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NOVEL FRACTIONAL DIFFERENTIAL OPERATOR AND ITS APPLICATION IN FLUID DYNAMICS

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Abstract. Theoretical analysis of unsteady incompressible viscous fluid has been carried with constant proportional Caputo fractional derivative namely constant proportional Caputo type with singular kernel. The modeled considered in this paper is the fundament problem of fluid dynamics. The resulting governing equations are modeled with hybrid fractional operator of singular kernel and its solution obtained by using Laplace transform method and expressed in terms of series. Some graphs are captured for fractional parameter α for large and small time and found that velocity shows dual trend for small and large values of time for different values of fractional parameter α. Further, compared the present results with the results obtained with new fractional operators and found that constant proportional Caputo type operator portrait better velocity decay. Moreover, for increasing time, momentum boundary layer thickness increases while for grater values of fractional parameter it reduces.

Key words: Constant proportional Caputo, Analytical solutions, Couette flow.
MSC: 76A05.

1. Introduction

The local conformable derivative formerly introduced in (2014) by Khalil et al. [1]. Unfortunately there were draw backs like some properties of this operator [2]. Later on it attracted many researchers who have been intensely studied with its possible applications in different branches of science for examples [3-11]. In [12], new class of conformable derivatives and its properties were introduced in (2015). These operators have applications in control theory. Since there are many fractional operators in the existing literature and some which have been extensively used by researchers, are Caputo, CF, and

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ABC during the last four to five years. Fractional operators are usually used to see the history or memory of the functions based on different kernels like, power law kernel, exponential kernel, Mittage-Leffler kernel etc. These kernels are usually appear in many physical processes like decay phenomena. Many researchers successfully found many applications in fluid dynamics, heat transfer, nanotechnology and many more with power law (C), exponential law (CF) and Mittage-Leffler law (ABC) fractional derivatives and can be seen in the references for instance [14-29].

In (2020) a new kind of fractional operator with power law recently suggested by Baleanu et al. [30]. It is combined with two types of fractional operators called proportional and Caputo in a single operator which also known as constant proportional Caputo type fractional derivative. The constant proportional Caputo type derivative is defined as follows:
The hybrid fractional operator by combining proportional and Caupto definitions is defined as [30].

$$ {}^{CPC}C_t^{\alpha} h(t) = [\Gamma(1-\alpha)]^{-1} \int_0^t \frac{\Big(L_1(\alpha)h(\tau) + L_o(\alpha)\acute{h}(\tau)\Big)}{(t-\tau)^{\alpha}} d\tau, \quad 0 < \alpha < 1. \quad (1) $$

This operator has application in fractional partial differential equation and control theory. In this work, solution of fractional PDE's expressed in terms of Mittage-Leffler function with limiting cases as follows The Laplace transform of constant proportional Caputo is given as [30]

$$ \text{Ł}\left\{{}^{CPC}C_t^{\alpha} h(t)\right\} = \left\{\frac{L_1}{s} + L_o\right\} s^{\alpha} h(s) - L_o s^{\alpha-1} h(0). \quad (2) $$

For the moment, there is no single result in the literature discussed with this new operator. Therefore, our interest here to apply the most recent fractional operator namely constant proportional Caputo fractional operator to the most fundamental problem of fluid dynamics. Solution obtained via Laplace transform in terms of Mittage-Leffler function and compared with the solutions of all the existing fractional operators. In this paper, we show that which operator is better in exhibiting decay of the velocity of the fluid. We have plotted some graphs for fractional parameters for small and large time. Also, drawn some comparison between present power law (C), exponential law (CF) and Mittage-Leffler law (ABC) and presented graphically.

2. TStatement of the problem and solution

Consider laminar flow of a viscous unsteady incompressible fluid between two infinite parallel plate which are kept at a distance L apart in $oxyz$ coordinate system as shown in Fig. 1. We assuming that upper plate is moving with constant velocity V_o in the direction of $x - Axis$ while lower plate is at rest and $x - Axis$ perpendicular to it and there is fluid flow properties contribution in the $z - Axis$. Also, neglecting pressure gradient and body force the dimensionless governing equation for the flow along with initial and boundary conditions are [26]

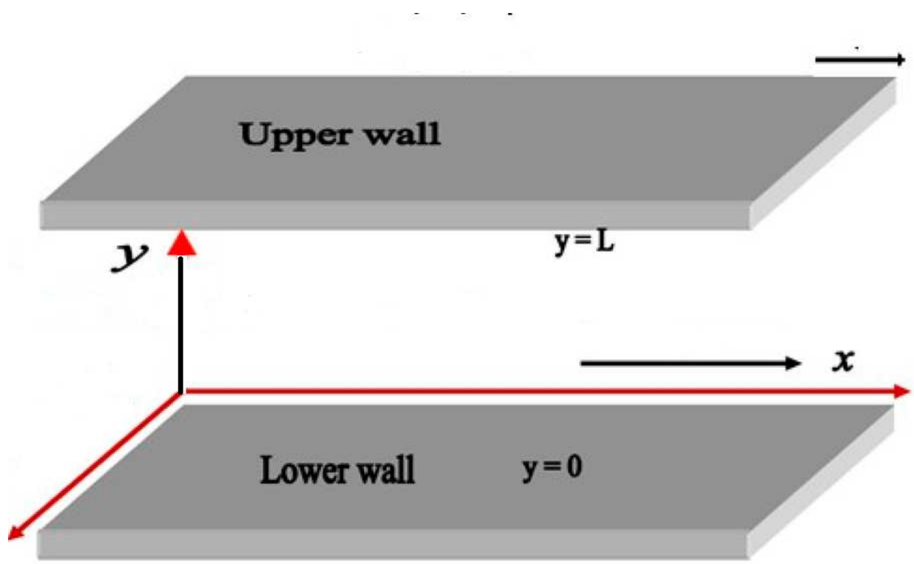


FIGURE 1. Geometry of the physical model

$$u_t(y,t) = u_{yy}(y,t), \tag{3}$$

$$u(y,0) = 0, \quad u(0,t) = 0, \quad u(1,t) = 1. \tag{4}$$

Fractional model using [30], we have

$$^{CPC}C_t^{\alpha} u(y,t) = u_{yy}(y,t), \qquad 0 < \alpha < 1. \tag{5}$$

By taking Laplace transform of Eq. (4) and (5) we have

$$\bar{u}_{yy}(y,s) = L_o \left[1 + \frac{L_1}{L_o} s^{-1}\right] s^{\alpha} \bar{u}(y,s), \tag{6}$$

initial and boundary conditions:

$$\bar{u}(0,s) = 0, \qquad \bar{u}(1,s) = \frac{1}{s}. \tag{7}$$

The solution of Eq. (6) subject Eq. (7) is given by or

$$\bar{u}(y,s) = \frac{sinh\left(y\sqrt{L_0\left[1 + \frac{L_2}{s}\right] s^{\alpha}}\right)}{s.sinh\left(\sqrt{L_0\left[1 + \frac{L_2}{s}\right] s^{\alpha}}\right)}, \; L_2 = \frac{L_1}{L_0}. \tag{8}$$

Eq. (8) can be written as in more suitable form so that we can find inverse Laplace transform analytically,

$$\bar{u}(y,s) = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=0}^{\infty} \frac{(y-2n-1)^p}{p!} \frac{[L_0]^{\frac{p}{2}-k}[L_1]^k}{k! s^{1+k-\frac{\alpha p}{2}}} \frac{\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+1-k)} - \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-(2n+1+y))^q}{q!} \frac{[L_0]^{\frac{q}{2}-m}[L_1]^m}{m! s^{1+m-\frac{\alpha q}{2}}} \frac{\Gamma(\frac{q}{2}+l)}{\Gamma(\frac{q}{2}+1-m)} \quad (9)$$

Taking Laplace inverse of Eq. (9) we get the final expression for solution in terms of series

$$u(y,t) = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=0}^{\infty} \frac{(y-2n-1)^p}{p!} \frac{[L_0]^{\frac{p}{2}-k}[L_1]^k}{k!} \frac{t^{k-\frac{\alpha p}{2}}\Gamma(\frac{p}{2}+1)}{\Gamma(1+k-\frac{\alpha p}{2})\Gamma(\frac{p}{2}+1-k)} - \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-(2n+1+y))^q}{q!} \frac{[L_0]^{\frac{q}{2}-m}[L_1(\alpha)]^m}{m!} \frac{t^{m-\frac{\alpha q}{2}}\Gamma(\frac{q}{2}+l)}{\Gamma(1+m-\frac{\alpha q}{2})\Gamma(\frac{q}{2}+1-m)}. \quad (10)$$

Now, we find the solution of the problem given in Eqs. (3) and (4), with different fractional operators by applying the Laplace transform method one by one which are listed below.

2.1. **Velocity field with power law kernel C.** Solution of Eq. (3) with initial and boundary conditions given in Eq. (4) with Caputo fractional derivative and Laplace transform is given below

$$u(y,t) = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(y-2n-1)^p t^{-\frac{\alpha p}{2}}}{p!\Gamma(1-\frac{\alpha p}{2})} - \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-(2n+1+y))^q t^{-\frac{\alpha q}{2}}}{q!\Gamma(1-\frac{\alpha q}{2})}. \quad (11)$$

2.2. **Velocity field with exponential kernel CF.** Solution of Eq. (3) with initial and boundary conditions given in Eq. (4) with CF fractional derivative and Laplace transform is given below

$$u(y,t) = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=0}^{\infty} \frac{(y-2n-1)^p a^{\frac{p}{2}}(-\alpha a)^k t^{k-1}}{p!} \frac{\Gamma(\frac{p}{2}+k)}{\Gamma(k)\Gamma(\frac{p}{2})} - \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-(2n+1+y))^q a^{\frac{q}{2}} t^{l-1}}{q!} \frac{\Gamma(\frac{q}{2}+l)}{\Gamma(l)\Gamma(\frac{q}{2})}. \quad (12)$$

2.3. **Velocity field with Mittage-Leffler kernel ABC.** Solution of Eq. (3) with initial and boundary conditions given in Eq. (4) with ABC fractional derivative and Laplace transform is given below

$$u(y,t)=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{(y-2n-1)^{p}a^{\frac{p}{2}}(-\alpha a)^{k}t^{\alpha k-1}}{p!}\frac{\Gamma(\frac{p}{2}+k)}{\Gamma(\alpha k)\Gamma(\frac{p}{2})}-$$
$$\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-(2n+1+y))^{q}a^{\frac{q}{2}}t^{\alpha l-1}}{q!}\frac{\Gamma(\frac{q}{2}+l)}{\Gamma(\alpha l)\Gamma(\frac{q}{2})}. \tag{13}$$

3. Numerical results and discussion

Novel fractional operator called constant proportional fractional operator applied in the fundamental problem of fluid dynamics. This new operator is a combination of constant proportional and Caputo type with singular kernel. Obtained solutions are expressed in terms of Mittage-Leffler function which has application in control theory as well. We have drawn some comparison with recent work through some graphs and presented graphically.

The α variations on velocity field is presented in figures 2 and 3. The fluid velocity shows rapid decay in its profile for the greater values of fractional parameters α for small time. Also noticed that momentum boundary layer increases between the layers. This happened due to the power law kernel fractional fractional operator which are used to explain the memory of the function at certain time t. The opposite trend was observed for large time. This means that velocity shows dual solutions for small and large time.

Lastly, in figure 4 and 5 we have drawn the comparison between all the existing fractional operators and constant proportional and Caputo fractional operator for large time $t = 15$, $t = 30$, 45 and $t = 60$. By increasing time fractional velocity falls back rapidly rather than all others fractional operators. It is also note that momentum boundary layer thickness of the velocity is also increases by increasing the value of time t. Figures 6 and 7 are plotted to see the impact for different values of alpha on velocities of different fractional models for $t = 45$ and $\alpha = 0.1, 0.2, 0.6, 0.99$ and observed that when $\alpha \to 1$ all the fractional models are coincident which is natural to convert into classical model.

Further, observation is that increasing the values of α momentum boundary layer thickness reduces.

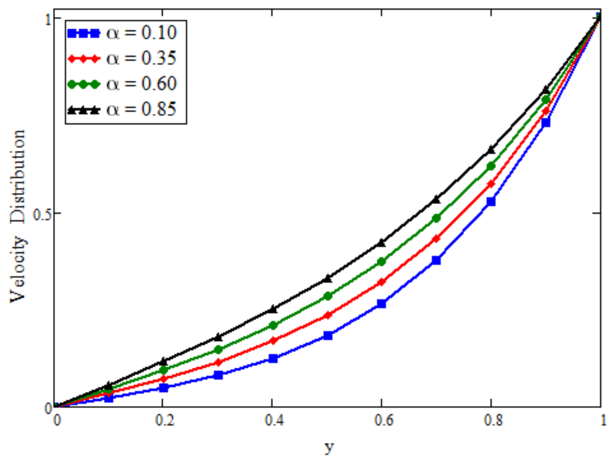


FIGURE 2. Profiles of velocity with constant proportional Caputo fractional derivative for small time $t = 0.1$

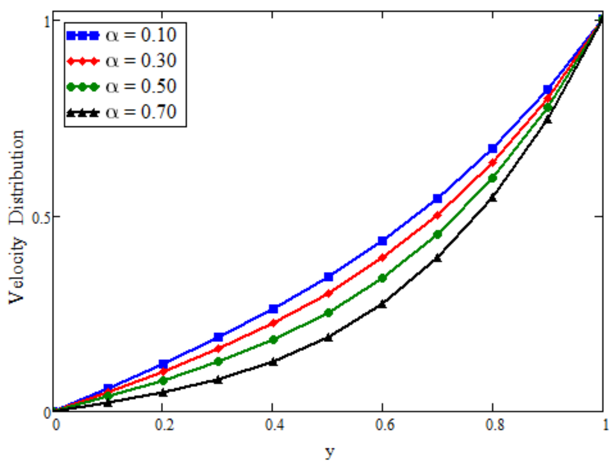


FIGURE 3. Profiles of velocity with constant proportional Caputo fractional derivative for large time $t = 2.5$

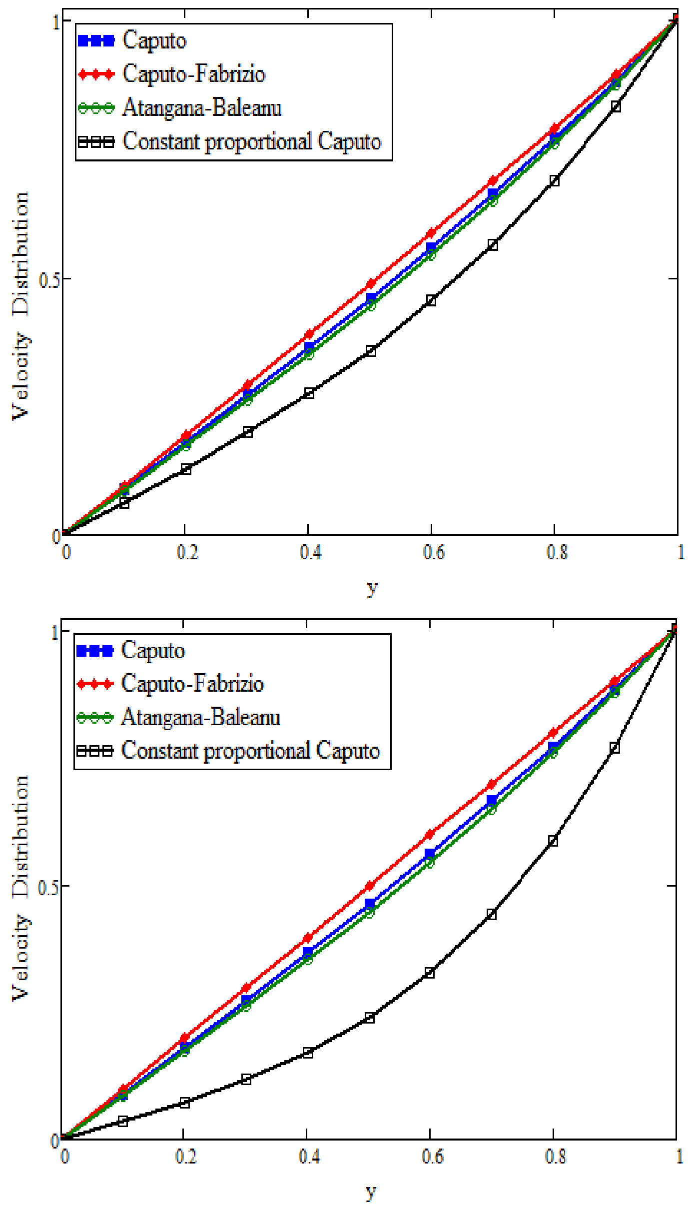


FIGURE 4. Comparison between velocities of power law (C), exponential law (CF) and Mittage-Leffler law (ABC), fractal fractional and constant proportional Caputo fractional derivatives for t=15 and t=30

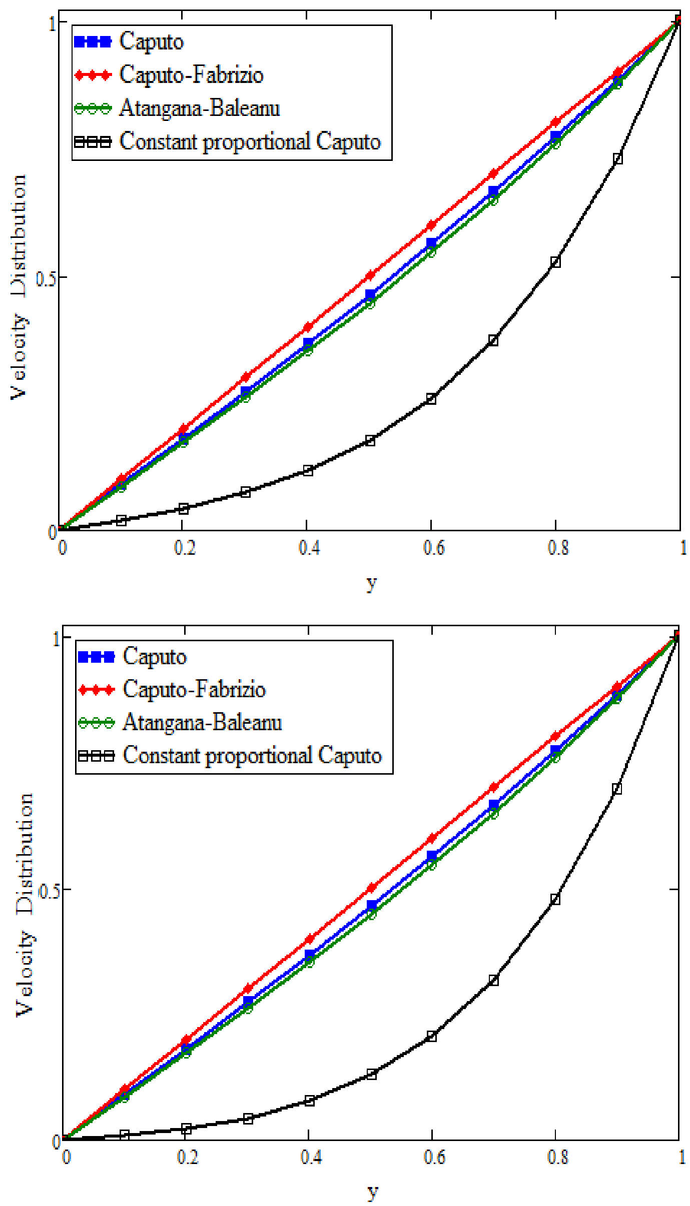
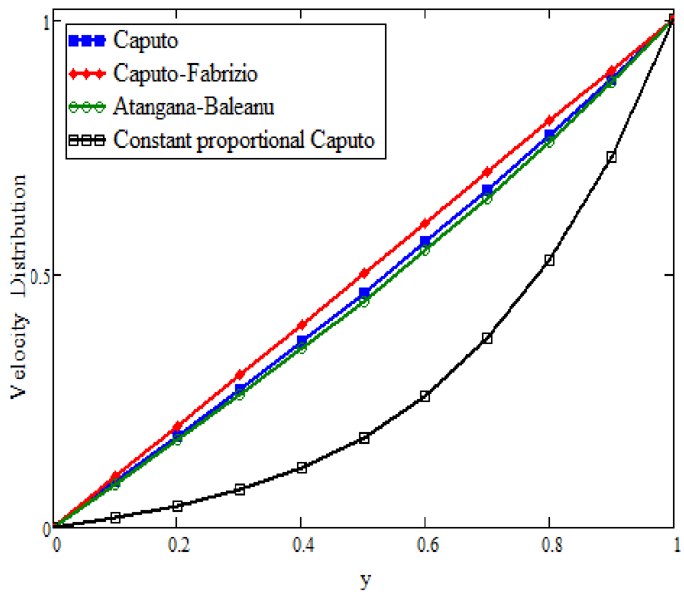


FIGURE 5. Comparison between velocities of power law (C), exponential law (CF) and Mittage-Leffler law (ABC), Fractal fractional and constant proportional Caputo fractional derivatives for t=45 and t=60



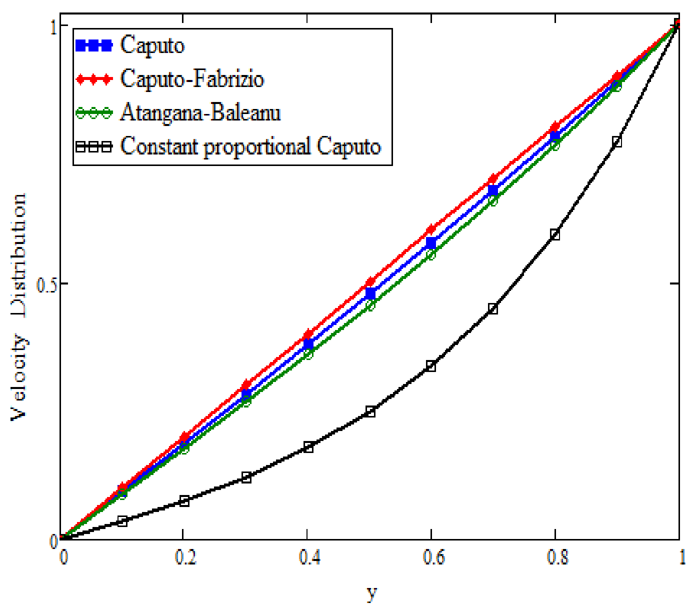


FIGURE 6. Comparison between velocities of power law (C), exponential law (CF) and Mittage-Leffler law (ABC) and constant proportional Caputo fractional derivatives for t=45 and $\alpha = 0.1$ and $\alpha = 0.2$

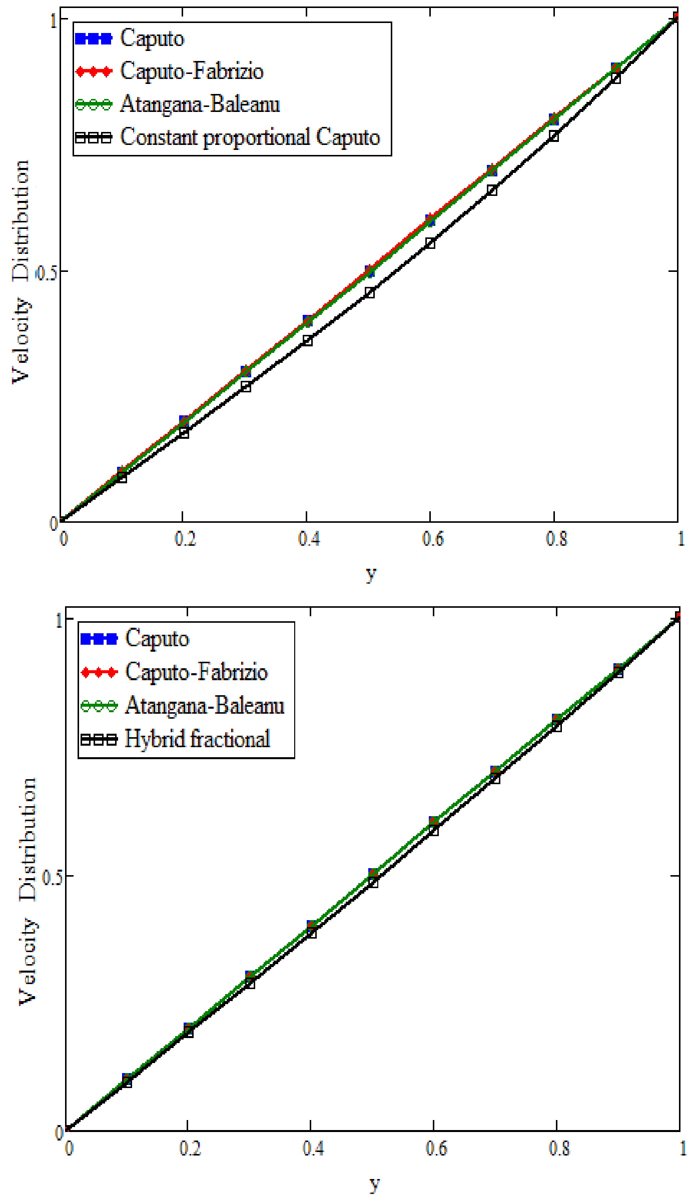


FIGURE 7. Comparison between velocities of power law (C), exponential law (CF) and Mittage-Leffler law (ABC) and constant proportional Caputo fractional derivatives for t=45 and $\alpha = 0.6$ and $\alpha = 0.9$

4. Conclusions

This paper deals with the application of constant proportional Caputo fractional derivative with singular kernel which is a linear combination of conformable fractional derivative and Caputo fractional derivative. Analytical solutions are obtained via Laplace transform. The main outcomes of the present study are:

- Fluid velocity field shows dual behavior for short and long time with different fractional α.
- Decay in the velocity is an increasing function for larger values of time.
- Momentum boundary layer thickness between all the fractional models reduces for increasing $\alpha \to 1$ hence coincide with ordinary model.
- In a comparison between constant proportional fractional and all other existing fractional operators, we concluded that new fractional operator exhibits memory of the velocity better than power law (C), exponential law (CF) and Mittage-Leffler law (ABC).

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References

[1] Khalil, R., Al Horani, M., Yousef, A., & Sababheh, M. (2014). A new definition of fractional derivative. *Journal of Computational and Applied Mathematics, 264*, 65-70.
[2] Hilfer, R., & Luchko, Y. (2019). Desiderata for fractional derivatives and integrals. *Mathematics, 7*(2), 149.
[3] Rahmat, M. R. S. (2019). A new definition of conformable fractional derivative on arbitrary time scales. *Advances in Difference Equations, 2019*(1), 1-16.
[4] Zhao, D., & Luo, M. (2017). General conformable fractional derivative and its physical interpretation. *Calcolo, 54*(3), 903-917.
[5] Abdeljawad, T. (2015). On conformable fractional calculus. *Journal of computational and Applied Mathematics, 279*, 57-66.
[6] Benkhettou, N., Hassani, S., & Torres, D. F. (2016). A conformable fractional calculus on arbitrary time scales. *Journal of King Saud University-Science, 28*(1), 93-98.
[7] Tasbozan, O., enesiz, Y., Kurt, A., & Baleanu, D. (2017). New analytical solutions for conformable fractional PDEs arising in mathematical physics by exp-function method. *Open Physics, 15*(1), 647-651.
[8] Salahshour, S., Ahmadian, A., Ismail, F., Baleanu, D., & Senu, N. (2015). A new fractional derivative for differential equation of fractional order under interval uncertainty. *Advances in Mechanical Engineering, 7*(12), 1-11.
[9] Atangana, A., Baleanu, D., & Alsaedi, A. (2015). New properties of conformable derivative. *Open Mathematics, 1*(open-issue),889-898.

[10] Jarad, F., Ugurlu, E., Abdeljawad, T., & Baleanu, D. (2017). On a new class of fractional operators. *Advances in Difference Equations, 2017*(1), 247.

[11] Yang, X. J. (2012). Advanced local fractional calculus and its applications.

[12] Anderson, D. R., & Ulness, D. J. (2015). Newly defined conformable derivatives. *Advances in Dynamical Systems and Applications, 10*(2), 109-137.

[13] Caputo, M. (1967). Linear models of dissipation whose Q is almost frequency independentII. *Geophysical Journal International, 13*(5), 529-539.

[14] Sobamowo, M. G. (2020). Transient free convection heat and mass transfer of Casson nanofluid over a vertical porous plate subjected to magnetic field and thermal radiation. *Engineering and Applied Science Letters, 3*(4), 35-54.

[15] Ghiasi, E. K., & Noeiaghdam, S. (2020). Truncating the series expansion for unsteady velocity-dependent Eyring-Powell fluid. *Engineering and Applied Science Letters, 3*(4), 28-34.

[16] Sobamowo, M. G., Kamiyo, O. M., Yinusa, A. A., & Akinshilo, T. A. (2020). Magneto-squeezing flow and heat transfer analyses of third grade fluid between two disks embedded in a porous medium using Chebyshev spectral collocation method. *Engineering and Applied Science Letters, 4*, 305-322.

[17] Simarmata, C. M. D., Susyanto, N., Hammadi, I. J., & Rahmaditya, C. (2020). A mathematical model of smoking behaviour in Indonesia with density-dependent death rate. *Open Journal of Mathematical Sciences, 4*(1), 118-125.

[18] Mehmood, S., Farid, G., & Farid, G. (2020). Fractional integrals inequalities for exponentially m-convex functions. *Open Journal of Mathematical Sciences, 4*(1), 78-85.

[19] Hassan, R., El-Agamy, M., Latif, M. S. A., & Nour, H. (2020). On Backlund transformation of Riccati equation method and its application to nonlinear partial differential equations and differential-difference equations. *Open Journal of Mathematical Sciences, 4*, 56-62.

[20] Imran, M. A., Khan, I., Ahmad, M., Shah, N. A., & Nazar, M. (2017). Heat and mass transport of differential type fluid with non-integer order time-fractional Caputo derivatives. *Journal of Molecular Liquids, 229*, 67-75.

[21] E. F. D. Goufo, A. Atangana, Dynamics of traveling waves of variable order hyperbolic Liouville equation: Regulation and control. Discrete and Continuous Dynamical Systems-S, (2019), 104-114.

[22] Jarad, F., & Abdeljawad, T. (2018). A modifi ed Laplace transform for certain generalized fractional operators. *Results in Nonlinear Analysis, 1*(2), 88-98.

[23] Singh, J., Kumar, D., Hammouch, Z., & Atangana, A. (2018). A fractional epidemiological model for computer viruses pertaining to a new fractional derivative. *Applied Mathematics and Computation, 316*, 504-515.

[24] Hammouch, Z., & Mekkaoui, T. (2018). Circuit design and simulation for the fractional-order chaotic behavior in a new dynamical system. *Complex & Intelligent Systems, 4*(4), 251-260.

[25] A. Atangana, et al., Fractional calculus with power law: The cradle of our ancestors, Fractional Dynamics (2015).

[26] Imran, M. A. (2020). Application of fractal fractional derivative of power law kernel (FFP0Dxa,) to MHD viscous fluid flow between two plates. *Chaos, Solitons & Fractals, 134*, 109691.

[27] Belmor, S., Jarad, F., Abdeljawad, T., & Kilin, G. (2020). A study of boundary value problem for generalized fractional differential inclusion via endpoint theory for weak contractions. *Advances in Difference Equations, 2020*(1), 1-11.

[28] Ahmed, I., Kumam, P., Jarad, F., Borisut, P., & Jirakitpuwapat, W. (2020). On Hilfer generalized proportional fractional derivative. *Advances in Difference Equations, 2020*(1), 1-18.

[29] Jarad, F., Abdeljawad, T., Rashid, S., & Hammouch, Z. (2020). More properties of the proportional fractional integrals and derivatives of a function with respect to another function. *Advances in Difference Equations, 2020*(1), 1-16.

[30] Baleanu, D., Fernandez, A., & Akgl, A. (2020). On a Fractional Operator Combining Proportional and Classical Differintegrals. *Mathematics, 8*(3), 360.