for some $i = 1, 2, 3$. Thus R has exactly three maximal co-ideals. □

4. Domination number

In a graph G, a set $S \subseteq V(G)$ is a *dominating set*, if every vertex not in S is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum size of a dominating set in G. A dominating set S in G is a *total dominating set* if $G[S]$ has no isolated vertex and S is an *independent dominating set* if $G[S]$ is independent. The minimum cardinality among the total dominating sets of G is called *total domination number* and denoted by $\gamma_t(G)$. Also, the *independent domination number* $\gamma_i(G)$ of graph G equals, the minimum cardinality among the independent dominating sets of G.

It is clearly, for any complete graph $\gamma(G) = 1$ and $\gamma_t(G) = 2$. Also, for any simple graph G, we have $\gamma_t(G) \geq 2$.

In this section, we express some of results about dominating set. Also, we compute domination number, total domination number and independent domination number for the graphs $MG(R)$ and $MG_2(R)$.

Proposition 13. *Let R be a semiring. Then $\gamma(MG(R)) = 1$ and $\gamma_t(MG(R)) = 2$.*

Proof. Let $x \in IM(R)$. Since x is adjacent to each vertex of $MG(R)$, thus $\{x\}$ is a dominating set and we have $\gamma(MG(R)) = 1$. Now, for any $y \in MG(R)$ that $y \neq x$, it is obvious that $\{x, y\}$ is a total dominating set. Therefore, $\gamma_t(MG(R)) = 2$. □

Theorem 14. *Let R be a c-semilocal semiring which is not c-local, then $\gamma(MG_2(R)) = 2$. Moreover, we have $\gamma_i(MG_2(R)) = 2$.*

Proof. Assume that $Co-Max(R) = \{m_1, ..., m_n\}$. Let $x \in \bigcap_{i=1}^{n-1} m_i \setminus m_n$ and $y \in m_n \setminus IM(R)$. It is obvious that $x, y \in MG_2(R)$. Also, all elements of m_i in which $1 \leq i \leq n-1$, are adjacent to x and all elements of m_n are adjacent to y. Thus $\{x, y\}$ is a dominating set of $MG_2(R)$. Now, if $\gamma(MG_2(R)) = 1$, then there exists $z \in MG_2(R)$ such that it is adjacent to every vertex of $MG_2(R)$. Since R is not c-local, hence $z \in IM(R)$ by [11, Proposition 3.1], which is impossible. Therefore $\gamma(MG_2(R)) = 2$. In this case, since $\{x, y\}$ is an independent set, so $\gamma_i(MG_2(R)) = 2$. □

Theorem 15. *Let R be a semiring with two maximal m_1 and m_2. If $|m_i \setminus m_j| \geq 2$, for each $1 \leq i, j \leq 2$ and $i \neq j$, then $\gamma_t(MG_2(R)) = 4$.*

Proof. If $Co - Max(R) = \{m_1, m_2\}$, then by [11, Theorem 3.7], $MG_2(R)$ is union of two disjoint cliques. On the other hand, a clique such as K with $|K| \geq 2$, has total dominating set of cardinality two. Thus $\gamma_t(MG_2(R)) = 4$, because by our assumption, $|m_i \setminus m_j| \geq 2$, for each $1 \leq i \neq j \leq 2$. □

Theorem 16. *Let R be a c-semilocal semiring with $|Co-Max(R)| \geq 3$. Then $\gamma_t(MG_2(R)) = 2$.*

Proof. Assume that $m_1, ..., m_n$ are maximal co-ideals of R. Let $x \in \bigcap_{i=1}^{n-1} m_i \setminus m_n$ and $y \in \bigcap_{i=2}^{n} m_i \setminus m_1$. Clearly, $\{x, y\}$ is a dominating set of $MG_2(R)$. By our hypothesis, since $n \geq 3$, there exists a maximal co-ideal m_k such that $m_k \neq m_1, m_n$ and $x, y \in m_k$. So x and y are adjacent and this implies $\gamma_t(MG_2(R)) = 2$. □

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