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EXISTENCE OF A SOLUTION FOR INTEGRAL URYSOHN TYPE EQUATIONS SYSTEM VIA FIXED POINTS TECHNIQUE IN COMPLEX VALUED EXTENDED b-METRIC SPACES

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Abstract. In this article, we obtain fixed point results and we give a common fixed point theorem for Ćirić type operators on complex valued extended b-metric spaces which may satisfy very general assumptions. Our results extend and generalize the results of Kiran *et al.* [1], as well as some known results in the literature. Then an illustrative application to Urysohn type integral equations system is given.

Key words: Single-valued mappings, complex valued extended b-metric spaces, common fixed point, nonlinear integral equations.
MSC: 46T99, 47H10, 54H25.

1. Introduction

Fixed point theory is a powerful tool in topology, functional analysis, and nonlinear analysis, in order to obtain results in dynamic optimization, solve partial or random differential equations, obtain results in the theory of differential and integral equations, notably in existence in differential and integral equations or inclusions.

In recent years, the generalizations of the contraction principle have been obtained by either weakening the contractive properties of the mapping and, possibly, by simultaneously endowing the space with a sufficiently rich structure to compensate for the relaxation of the contractiveness, or by extending

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the structure of the space. Many notions of metric-type was introduced (complex metric space, b-metric space, extended b-metric space, complex b-metric space) and in such spaces several fixed point theorems are obtained. See for example, [2, 3, 4, 5, 6, 7, 8, 9].

In this paper we will consider generalized contractive mappings of Ćirić type in complex valued extended b-metric spaces and the main result consists on the existence of common fixed point.

The class of contractive mappings were first introduced in the context of metric spaces (see [10]). A self map $T : X \to X$ on a metric space (X, δ) is said to be a Ćirić mapping if, for some $\gamma \in (0, 1)$, it satisfies the following inequality, for all x, and y in X,

$$\delta(Tx, Ty) \preceq \gamma \max \left\{ \delta(x, y), \delta(x, Tx), \delta(y, Ty), \frac{1}{2} \left(\delta(x, Ty) + \delta(y, Tx) \right) \right\}.$$

The notion of b-metric was introduced by Bakhtin and Czerwik in [11, 12] in order to address problems formulated in spaces whose associated notion of metric requires a relaxed version of the triangle inequality. In these and other articles (see, for example, [4, 11, 13, 14, 15, 16]), fixed point theorems have been proved and applications have been considered.

Let us recall some notation and definitions that will play a key role in the derivation of our results.

Definition 1 ([11],[14])**.** *Let X be a non empty set and let $s \geq 1$ be a given real number. A functional $\delta : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied:*

(*i*) $\delta(x, y) = 0$ *if and only if* $x = y$,
(*ii*) $\delta(x, y) = \delta(y, x)$,
(*iii*) $\delta(x, z) \leq s[\delta(x, y) + \delta(y, z)]$,

for all $x, y, z \in X$. A pair (X, δ) is called a b-metric space.

Example 1 ([16])**.** *Let (X, δ) be a metric space and $\rho(x, y) = (\delta(x, y))^p$, where $p \geq 1$ is a real number. Then, (X, δ) is a b-metric space with $s = 2^{p-1}$.*

It is clear that a b-metric space becomes a metric space if we take $s = 1$. Hence, we conclude that the class of b-metric spaces is larger than that of metric spaces.

In [17], Karman and co-authors introduced the concept of extended b-metric space that generalizes the concept of b-metric space.

Definition 2. *Let X be a nonempty, and $\theta : X \times X \to [1, +\infty)$. A function $\delta_\theta : X \times X \to [0, +\infty)$ is an extended δ-metric if, for all $x, y, z \in X$, it satisfies:*

(*i*) $\delta_\theta(x, y) = 0$ *if and only if* $x = y$,
(*ii*) $\delta_\theta(x, y) = \delta_\theta(y, x)$,
(*iii*) $\delta_\theta(x, z) \leq \theta(x, z)[\delta_\theta(x, y) + \delta_\theta(y, z)]$.

The pair (X, δ_θ) *is called a extended b-metric space.*

Remark 1. *If* $\theta(x, y) = s$ *for* $s \geq 1$*, then* (X, b_θ) *satisfies the definition of a b-metric space.*

Example 2. *Let* $X = [0, +\infty)$, $\theta : X \times X \to [1, +\infty)$ *defined by* $\delta_\theta(x, y) = (x - y)^2$ *and* $\theta(x, y) = x + y + 2$ *Then,* (X, δ_θ) *is an extended b-metric space.*

Example 3. *Let* $X = C([a, b], \mathbb{R})$ *be the space of all continuous real valued functions defined on* $[a, b]$*. Let* $\delta_\theta(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|^2$*, and* $\theta : X \times X \to [1, +\infty)$ *defined by* $\theta(x, y) := |x(t)| + |y(t)| + 2$*, then* (X, δ_θ) *is a complete extended b-metric space.*

Definition 3. *Let* (X, δ_θ) *be an extended b-metric space.*

(i) *A sequence* $\{x_n\}_{n \in \mathbb{N}}$ *in* X *converges to* $x \in X$ *if, for every* $\varepsilon > 0$*, there exists* $N = N(\varepsilon) \in \mathbb{N}$ *such that*

$$b_\theta(x_n, x) < \varepsilon,$$

for all $n \geq N$*. Alternatively we may write* $\lim_{n \to \infty} x_n = x$*.*

(ii) *A sequence* $\{x_n\}_{n \in \mathbb{N}}$ *in* X *is Cauchy, if for every* $\epsilon > 0$ *there exists* $N = N(\varepsilon) \in \mathbb{N}$ *such that*

$$\delta_\theta(x_m, x_n) < \varepsilon,$$

for all $m, n \geq N$*.*

Definition 4. *An extended b-metric space* (X, δ_θ) *is complete if every Cauchy sequence in* X *is convergent.*

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$$z_1 \preceq z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \preceq z_2$ if one of the following conditions is satisfied:

$$(C_1)\ Re(z_1) = Re(z_2) \text{ and } Im(z_1) < Im(z_2)$$

$$(C_2)\ Re(z_1) < Re(z_2) \text{ and } Im(z_1) = Im(z_2)$$

$$(C_3)\ Re(z_1) < Re(z_2) \text{ and } Im(z_1) < Im(z_2)$$

$$(C_4)\ Re(z_1) = Re(z_2) \text{ and } Im(z_1) = Im(z_2)$$

In particular, we write $z_1 \preceq z_2$ if $z_1 \neq z_2$ and one of (C_1), (C_2) and (C_3) is satisfied and we write $z_1 \prec z_2$ if only (C_3) is satisfied.

Definition 5. [18] *Let X be a nonempty set. A mapping $\delta : X \times X \to \mathbb{C}$ is called a complex valued metric on X if, for all $x, y, z \in X$, the following conditions holds:*
(CM_1) $0 \preceq \delta(x,y)$ *and* $\delta(x,y) = 0$ *if and only if* $x = y$,
(CM_2) $\delta(x,y) = \delta(y,x)$,
(CM_3) $\delta(x,y) \preceq \delta(x,z) + \delta(z,y)$.
Then δ is called a complex valued metric on X and (X, δ) is called a complex valued metric space.

For some examples in this space we refer to [18, 13].

Definition 6. [4] *Let X be a nonempty set and let $s \geq 1$ be a given real number. A mapping $\delta : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if, for all $x, y, z \in X$, the following conditions holds:*
(CBM_1) $0 \preceq \delta(x,y)$ *and* $\delta(x,y) = 0$ *if and only if* $x = y$,
(CBM_2) $\delta(x,y) = \delta(y,x)$,
(CBM_3) $\delta(x,y) \preceq s[\delta(x,z) + \delta(z,y)]$.
Then δ is called a complex valued b-metric on X and (X, δ) is called a complex valued b-metric space.

Definition 7. [4] *Let X be a nonempty set and let $\theta : X \times X \to [1, +\infty)$ be a function. Then $\delta_\theta : X \times X \to \mathbb{C}$ is called a complex valued extended b-metric on X if, for all $x, y, z \in X$, the following conditions holds:*
(CBM_1) $0 \preceq \delta_\theta(x,y)$ *and* $\delta_\theta(x,y) = 0$ *if and only if* $x = y$,
(CBM_2) $\delta_\theta(x,y) = \delta_\theta(y,x)$,
(CBM_3) $\delta_\theta(x,z) \preceq \theta(x,z)[\delta_\theta(x,z) + \delta_\theta(z,y)]$.
Then δ_θ is called a complex valued extended b-metric on X and (X, δ_θ) is called a complex valued extended b-metric space.

In order to simplify the notations throughout this paper we will denote the complex valued extended b-metric space with CV extended b-metric space.

Next, let us recall some properties for complex valued extended b-metric spaces.

Example 4. [4] *Let X be a nonempty set and $\theta : X \times X \to [1, +\infty)$ be defined as*

$$\theta(x,y) = \frac{1 + x + y}{1 + x}$$

Further, let:
(i) $\delta_\theta(x,y) = \dfrac{i}{xy}$ *for all* $x, y \in (0,1]$,
(ii) $\delta_\theta(x,y) = 0 \leftrightarrow x = y \quad \forall x, y \in [0,1]$,
(iii) $\delta_\theta(x,0) = d_\theta(0,x) = \dfrac{i}{x}$ *for all* $x \in (0,1]$.
Then the pair (X, δ_θ) is known as complex valued extended b-metric space.

Definition 8. *Let (X, δ_θ) be a CV extended b-metric space and $S, T : X \to X$ be a multivalued mappings*
(i) A point $x \in X$ is called a fixed point of T if $x = Tx$,
(ii) A point $x \in X$ is called a common fixed point of S and T if $x = Tx$ and $x = Sx$.

In this paper, we prove first two new fixed point theorems in CV extended b-metric spaces under Ćirić type contractive condition for single-valued mappings, with the condition of continuity of the operator and with miss of it. Then we prove that we can obtain in similar conditions common fixed points theorems, taking into account the continuity of the operators. The fixed point results generalize and extend other known results from the related literature. The application section is dedicated to prove the existence of the solution for a integral Urysohn type equations system.

2. Fixed points results for Ćirić type operators

First of all let us give some lemmas for the case of CV extended b-metric spaces, which will be useful in proving our first fixed point result.

Lemma 1. *Every sequence $\{x_n)\}_{n\in\mathbb{N}}$ of elements from a CV extended b-metric space (X, δ_θ), satisfying the property*

$$\text{there exists } \gamma \in [0,1) \text{ such that } \delta_\theta(x_{n+1}, x_n) \preceq \gamma\delta_\theta(x_n, x_{n-1})$$

where for each $x_0 \in X$, $\gamma \lim_{n,m\to+\infty} \theta(x_n, x_m) < 1$. Then $\{x_n\}$ is a Cauchy sequence.

Lemma 2. *For every sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements from a CV extended b-metric space (X, δ_θ), the inequality*

$$\delta_\theta(x_0, x_k) \preceq \sum_{i=0}^{k-1} \delta_\theta(x_i, x_{i+1}) \prod_{l=0}^{i} \theta(x_l, x_k)$$

holds for every $k \in \mathbb{N}$.

Lemma 3. *Every sequence $\{x_n)\}_{n\in\mathbb{N}}$ of elements from a CV extended b-metric space (X, δ_θ), satisfying the property*

there exists $\gamma \in [0,1)$ such that $\delta_\theta(x_{n+1}, x_n) \preceq \gamma\delta_\theta(x_n, x_{n-1})$ for every $n \in \mathbb{N}$,

is a Cauchy sequence.

Remark 2. *The previous lemmas are given for the case of extended b-metric spaces by Q. Kiran and all in* [1]. *We remark that, using similar steps for the proof, we get same conclusions for the case of CV extended b-metric spaces.*

Let us give our first main fixed point result of this section.

Theorem 4. *Let (X, δ_θ) be a complete CV extended b-metric space such that the metric δ_θ is continuous, let $\theta : X \times X \to [1, +\infty)$ and $T : X \to X$ be a single valued continuous mapping such that:*

$$\delta_\theta(Tx, Ty) \preceq \lambda \max\{\delta_\theta(x, y), \delta_\theta(x, Tx), \delta_\theta(y, Ty), \frac{1}{2}[\delta_\theta(x, Ty) + \delta_\theta(y, Tx)]\} \tag{1}$$

for all $x, y \in X$, and $0 < \lambda < 1$, such that for each $x_0 \in X$ and any convergent sequence $\{x_n\}$, $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{\lambda}$, then T has a unique fixed point.

Proof. Fix $x_0 \in X$ let $x_1 = Tx_0$ and $x_2 = Tx_1$ Thus from (1) we have:

$$\delta_\theta(x_1, x_2) = \delta_\theta(Tx_0, Tx_1) \preceq \lambda \max\{\delta_\theta(x_0, x_1), \delta_\theta(x_0, Tx_0), \delta_\theta(x_1, Tx_1), \frac{1}{2}[\delta_\theta(x_0, Tx_1) + \delta_\theta(x_1, Tx_0)]\}.$$

This gives:

$$\delta_\theta(x_1, x_2) \preceq \lambda \max\{d_\theta(x_0, x_1), \delta_\theta(x_0, Tx_0), \delta_\theta(x_1, Tx_1), \frac{1}{2}[\delta_\theta(x_0, Tx_1)+\delta_\theta(x_1, Tx_0)]\}.$$

Which means:

$$\delta_\theta(x_1, x_2) \preceq \lambda \max\{\delta_\theta(x_0, x_1), \delta_\theta(x_0, x_1), \delta_\theta(x_1, x_2), \frac{1}{2}[\delta_\theta(x_0, x_2) + \delta_\theta(x_1, x_1)]\}.$$

Then we have:

Case I.

If $\max\{\delta_\theta(x_0, x_1), \delta_\theta(x_0, x_1), \delta_\theta(x_1, x_2), \frac{1}{2}[\delta_\theta(x_0, x_2) + \delta_\theta(x_1, x_1)]\} = \delta_\theta(x_1, x_2)$, we get:

$$|\delta_\theta(x_1, x_2)| \preceq \lambda|\delta_\theta(x_1, x_2)|.$$

This leads to $\lambda \geq 1$. Contradiction.

Case II.

If $\max\{\delta_\theta(x_0, x_1), \delta_\theta(x_0, x_1), \delta_\theta(x_1, x_2), \frac{1}{2}[\delta_\theta(x_0, x_2) + \delta_\theta(x_1, x_1)]\} = \delta_\theta(x_0, x_1)$ we have:

$$|\delta_\theta(x_1, x_2)| \leq \lambda|\delta_\theta(x_0, x_1)|.$$

Inductively, we can find a sequence $\{x_n\} \in X$ such that:

$$|\delta_\theta(x_n, x_{n+1})| \leq \lambda|\delta_\theta(x_{n-1}, x_n)|.$$

Thus, the conditions of Lemma 3 hold for all terms of the sequence $\{x_n\}_{n\in\mathbb{N}}$ and hence, the generated sequence is Cauchy.

By completeness of X, there exists some $x^* \in X$ such that

$$\lim_{p\to\infty} x_n = x^*.$$

Case III.

If $\max\{\delta_\theta(x_0,x_1),\delta_\theta(x_0,x_1),\delta_\theta(x_1,x_2),\frac{1}{2}[\delta_\theta(x_0,x_2)+\delta_\theta(x_1,x_1)]\} = \frac{1}{2}\delta_\theta(x_0,x_2)$ we have:

$$\delta_\theta(x_1,x_2) \preceq \frac{\lambda c_2}{2}\delta_\theta(x_0,x_2),$$

$$\delta_\theta(x_1,x_2) \preceq \frac{\lambda c_2\theta(x_0,x_2)}{2}[\delta_\theta(x_0,x_1)+\delta_\theta(x_1,x_2)],$$

$$\delta_\theta(x_1,x_2) \preceq \frac{\lambda c_2\theta(x_0,x_2)}{2-\lambda c_2\theta(x_0,x_2)}\delta_\theta(x_0,x_1).$$

Proceeding further we get:

$$\delta_\theta(x_{n+1},x_{n+2}) \preceq \frac{\gamma\theta(x_n,x_{n+2})}{2-\gamma\theta(x_n,x_{n+2})}\delta_\theta(x_n,x_{n+1}). \tag{2}$$

it follows that

$$\delta_\theta(x_{n+1},x_{n+2}) \preceq \eta\delta_\theta(x_n,x_{n+1}).$$

Now, we show that there exists $N_\eta \in \mathbb{N}$ such that, for all $n > N_\eta$, $\eta = \eta(N_\eta) < 1$.

Since $\gamma \lim_{n,m\to+\infty}\theta(x_n,x_m) < 1$, we have that: $2-\gamma\lim_{n,m\to\infty}\theta(x_n,x_m) > 1$. From this, it follows that:

$$\gamma\lim_{n,m\to+\infty}\theta(x_n,x_m) \leq 2-\gamma\lim_{n,m\to+\infty}\theta(x_n,x_m),$$

and, thus, $\eta < 1$.

From Lemma 3 we get that $\{x_n\}$ is Cauchy. We conclude that, for all the three cases, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $\delta_\theta(x_n,x^*) \to 0$ as $n \to \infty$. From the continuity of T it follows $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$, and, from the uniqueness of the limit, we conclude that $x^* = Tx^*$.

It remains to show the uniqueness of x^*. Assume that $y^* \in X$ is another fixed point of T. Then

$$\begin{aligned}
&\delta_\theta(x^*,y^*) = \delta_\theta(Tx^*,Ty^*)\\
&\preceq \gamma\max\{\delta_\theta(x^*,y^*),\delta_\theta(x^*,Tx^*),\delta_\theta(y^*,Ty^*),\frac{1}{2}(\delta_\theta(x^*,Ty^*)+\delta_\theta(y^*,Tx^*))\}\\
&\preceq \gamma\max\{\delta_\theta(x^*,y^*),\delta_\theta(x^*,x^*),\delta_\theta(y^*,y^*),\frac{1}{2}(\delta_\theta(x^*,y^*)+\delta_\theta(y^*,x^*))\}\\
&= \gamma\delta_\theta(x^*,y^*).
\end{aligned}$$

This contradiction leads us to say that T has a unique fixed point.

□

The following result follow the conditions of the previous one but we lift the continuity assumption as follow.

Theorem 5. *Let (X, δ_θ) be a complete CV extended b-metric space such that the metric δ_θ is continuous, let $\theta : X \times X \to [1, +\infty)$ and $T : X \to X$ be a single valued map such that:*

$$\delta_\theta(Tx, Ty) \preceq \lambda \max\{\delta_\theta(x, y), \delta_\theta(x, Tx), \delta_\theta(y, Ty), \frac{1}{2}[\delta_\theta(x, Ty) + \delta_\theta(y, Tx)]\}, \tag{3}$$

for all $x, y \in X$, and $0 < \lambda < 1$, such that for each $x_0 \in X$ and any convergent sequence $\{x_n\}$, $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{\lambda}$, then T has a unique fixed point.

Proof. Let the Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ be constructed as in Theorem (5). Since X is complete, there exists $x^* \in X$ such that $\delta_\theta(x_n, x^*) \to 0$ as $n \to \infty$.

Since, now, T is not continuous, let us assume that $\delta_\theta(x^*, Tx^*) = r > 0$. Then, we may write the following estimates

$$\begin{aligned}
r = \delta_\theta(x^*, Tx^*) &\preceq \theta(x^*, Tx^*)\left(\delta_\theta(x^*, x_{n+1}) + \delta_\theta(x_{n+1}, Tx^*)\right)\\
&\preceq \theta(x^*, Tx^*)\delta_\theta(x^*, x_{n+1}) + \theta(x^*, Tx^*)\delta_\theta(Tx_n, Tx^*)\\
&\preceq \theta(x^*, Tx^*)\delta_\theta(x^*, x_{n+1}) + \theta(x^*, Tx^*)\gamma \max\{\delta_\theta(x_n, x^*), \delta_\theta(x_n, Tx_n),\\
&\qquad \delta_\theta(x^*, Tx^*), \frac{1}{2}(\delta_\theta(x_n, Tx^*) + T_\theta(x^*, Tx_n))\}\\
&\preceq \theta(x^*, Tx^*)\delta_\theta(x^*, x_{n+1}) + \theta(x^*, Tx^*)\gamma \max\{\delta_\theta(x_n, x^*), \delta_\theta(x_n, x_{n+1}),\\
&\qquad \delta_\theta(x^*, Tx^*), \frac{1}{2}(\delta_\theta(x_n, Sx^*) + \delta_\theta(x^*, x_{n+1}))\}\\
&\preceq \theta(x^*, Tx^*)\delta_\theta(x^*, x_{n+1}) + \gamma\theta(x^*, Tx^*)\delta_\theta(x^*, Tx^*)\\
&\preceq \theta(x^*, Tx^*)\delta_\theta(x^*, x_{n+1}) + \theta(x^*, Tx^*)\gamma r.
\end{aligned}$$

The inequality before the last one follows by showing that $\delta_\theta(x^*, Tx^*)$ is the maximal element in the left hand side with a simple contradiction argument, involving choosing n sufficiently large, forcing all other terms being smaller than r.

From the last inequality, we obtain:

$$r \leq \theta(x^*, Tx^*)(\delta_\theta(x^*, x_{n+1}) + \gamma r).$$

Then, since this inequality has to hold for all situations, by considering $\lim_{n\to\infty} \delta_\theta(x^*, x_{n+1}) = 0$ and $\theta(x^*, Tx^*) = 1$, it follows that $\gamma \geq 1$, and hence, a contradiction. Then $x^* = Tx^*$.

In the same way, we get $x^* = Tx^*$. Hence, x^* is a fixed point for the pair T. For the uniqueness of the fixed point x^*, we use similar steps as in the proof of Theorem (5). □

Further, we discuss the existence and uniqueness of common fixed point for the pair (S, T) in similar conditions of the Theorem 5. Similar with the previous results, we assume that the operators T and S are continuous, while, in a second result, we lift this assumption.

Theorem 6. *Let (X, δ_θ) be a complete CV extended b-metric space such that the metric δ_θ is continuous, and let $S, T : X \to X$ be two continuous self operators such that:*

$$\delta_\theta(Sx, Ty) \preceq \gamma \max\left\{\delta_\theta(x, y), \delta_\theta(x, Sx), \delta_\theta(y, Ty), \frac{1}{2}(\delta_\theta(x, Ty) + \delta_\theta(y, Sx))\right\}, \tag{4}$$

for all $x, y \in X$, where $0 < \gamma < 1$, is such that, for each $x_0 \in X$, and any convergent sequence $\{x_n\}$, $\gamma \lim_{n,m\to+\infty} \theta(x_n, x_m) < 1$.

Then, the pair (S, T) has a unique common fixed point.

Proof. Let x_0 be arbitrary point in X, and define a sequence $\{x_n\}$ as follows

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1},\ n = 0, 1, 2, ... \tag{5}$$

Then, by (11) and (5), we get:

$$\begin{aligned}
&\delta_\theta(x_{2n+1}, x_{2n+2}) = \delta_\theta(Sx_{2n}, Tx_{2n+1})\\
&\preceq \gamma \max\{\delta_\theta(x_{2n}, x_{2n+1}), \delta_\theta(x_{2n}, Sx_{2n}), \delta_\theta(x_{2n+1}, Tx_{2n+1})\\
&\qquad \frac{1}{2}(\delta_\theta(x_{2n}, Tx_{2n+1}) + \delta_\theta(x_{2n+1}, Sx_{2n}))\}\\
&\preceq \gamma \max\{\delta_\theta(x_{2n}, x_{2n+1}), \delta_\theta(x_{2n}, x_{2n+1}), \delta_\theta(x_{2n+1}, x_{2n+2})\\
&\qquad \frac{1}{2}(d_\theta(x_{2n}, x_{2n+2}) + \delta_\theta(x_{2n+1}, x_{2n+1}))\}\\
&= \gamma \max\{\delta_\theta(x_{2n}, x_{2n+1}), \delta_\theta(x_{2n+1}, x_{2n+2} + \frac{1}{2}\delta_\theta(x_{2n}, x_{2n+2})\}.
\end{aligned}$$

Case I.
If $\max\{\delta_\theta(x_{2n}, x_{2n+1}), \delta_\theta(x_{2n+1}, x_{2n+2}), \frac{1}{2}\delta_\theta(x_{2n}, x_{2n+2})\} = \delta_\theta(x_{2n+1}, x_{2n+2})$, then we have

$$\delta_\theta(x_{2n+1}, x_{2n+2}) \preceq \gamma\delta_\theta(x_{2n+1}, x_{2n+2}).$$

This leads to $\gamma \geq 1$. Contradiction.

Case II.
If $\max\{\delta_\theta(x_{2n}, x_{2n+1}), \delta_\theta(x_{2n+1}, x_{2n+2}), \frac{1}{2}\delta_\theta(x_{2n}, x_{2n+2})\} = \delta_\theta(x_{2n}, x_{2n+1})$, then we have:

$$\delta_\theta(x_{2n+1}, x_{2n+2}) \preceq \gamma\delta_\theta(x_{2n}, x_{2n+1}). \tag{6}$$

In a similar way, we obtain:

$$\delta_\theta(x_{2n+2}, x_{2n+3}) \preceq \gamma\delta_\theta(x_{2n+1}, x_{2n+2}). \tag{7}$$

Thus, the conditions of Lemma 3 hold for all terms of the sequence $\{x_n\}_{n\in\mathbb{N}}$ and hence, the generated sequence is Cauchy.

Case III.
If $\max\{\delta_\theta(x_{2n},x_{2n+1}),\delta_\theta(x_{2n+1},x_{2n+2}),\frac{1}{2}\delta_\theta(x_{2n},x_{2n+2})\}=\frac{1}{2}\delta_\theta(x_{2n},x_{2n+2})$, then we have:

$$\frac{1}{2}\delta_\theta(x_{2n},x_{2n+2})=\frac{1}{2}\delta_\theta(x_{2n},x_{2n+2})\leq\frac{1}{2}\theta(x_{2n},x_{2n+2})\delta_\theta(x_{2n},x_{2n+1}). \quad (8)$$

In this case, we obtain:

$$\delta_\theta(x_{2n+1},x_{2n+2})\preceq\frac{\theta(x_{2n},x_{2n+2})\gamma}{2}(\delta_\theta(x_{2n},x_{2n+1})+\delta_\theta(x_{2n+1},x_{2n+2})).$$

Hence

$$(1-\frac{\theta(x_{2n},x_{2n+2})\gamma}{2})\delta_\theta(x_{2n+1},x_{2n+2})\preceq\frac{\gamma\theta(x_{2n},x_{2n+2})}{2}\delta_\theta(x_{2n},x_{2n+1}).$$

Thus, we conclude that:

$$\delta_\theta(x_{2n+1},x_{2n+2})\preceq\frac{\gamma\theta(x_{2n},x_{2n+2})}{2-\gamma\theta(x_{2n},x_{2n+2})}\delta_\theta(x_{2n},x_{2n+1}). \quad (9)$$

By considering an additional iteration, we find that:

$$\delta_\theta(x_{2n+2},x_{2n+3})\preceq\frac{\gamma\theta(x_{2n+1},x_{2n+3})}{2-\gamma\theta(x_{2n+1},x_{2n+3})}\delta_\theta(x_{2n+1},x_{2n+2}). \quad (10)$$

Thus, from (9), and (10) it follows that $\delta_\theta(x_{n+1},x_{n+2})\preceq\eta(n)\delta_\theta(x_n,x_{n+1})$ where $\eta(n)$ is defined by $\eta(n):=\dfrac{\gamma\theta(x_n,x_{n+2})}{2-\gamma\theta(x_n,x_{n+2})}$. Now, we show that, there exists $N_\eta\in\mathbb{N}$ such that $\forall\ n>N_\eta$, $\eta=\eta(N_\eta)<1$. Since $\gamma\lim_{n,m\to+\infty}\theta(x_n,x_m)<1$, we have that: $2-\gamma\lim_{n,m\to\infty}\theta(x_n,x_m)>1$. From this, it follows that:

$$\gamma\lim_{n,m\to+\infty}\theta(x_n,x_m)\preceq 2-\gamma\lim_{n,m\to+\infty}\theta(x_n,x_m),$$

and, thus, $\eta<1$. By applying Lemma 3, we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

We conclude that, for all the three cases, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, there exists $x^*\in X$ such that $\delta_\theta(x_n,x^*)\to 0$ as $n\to\infty$. Then, it follows that $\delta_\theta(x_{2n},x^*)\to 0$ as $n\to\infty$.

From the continuity of S, we have that $x_{2n+1}=Sx_{2n}\to Sx^*$ as $n\to\infty$, and, from the uniqueness of the limit, we conclude that $x^*=Sx^*$.

At the same time, we have $\delta_\theta(x_{2n+1},x^*)\to 0$ as $n\to\infty$. From the continuity of T, it follows $x_{2n+2}=Tx_{2n+1}\to Tx^*$ as $n\to\infty$, and from the uniqueness of the limit, we obtain $x^*=Tx^*$.

Thus, we conclude that x^* is a common fixed point of the pair (S,T).

It remains to show the uniqueness of x^*. Assume that $y^* \in X$ is another common fixed point for the pair (S,T). Then,

$$\delta_\theta(x^*,y^*) = \delta_\theta(Sx^*,Ty^*)$$
$$\preceq \gamma \max\{\delta_\theta(x^*,y^*), \delta_\theta(x^*,Sx^*), \delta_\theta(y^*,Ty^*), \frac{1}{2}(\delta_\theta(x^*,Ty^*) + \delta_\theta(y^*,Sx^*))\}$$
$$\preceq \gamma \max\{\delta_\theta(x^*,y^*), \delta_\theta(x^*,x^*), \delta_\theta(y^*,y^*), \frac{1}{2}(\delta_\theta(x^*,y^*) + \delta_\theta(y^*,x^*))\}$$
$$= \gamma\delta_\theta(x^*,y^*).$$

This implies that $x^* = y^*$ which complete the proof. □

Using Theorem 6, we give another fixed point result, more general, by omitting the continuity condition of the mappings S and T.

Theorem 7. *Let (X, d_θ) be a complete CV extended b-metric space such that δ_θ is continuous and let $S,T : X \to X$ be two self operators such that:*

$$\delta_\theta(Sx,Ty) \preceq \gamma \max\left\{\delta_\theta(x,y), \delta_\theta(x,Sx), \delta_\theta(y,Ty), \frac{1}{2}(\delta_\theta(x,Ty) + \delta_\theta(y,Sx))\right\}, \tag{11}$$

for all $x, y \in X$, where $0 < \gamma < 1$, is such that, for each $x_0 \in X$, and any convergent sequence $\{x_n\}$, $\gamma \lim_{n,m\to+\infty} \theta(x_n,x_m) < 1$.

Then, the pair (S,T) has a unique common fixed point.

Proof. Let the Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ be constructed as in Theorem 6. Since X is complete, there exists $x^* \in X$ such that $d_\theta(x_n,x^*) \to 0$ as $n \to \infty$.

Since S and T are not continuous, let us assume that $\delta_\theta(x^*,Sx^*) = r > 0$. Then, we may write the following estimates

$$r = \delta_\theta(x^*,Sx^*) \preceq \theta(x^*,Sx^*)\left(\delta_\theta(x^*,x_{2n+2}) + \delta_\theta(x_{2n+2},Sx^*)\right)$$
$$\preceq \theta(x^*,Sx^*)\delta_\theta(x^*,x_{2n+2}) + \theta(x^*,Sx^*)\delta_\theta(Tx_{2n+1},Sx^*)$$
$$\preceq \theta(x^*,Sx^*)\delta_\theta(x^*,x_{2n+2}) + \theta(x^*,Sx^*)\gamma \max\{\delta_\theta(x_{2n+1},x^*), \delta_\theta(x_{2n+1},Tx_{2n+1}),$$
$$\delta_\theta(x^*,Sx^*), \frac{1}{2}(\delta_\theta(x_{2n+1},Sx^*) + \delta_\theta(x^*,Tx_{2n+1}))\}$$
$$\preceq \theta(x^*,Sx^*)\delta_\theta(x^*,x_{2n+2}) + \theta(x^*,Sx^*)\gamma \max\{\delta_\theta(x_{2n+1},x^*), \delta_\theta(x_{2n+1},x_{2n+2}),$$
$$\delta_\theta(x^*,Sx^*), \frac{1}{2}(\delta_\theta(x_{2n+1},Sx^*) + \delta_\theta(x^*,x_{2n+2}))\}$$
$$\preceq \theta(x^*,Sx^*)\delta_\theta(x^*,x_{2n+2}) + \gamma\theta(x^*,Sx^*)\delta_\theta(x^*,Sx^*)$$
$$\preceq \theta(x^*,Sx^*)\delta_\theta(x^*,x_{n+2}) + \theta(x^*,Sx^*)\gamma r.$$

The inequality before the last one follows by showing that $\delta_\theta(x^*,Sx^*)$ is the maximal element in the " max " of the left hand side with a simple contradiction argument involving choosing k sufficiently large forcing all other terms being smaller than r.

From the last inequality, we obtain:

$$r \leq \theta(x^*, Sx^*)(|\delta_\theta(x^*, x_{2n+2})| + \gamma r).$$

Then, since this inequality has to hold for all situations, by considering $\lim_{k\to\infty} \delta_\theta(x^*, x_{2n+2}) = 0$ and $\theta(x^*, Sx^*) = 1$, it follows that $\gamma \geq 1$, and hence, a contradiction. Then, $x^* = Sx^*$.

In the same way, we get $x^* = Tx^*$. Hence, x^* is a common fixed point for the pair (S, T). For the uniqueness of the common fixed point x^*, we use similar steps as in the proof of Theorem 6. □

3. Existence of a solution for a system Urysohn integral type equations

Let us consider the following system of Urysohn type integral equations.

$$\begin{cases} x(t) = f(t) + \int_a^b F_1(t, s, x(s))ds \\ y(t) = f(t) + \int_a^b F_2(t, s, y(s))ds \end{cases} . \tag{12}$$

where,

(i) $x(t)$ and $y(t)$ are unknown variables for each $t \in [a, b]$, $a > 0$,

(ii) $f(t)$ is the deterministic free term defined for $t \in [a, b]$,

(iii) $F_1(t, s)$ and $F_2(t, s)$ are deterministic kernels defined for $t, s \in [a, b]$.

Let $X = (C[a, b], \mathbb{R}^n), \beta > 1$ and $d_\theta : X \times X \to \mathbb{R}^n$ defined by

$$d_\theta(x, y) = \|x(t) - y(t)\|_\infty = \max_{t\in[a,b]} |x(t) - y(t)|^2 \sqrt{1 - \beta^3} e^{i \cos^{-1} \beta},$$

for all $x, y \in X$, $i = \sqrt{-1} \in \mathbb{C}$ where $\theta : X \times X \to [1, +\infty)$ is defined as $\theta(x, y) = |x(t)| + |y(t)| + 1$.

Obviously $(C[a, b], \mathbb{R}^n, \|\cdot\|_\infty)$ is a complete complex valued extended b-metric space.

Further, we will have a Urysohn integral type system as (12) under the following conditions:

(u_1) $f(t) \in X$,

(u_2) $F_1, F_2 : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions satisfying

$$|F_1(t, s, u(s)) - F_2(t, s, v(s))| \preceq \frac{1}{\sqrt{(b-a)e^\beta}} M(u, v),$$

where,

$$M(u, v) = \max\{d(u, v), d(u, Tu), d(v, Tv), \frac{1}{2}(d(u, Tv) + d(v, Tu))\}.$$

This section is dedicated to study of existence of a unique solution to the system (12) applying Theorem 6. Further, we give the main theorem of this section.

Theorem 8. *$(C[a,b],\mathbb{R}^n,\|.\|_\infty)$ be a complete complex valued extended b-metric space, then the system (12) under the conditions (u_1) and (u_2) has a unique common solution.*

Proof. For $x,y\in(C[a,b],\mathbb{R}^n)$ and $t\in[a,b]$, we define the continuous mappings $S,T:X\to X$ by

$$Sx(t)=f(t)+\int_a^b F_1(t,s,x(s))ds,$$
$$Ty(t)=f(t)+\int_a^b F_2(t,s,y(s))ds.$$

Then we have

$$\begin{aligned}
|Sx(t)-Ty(t)|^2 &= \int_a^b |F_1(t,s,x(s))-F_2(t,s,y(s))|^2\,ds\\
&\preceq \int_a^b \frac{1}{(b-a)e^\beta}\,|M(x,y)|^2\,ds\\
&= \frac{1}{(b-a)e^\beta}\int_a^b \frac{e^{-i\cos^{-1}\beta}}{\sqrt{1-\beta^3}}\,|M(x,y)|^2\sqrt{1-\beta^3}e^{i\cos^{-1}\beta}ds\\
&\preceq \frac{1}{(b-a)e^\beta}\frac{e^{-i\cos^{-1}\beta}}{\sqrt{1-\beta^3}}\,\|M(x,y)\|_\infty\int_a^b ds\\
&= \frac{1}{e^\beta}\frac{e^{-i\cos^{-1}\beta}}{\sqrt{1-\beta^3}}\,\|M(x,y)\|_\infty .
\end{aligned}$$

Then we get

$$|Sx(t)-Ty(t)|^2\sqrt{1-\beta^3}e^{i\cos^{-1}\beta}\preceq\frac{1}{e^\beta}\,\|M(x,y)\|_\infty ,$$

or, equivalently

$$\|Sx(t)-Ty(t)\|_\infty\preceq\frac{1}{e^\beta}\,\|M(x,y)\|_\infty .$$

Then we have

$$d(Sx,Ty)\preceq\gamma M(x,y).$$

For $0<\gamma=\frac{1}{e^\beta}<1$ and $\lim\limits_{n,m\to\infty}\theta(x_n,x_m)=1<e^\beta$ we have accomplished all the conditions of Theorem 6. Therefore the system (12) has a unique common solution on X. □

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