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A NEW EFFICIENT METHOD FOR TIME-FRACTIONAL SINE-GORDON EQUATION WITH THE CAPUTO AND CAPUTO-FABRIZIO OPERATORS

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Abstract. In this work, a new efficient method called, Elzaki's fractional decomposition method (EFDM) has been used to give an approximate series solutions to time-fractional Sine-Gordon equation. The time-fractional derivatives are described in the Caputo and Caputo-Fabrizio sense. The EFDM is based on the combination of two different methods which are: the Elzaki transform method and the Adomian decomposition method. To demonstrate the accuracy and efficiency of the proposed method, a numerical example is provided. The obtained results indicate that the EFDM is simple and practical for solving the fractional partial differential equations which appear in various fields of applied sciences.

Key words: time-fractional Sine-Gordon equation, Caputo fractional derivative operator, Caputo-Fabrizio fractional derivative operator, Elzaki transform, Adomian decomposition method, approximate series solution.
MSC: Primary 34A08, 26A33, 34K28, 35C10.

1. Introduction

In recent years, the study of fractional partial differential equations (FPDEs) has drawn much attention by mathematical and physical researchers, due to their commonly appearance in different kinds of sciences, such as mathematics, physics, chemistry, optimal control theory, finance, biology, engineering, aerodynamics, electrodynamics and so forth [1–14].

Nowadays, many methods have been devloped to solve FPDEs, among them: backward Euler method (BEM) [15], B-spline collocation method (BSCM) [16–18], Adomian decomposition method (ADM) [19], homotopy analysis method (HAM) [20], fractional reduced differential transform method (FRDTM) [21],

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fractional residual power series method (FRPSM) [22], generalized Taylor fractional series method (GTFSM) [23].

In this article, we propose a new efficient method, called Elzaki's fractional decomposition method (EFDM) to solve the time-fractional Sine-Gordon equation in the form

$$D_t^{\alpha}u(x,t) - a^2 u_{xx}(x,t) - b\sin(u(x,t)) = 0, \tag{1}$$

with the initial conditions

$$u(x,0) = f(x), u_t(x,0) = g(x), \tag{2}$$

where D_t^{α} is the time-fractional derivative operator in the sense of Caputo of order $\alpha, 1 < \alpha \leq 2, a$ and b are constants. And in the form

$$\mathcal{D}_t^{(\mu)}u(x,t) - a^2 u_{xx}(x,t) - b\sin(u(x,t)) = 0, \tag{3}$$

with the initial conditions

$$u(x,0) = f(x), u_t(x,0) = g(x), \tag{4}$$

where $\mathcal{D}_t^{(\mu)} = \mathcal{D}_t^{(\alpha+1)}$ is the time-fractional derivative operator in the sense of Caputo-Fabrizio of order $\mu = \alpha + 1$, $0 < \alpha \leq 1$. When $\alpha = \mu = 2$, equations (1) and (3) reduces to the standard Sine-Gordon equation.

The sine-Gordon equation is one of the most crucial nonlinear hyperbolic partial differential equation in 1+1 dimensions involving the d'Alembert operator and the sine of the unknown function, where it was first discovered in the nineteenth century in the course of study of various problems of differential geometry. The Sine-Gordon equation appears in many physical applications in relativistic field theory, the propagation of fluxons in Josephson junctions (a junction between two superconductors), the motion of rigid pendula attached to a stretched wire, dislocations in crystals, mechanical transmission line and so on [24–26].

This article is structured as follows: In section 2, we give the basic definitions and properties of the fractional calculus and Elzaki transform. In section 3, we describe the Elzaki's fractional decomposition method (EFDM) to solve time-fractional Sine-Gordon equation in the sense of the Caputo and Caputo-Fabrizio (1) and (3). In section 4, we establish the convergence of numerical scheme based on EFDM. In section 5, we apply the EFDM to solve a numerical example to demonstrate the accuracy and effectiveness of this method. Section 6, is devoted to the conclusions on this work.

2. Definitions and properties

In this section, we give the basic definitions and properties of the fractional calculus and Elzaki transform.

Definition 1. *[27] Let $f \in L^1(\mathbb{R}^+)$. The Riemann-Liouville fractional integral of order $\alpha \geq 0$ is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, \tag{5}$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2. *[27] Let $f^{(n)} \in L^1(\mathbb{R}^+)$. The Caputo fractional derivative of order $\alpha \geq 0$ is defined as*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, \tag{6}$$

where $n-1 < \alpha \leq n$, $n = [\alpha]+1$ with $[\alpha]$ being the integer part of α.

In equation (5) if transformations happen as follows

$$(t-\xi)^{n-\alpha-1} \longrightarrow \exp\left[-\frac{\alpha\,(t-\xi)}{1-\alpha}\right] \text{ and } \frac{1}{\Gamma(n-\alpha)} \longrightarrow \frac{M(\alpha)}{1-\alpha},$$

the new definition of fractional operator is expressed by Caputo and Fabrizio.

Definition 3. *[28] Let $f \in H^1(\mathbb{R}^+)$, then the Caputo-Fabrizio fractional derivative of order $\alpha, 0 < \alpha \leq 1$ is defined as*

$$\begin{aligned} \mathcal{D}^{(\alpha)} f(t) &= \frac{M(\alpha)}{1-\alpha} \int_0^t f'(\xi) \exp\left[-\frac{\alpha\,(t-\xi)}{1-\alpha}\right] d\xi \\ &= \frac{M(\alpha)}{1-\alpha} \left(f'(t) * \exp\left[-\frac{\alpha t}{1-\alpha}\right]\right), \end{aligned} \tag{7}$$

where $*$ denotes the convolution and $M(\alpha)$ is a normalization function that satisfies $M(0) = M(1) = 1$.

From equation (7) it follows that if $f(t) = C$ is a constant, then $\mathcal{D}^{(\alpha)} C = 0$ as in the sense of Caputo [27].

If $f \notin H^1(\mathbb{R}^+)$, then its fractional derivative is redefined as [28]

$$\mathcal{D}^{(\alpha)} f(t) = \frac{\alpha M(\alpha)}{1-\alpha} \int_0^t (f(t) - f(\xi)) \exp\left[-\frac{\alpha\,(t-\xi)}{1-\alpha}\right] d\xi, t > 0.$$

For $n \geq 1$ and $0 < \alpha \leq 1$, the fractional derivative of order $(\alpha+n)$ is defined by

$$\mathcal{D}^{(\alpha+n)} f(t) = \mathcal{D}^{(\alpha)}(\mathcal{D}^{(n)} f(t)). \tag{8}$$

The above Caputo-Fabrizio fractional derivative was later modified by Jorge Losada and Juan José Nieto [29] as

$$\mathcal{D}^{(\alpha)}f(t)=\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int\limits_0^t f'(\xi)\exp\left[-\frac{\alpha\,(t-\xi)}{1-\alpha}\right]d\xi, t>0. \tag{9}$$

The fractional integral corresponding to the derivative in equation (9) was defined by Jorge Losada and Juan José Nieto in 2015, as follows.

Definition 4. *[29] Let* $0<\alpha\leq 1$*. The fractional integral of order* α *of* f *is defined by*

$$\mathcal{I}^{(\alpha)}f(t)=\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}f(t)+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int\limits_0^t f(\xi)d\xi, t>0. \tag{10}$$

From the definition in equation (10), the fractional integral of Caputo-Fabrizio type of a function f of order $0<\alpha\leq 1$ is an average between function f and its one order integral, i.e.,

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}+\frac{2\alpha}{(2-\alpha)M(\alpha)}=1.$$

Therefore,

$$M(\alpha)=\frac{2}{2-\alpha}, 0<\alpha\leq 1.$$

Due to this, Losada and Nieto remarked that Caputo-Fabrizio fractional derivative can redefined as

Definition 5. *[29] Let* $0<\alpha\leq 1$*. The fractional Caputo-Fabrizio derivative of order* α *of a function* f *is given by*

$$\mathcal{D}^{(\alpha)}f(t)=\frac{1}{1-\alpha}\int\limits_0^t f'(\xi)\exp\left[-\frac{\alpha\,(t-\xi)}{1-\alpha}\right]d\xi, t>0, \tag{11}$$

and its fractional integral is defined as

$$\mathcal{I}^{(\alpha)}f(t)=(1-\alpha)f(t)+\alpha\int\limits_0^t f(\xi)d\xi, t>0.$$

Definition 6. *[30] The Elzaki transform is defined over the set of functions*

$$A=\left\{f(t)/\exists M,k_1,k_2>0,|f(t)|<M\exp\left(\frac{|t|}{k_j}\right),\ \textit{if}\ t\in(-1)^j\times[0,\infty)\right\},$$

by the following integral

$$\mathcal{E}\left[f(t)\right]=T(v)=v\int_0^\infty f(t)\exp\left(-\frac{t}{v}\right)dt, t>0,$$

where v is the factor of variable t.

Property 1: The Elzaki transform is a linear operator. That is, if λ and μ are non-zero constants, then

$$\mathcal{E}\left[\lambda f(t) \pm \mu g(t)\right] = \lambda \mathcal{E}\left[f(t)\right] \pm \mu \mathcal{E}\left[g(t)\right].$$

Property 2: If $f^{(n)}(t)$ is the $n-$th derivative of the function $f(t) \in A$ with respect to "t" then its Elzaki transform is given by

$$\mathcal{E}\left[f^{(n)}(t)\right] = \frac{1}{v^n} T(v) - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0).$$

Property 3: Suppose $T(v)$ and $G(v)$ are the Elzaki transforms of $f(t)$ and $g(t)$, respectively, both defined in the set A. Then

$$\mathcal{E}\left[\left(f * g\right)(t)\right] = \frac{1}{v} T(v) G(v).$$

Property 4: Some special Elzaki transforms

$$\begin{aligned} \mathcal{E}(1) &= v^2, \\ \mathcal{E}(t) &= v^3, \\ \mathcal{E}\left(\frac{t^n}{n!}\right) &= v^{n+2}, n = 0, 1, 2, ... \\ \mathcal{E}\left(\sin(at)\right) &= \frac{av^3}{1 + a^2 v^2}. \end{aligned}$$

Property 5: The Elzaki transform of t^α is given by

$$\mathcal{E}\left(t^\alpha\right) = v^{\alpha+2} \Gamma\left(\alpha + 1\right).$$

Theorem 1. *[31] Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n - 1 < \alpha \leq n$ and $T(v)$ be the Elzaki transform of the function $f(t)$, then the Elzaki transform denoted by $T_\alpha(v)$ of the Caputo fractional derivative of $f(t)$ of order α, is given by*

$$\mathcal{E}\left[D^\alpha f(t)\right] = T_\alpha(v) = \frac{1}{v^\alpha} T(v) - \sum_{k=0}^{n-1} v^{2-\alpha+k} f^{(k)}(0). \tag{12}$$

Proof. See. Khalouta, A., & Kadem, A. (2020). Theorem 2.7. [31] □

Theorem 2. *The Elzaki transform of the Caputo-Fabrizio fractional derivative of the function $f(t)$ of order $\alpha + n$, where $0 < \alpha \leq 1$ and $n \in \mathbb{N} \cup \{0\}$, is given by*

$$\mathcal{E}\left[\mathcal{D}^{(\alpha+n)} f(t)\right] = \frac{1}{1 - \alpha(1 - v)} \left[\frac{1}{v^n} \mathcal{E}\left(f(t)\right) - \sum_{k=0}^{n} v^{2-n+k} f^{(k)}(0)\right]. \tag{13}$$

Proof. By the Definition of the 5 and relation (8), we have

$$\begin{aligned}
\mathcal{E}\left[\mathcal{D}^{(\alpha+n)}f(t)\right] &= \mathcal{E}\left[\mathcal{D}^{(\alpha)}(\mathcal{D}^{(n)}f(t))\right] \\
&= \frac{1}{1-\alpha}v\int_0^{+\infty}\exp\left(-\frac{t}{v}\right)\int_0^t f^{(n)}(\xi)\exp\left[-\frac{\alpha(t-\xi)}{1-\alpha}\right]d\xi \\
&= \frac{1}{1-\alpha}v\int_0^{+\infty}\exp\left(-\frac{t}{v}\right)\left(f^{(n)}(t)*\exp\left[-\frac{\alpha t}{1-\alpha}\right]\right) \\
&= \frac{1}{1-\alpha}\mathcal{E}\left(f^{(n)}(t)*\exp\left[-\frac{\alpha t}{1-\alpha}\right]\right).
\end{aligned}$$

Hence, from the Properties (2), (3) and (4) of the Elzaki transform, we have

$$\begin{aligned}
\mathcal{E}\left[\mathcal{D}^{(\alpha+n)}f(t)\right] &= \frac{1}{1-\alpha}\frac{1}{v}\mathcal{E}\left(f^{(n)}(t)\right)\mathcal{E}\left(\exp\left[-\frac{\alpha t}{1-\alpha}\right]\right) \\
&= \frac{v}{1-\alpha(1-v)}\left[\frac{1}{v^n}\mathcal{E}\left(f(t)\right)-\sum_{k=0}^{n-1}v^{2-n+k}f^{(k)}(0)\right] \\
&= \frac{v}{1-\alpha(1-v)}\left[\frac{1}{v^{n+1}}\mathcal{E}\left(f(t)\right)-\sum_{k=0}^{n}v^{1-n+k}f^{(k)}(0)\right] \\
&= \frac{1}{1-\alpha(1-v)}\left[\frac{1}{v^n}\mathcal{E}\left(f(t)\right)-\sum_{k=0}^{n}v^{2-n+k}f^{(k)}(0)\right].
\end{aligned}$$

The proof is complete. □

3. Description of the Elzaki's fractional decomposition method (EFDM)

In this section,we propose an algorithm based on EFDM to solve time-fractional Sine-Gordon equation with the Caputo and Caputo-Fabrizio operators.

Theorem 3. *Consider the following time-fractional Sine-Gordon equation with the Caputo and Caputo-Fabrizio operators (1) and (3). The EFDM gives the solutions of (1) and (3) in the form of infinite series as follows*

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t). \tag{14}$$

Proof. Consider the following time-fractional Sine-Gordon equation with the Caputo and Caputo-Fabrizio operators (1) and (3).

1) In the case of the Caputo operator.

First, we apply the Elzaki transform on both sides of (1) and using the Theorem 1, we obtain

$$\mathcal{E}\left[u(x,t)\right] = v^2 f(x) + v^3 g(x) + v^{\alpha}\mathcal{E}\left[a^2 u_{xx}(x,t) + b\sin(u(x,t))\right]. \tag{15}$$

Then, we take the inverse Elzaki transform on both sides of (15), we have

$$u(x,t) = f(x) + tg(x) + \mathcal{E}^{-1}\left(v^{\alpha}\mathcal{E}\left[a^2 u_{xx}(x,t) + b\sin(u(x,t))\right]\right). \tag{16}$$

Now, we represent the solution in an infinite series form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{17}$$

and the nonlinear terms $\sin(u(x,t))$ can be decomposed as

$$\sin(u(x,t)) = \sum_{n=0}^{\infty} A_n, \tag{18}$$

where A_n are the Adomian polynomials [32], can be calculated by the following formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i(x,t\right)\right]_{\lambda=0}. \tag{19}$$

The first few components of A_n are given by

$$\begin{aligned}
A_0 &= \sin(u_0(x,t)),\\
A_1 &= u_1(x,t)\cos(u_0(x,t)),\\
A_2 &= u_2(x,t)\cos(u_0(x,t)) - \frac{1}{2!}u_1^2(x,t)\sin(u_0(x,t)),\\
A_3 &= u_3(x,t)\cos(u_0(x,t)) - u_2(x,t)u_1(x,t)\sin(u_0(x,t)) - \frac{1}{3!}u_1^3(x,t)\cos(u_0(x,t)),\\
&\vdots
\end{aligned}$$

Substituting (17) and (18) in (16), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + tg(x) + \mathcal{E}^{-1}\left(v^{\alpha}\mathcal{E}\left[a^2 u_{nxx}(x,t) + b\sum_{n=0}^{\infty} A_n\right]\right). \tag{20}$$

By comparing both sides of (20), we get the following relation

$$\begin{aligned} u_0(x,t) &= f(x)+tg(x), \\ u_1(x,t) &= \mathcal{E}^{-1}\left(v^{\alpha}\mathcal{E}\left[a^2u_{0xx}(x,t)+bA_0\right]\right), \\ u_2(x,t) &= \mathcal{E}^{-1}\left(v^{\alpha}\mathcal{E}\left[a^2u_{1xx}(x,t)+bA_1\right]\right), \\ u_3(x,t) &= \mathcal{E}^{-1}\left(v^{\alpha}\mathcal{E}\left[a^2u_{2xx}(x,t)+bA_2\right]\right), \\ &\vdots \\ u_n(x,t) &= \mathcal{E}^{-1}\left(v^{\alpha}\mathcal{E}\left[a^2u_{(n-1)xx}(x,t)+bA_{n-1}\right]\right). \end{aligned} \tag{21}$$

Then, the solution of (1) is given in the form of infinite series as follows

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t).$$

2) In the case of the Caputo-Fabrizio operator.

First, we apply the Elzaki transform on both sides of (3) and using the Theorem 2, we obtain

$$\mathcal{E}\left[u(x,t)\right]=v^2f(x)+v^3g(x)+v\left(1-\alpha(1-v)\right)\mathcal{E}\left[a^2u_{xx}(x,t)+b\sin(u(x,t))\right]. \tag{22}$$

Then, we take the inverse Elzaki transform on both sides of (22), we have

$$u(x,t)=f(x)+tg(x)+\mathcal{E}^{-1}\left(v\left(1-\alpha(1-v)\right)\mathcal{E}\left[a^2u_{xx}(x,t)+b\sin(u(x,t))\right]\right). \tag{23}$$

Now, we represent the solution in an infinite series form

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \tag{24}$$

and the nonlinear terms $\sin(u(x,t))$ can be decomposed as

$$\sin(u(x,t))=\sum_{n=0}^{\infty}A_n. \tag{25}$$

Substituting (24) and (25) in (23), we obtain

$$\sum_{n=0}^{\infty}u_n(x,t)=f(x)+tg(x)+\mathcal{E}^{-1}\left(v\left(1-\alpha(1-v)\right)\mathcal{E}\left[a^2u_{nxx}(x,t)+b\sum_{n=0}^{\infty}A_n(t)\right]\right). \tag{26}$$

By comparing both sides of (26), we get the following relation

$$\begin{aligned}
u_0(x,t) &= f(x)+tg(x),\\
u_1(x,t) &= \mathcal{E}^{-1}\left(v\left(1-\alpha(1-v)\right)\mathcal{E}\left[a^2u_{0xx}(x,t)+bA_0\right]\right),\\
u_2(x,t) &= \mathcal{E}^{-1}\left(v\left(1-\alpha(1-v)\right)\mathcal{E}\left[a^2u_{1xx}(x,t)+bA_1\right]\right),\\
u_3(x,x) &= \mathcal{E}^{-1}\left(v\left(1-\alpha(1-v)\right)\mathcal{E}\left[a^2u_{2xx}(x,t)+bA_2\right]\right),\\
&\vdots\\
u_n(x,x) &= \mathcal{E}^{-1}\left(v\left(1-\alpha(1-v)\right)\mathcal{E}\left[a^2u_{(n-1)xx}(x,t)+bA_{n-1}\right]\right). \qquad (27)
\end{aligned}$$

Then, the solution of (3) is given in the form of infinite series as follows

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t).$$

The proof is complete. □

4. Convergence analysis

In this section, we establish the convergence of our numerical scheme based on EFDM.

Theorem 4. *Let $(\mathcal{B},\|.\|)$ be a Banach space, Then the series solutions of (1) and (3) converges to $S\in\mathcal{B}$, if there exists γ, $0<\gamma<1$ such that*

$$\|u_n\|\leq\gamma\|u_{n-1}\|,\forall n\in\mathbb{N}.$$

Proof. Define the sequences $\{S_n\}_{n\geq 0}$ of partial sums of the series given by the recursive relation (21) or (27) as

$$S_n(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+...+u_n(x,t),$$

and we need to show that $\{S_n\}_{n\geq 0}$ are a Cauchy sequences in Banach space $\mathcal{B}$. For this purpose, we consider

$$\|S_{n+1}-S_n\|\leq\|u_{n+1}\|\leq\gamma\|u_n\|\leq\gamma^2\|u_{n-1}\|\leq...\leq\gamma^{n+1}\|u_0\|. \qquad (28)$$

For every $n,m\in\mathbb{N}$, $n\geq m$, by using (28) and the triangle inequality successively, we have

$$\begin{aligned}
\|S_n-S_m\| &= \|S_{m+1}-S_m+S_{m+2}-S_{m+1}+...+S_n-S_{n-1}\|\\
&\leq \|S_{m+1}-S_m\|+\|S_{m+2}-S_{m+1}\|+...+\|S_n-S_{n-1}\|\\
&\leq \gamma^{m+1}\|u_0\|+\gamma^{m+2}\|u_0\|+...+\gamma^n\|u_0\|\\
&= \gamma^{m+1}\left(1+\gamma+...+\gamma^{n-m-1}\right)\|u_0\|\\
&\leq \gamma^{m+1}\left(\frac{1-\gamma^{n-m}}{1-\gamma}\right)\|u_0\|.
\end{aligned}$$

Since $0 < \gamma < 1$, so $1 - \gamma^{n-m} \leq 1$ then

$$\|S_n - S_m\| \leq \frac{\gamma^{m+1}}{1-\gamma} \|u_0\| .$$

Since u_0 is bounded, then

$$\lim_{n,m \longrightarrow \infty} \|S_n - S_m\| = 0.$$

Therefore, the sequences $\{S_n\}_{n\geq 0}$ are Cauchy sequences in the Banach space $\mathcal{B}$, so the series solution defined in (14) converges.

This completes the proof. □

Theorem 5. *The maximum absolute truncation error of the series solution (14) of (1) or (3) is estimated to be*

$$\sup_{(x,t)\in\Omega} \left| u_n(x,t) - \sum_{k=0}^{m} u_k(x,t) \right| \leq \frac{\gamma^{m+1}}{1-\gamma} \sup_{(x,t)\in\Omega} |u_0(x,t)| , \tag{29}$$

where the region $\Omega \subset \mathbb{R} \times \mathbb{R}^+$.

Proof. From the Theorem 4, we have

$$\|S_n - S_m\| \leq \frac{\gamma^{m+1}}{1-\gamma} \sup_{(x,t)\in\Omega} |u_0(x,t)| . \tag{30}$$

But we assume that $S_n = \sum\limits_{k=0}^{n} u_k(x,t)$ and since $n \to \infty$, we obtain $S_n \to u_n(x,t)$, so (30) can be rewritten as

$$\|u_n(x,t) - S_m\| = \left\| u_n(x,t) - \sum_{k=0}^{m} u_k(x,t) \right\| \leq \frac{\gamma^{m+1}}{1-\gamma} \sup_{(x,t)\in\Omega} |u_0(x,t)| .$$

So, the maximum absolute truncation error in the region $\Omega \subset \mathbb{R} \times \mathbb{R}^+$ is

$$\sup_{(x,t)\in\Omega} \left| u_n(x,t) - \sum_{k=0}^{m} u_k(x,t) \right| \leq \frac{\gamma^{m+1}}{1-\gamma} \sup_{(x,t)\in\Omega} |u_0(x,t)| .$$

This completes the proof. □

5. Application

In this section, we verify the accuracy and effectiveness of the EFDM described in Section 3, by solving an example of time-fractional Sine-Gordon equation with the Caputo and Caputo-Fabrizio operators.

Example 1. *Let us consider the time-fractional Sine-Gordon equation with the Caputo operator*

$$D_t^{\alpha} u(x,t) - u_{xx}(x,t) - \sin(u(x,t)) = 0, 1 < \alpha \leq 2, \tag{31}$$

with the initial conditions

$$u(x,0) = \frac{\pi}{2}, u_t(x,0) = 0. \tag{32}$$

Following the description of the FEDM presented in Section 3, gives

$$\begin{aligned} u_0(x,t) &= \frac{\pi}{2}, \\ u_1(x,t) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ u_2(x,t) &= 0, \\ u_3(x,t) &= -\frac{\Gamma(2\alpha+1)t^{3\alpha}}{2\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}, \\ &\vdots \end{aligned}$$

and so on.

Then, the series solution of (31) and (32), is given by

$$u(x,t) = \frac{\pi}{2} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)t^{3\alpha}}{2\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + ... \tag{33}$$

When $\alpha = 2$ in (33), we have

$$u(x,t) = \frac{\pi}{2} + \frac{t^2}{2} - \frac{t^6}{240} + ...$$

which is in complete agreement with the result by the ADM [33].

Now, we consider the time-fractional Sine-Gordon equation with the Caputo-Fabrizio operator

$$\mathcal{D}_t^{(\mu)} u(x,t) - u_{xx}(x,t) - \sin(u(x,t)) = 0, \mu = \alpha + 1, 0 < \alpha \leq 1, \tag{34}$$

with the initial conditions

$$u(x,0) = \frac{\pi}{2}, u_t(x,0) = 0. \tag{35}$$

Following the description of the FEDM presented in Section 3, gives

$$
\begin{aligned}
u_0(x,t) &= \frac{\pi}{2},\\
u_1(x,t) &= (1-\alpha)t + \alpha\frac{t^2}{2},\\
u_2(x,t) &= 0,\\
u_3(x,t) &= -\frac{(1-\alpha)^3}{6}t^3 - \frac{\alpha(1-\alpha)^2}{6}t^4 - \frac{\alpha^2(1-\alpha)}{20}t^5 - \frac{\alpha^3}{240}t^6,\\
&\vdots
\end{aligned}
$$

and so on.

Then, the series solution of (34) and (35), is given by

$$
\begin{aligned}
u(x,t) &= \frac{\pi}{2} + (1-\alpha)t + \frac{\alpha}{2}t^2 - \frac{(1-\alpha)^3}{6}t^3 - \frac{\alpha(1-\alpha)^2}{6}t^4\\
&\quad -\frac{\alpha^2(1-\alpha)}{20}t^5 - \frac{\alpha^3}{240}t^6 + ... \qquad (36)
\end{aligned}
$$

When $\alpha = 1$ in (36), we have

$$u(x,t) = \frac{\pi}{2} + \frac{t^2}{2} - \frac{t^6}{240} + ...$$

which is in complete agreement with the result by the ADM [33].

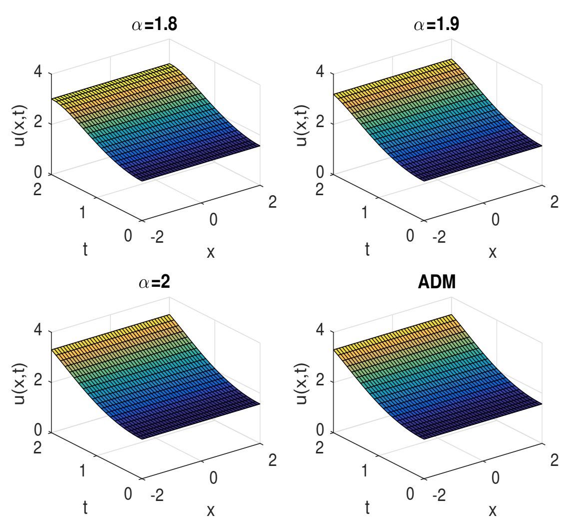

FIGURE 1. 3D plots of the 4−term approximate solutions for various values of α for equation (31) and ADM-solution [33].

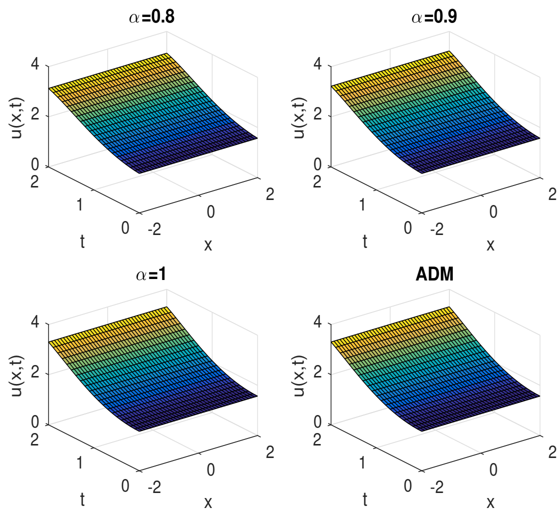

FIGURE 2. 3D plots of the 4−term approximate solutions for various values of α for equation (34) and ADM-solution [33].

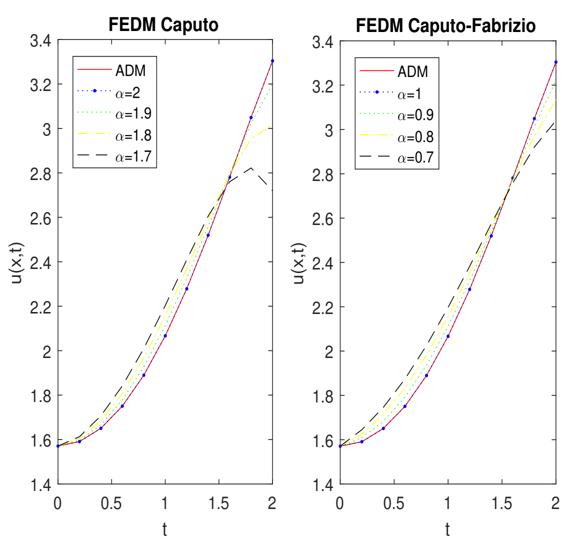

FIGURE 3. 2D plots of the 4−term approximate solutions for various values of α and ADM-solution [33].

Table 1. The numerical values of the 4-term approximate solutions for various values of α for equation (31) and ADM-solution [33].

t	$\alpha = 1.6$	$\alpha = 1.7$	$\alpha = 1.8$	$\alpha = 1.9$	$\alpha = 1$	ADM-solution
0.1	1.5884	1.5837	1.5802	1.5777	1.5758	1.5758
0.3	1.6726	1.6544	1.6391	1.6263	1.6158	1.6158
0.5	1.8007	1.7696	1.7419	1.7173	1.6957	1.6957
0.7	1.9621	1.9214	1.8832	1.8478	1.8153	1.8153
0.9	2.1484	2.1033	2.0586	2.0152	1.9736	1.9736

Table 2. The numerical values of the 4-term approximate solutions for various values of α of equation (34) and ADM-solution [33].

t	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	ADM-solution
0.1	1.6138	1.6043	1.5948	1.5853	1.5758	1.5758
0.3	1.7174	1.6921	1.6667	1.6413	1.6158	1.6158
0.5	1.8432	1.8068	1.7701	1.7330	1.6957	1.6957
0.7	1.9890	1.9468	1.9037	1.8598	1.8153	1.8153
0.9	2.1508	2.1090	2.0654	2.0202	1.9736	1.9736

6. Conclusion

In this article, a new efficient method called Elzaki's fractional decomposition method (EFDM) is presented to solve the time-fractional Sine-Gordon equation with the Caputo and Caputo-Fabrizio operators. The EFDM gives an infinite series which converges rapidly to the exact solution. A numerical example was used to demonstrate the accuracy and effectiveness of this method. Series solutions of the time-fractional Sine-Gordon equation are successfully obtained using the proposed method, and the results are compared with the results of the existing methods in literature. Graphics and numerical results show that this method is very efficient and practical to solve this type of equations. On the basis of the results obtained, we can conclude that the proposed method is a powerful mathematical tool to solve a wide range of fractional partial differential equations which appear in various fields of applied sciences.

In future works, we will try to propose new methods to study the solutions of nonlinear fractional partial differential equations in particular, nonhomogeneous space and time fractional Sine-Gordon equation.

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