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NUMERICAL SOLUTIONS OF THE FRACTIONAL SIS EPIDEMIC MODEL VIA A NOVEL TECHNIQUE

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ABSTRACT. This article introduces a novel technique called modified fractional Taylor series method (MFTSM) to find numerical solutions for the fractional SIS epidemic model. The fractional derivative is considered in the sense of Caputo. The most important feature of the MFTSM is that it is very effective, accurate, simple, and more computational than the methods found in literature. The validity and effectiveness of the proposed technique are investigated and verified through numerical example.

Key words: Fractional SIS epidemic model, Caputo fractional derivative, modified fractional Taylor series method, numerical solution.
MSC: 34A08, 26A33, 34K28, 35C10.

1. Introduction

The fractional SIS epidemic model is given as

$$\begin{cases} D^\alpha S(t) = -rS(t)I(t) + \mu I(t), \\ D^\alpha I(t) = rS(t)I(t) - \mu I(t), \end{cases} \tag{1}$$

subject to

$$S(0) = S_0, I(0) = I_0, \tag{2}$$

where D^α is the fractional derivative operator in the Caputo sense of order $\alpha, 0 < \alpha \leq 1$, $S(t)$ is the susceptible population, $I(t)$ is the infected population, $r > 0$ is the infectivity coefficient, and $\mu > 0$ is the recovery coefficient, while $S_0 > 0$ and $I_0 > 0$ are given constants.

When $\alpha = 1$, the system (1) reduces to the classical SIS epidemic model. Recently, many methods have been developed to solve the classical SIS epidemic

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model. Among these methods are: Lie group (LG) [1], homotopy analysis method (HAM) [2], differential transformation method (DTM) [3].

The fractional SIS epidemic model described in system of first nonlinear fractional order ordinary differential equations. Nowadays, there is increasing attention to nonlinear fractional differential equations and their applications in different research fields, where these equations appear in many branches of physics, engineering and applied mathematics such as: fluid mechanics, viscoelasticity, diffusion, signal processing, electromagnetism, electrochemistry, nonlinear control theory, nonlinear biological systems, biomedical, chemical reaction theory, and so on [4–20].

In all these scientific fields, it is necessary to determine the exact or at least approximate solutions for the nonlinear fractional differential equations. In general, there is no specific method that gives an exact solution to these equations due to the complex nonlinear parts involved, most of these methods are only approximate. For this reason, many analytical and numerical methods have been proposed recently to solve the nonlinear fractional differential equations. For example, fractional natural decomposition method (FNDM) [21], natural variational iteration method (NVIM) [22], fractional Elzaki projected differential transform method (FEPDTM) [23], fractional residual power series method (FRPSM) [24], fractional reduced differential transform method (FRDTM) [25].

The main goal of this article is to propose a new technique called modified fractional Taylor series method (MFTSM) to find numerical solutions for the fractional SIS epidemic model (1) subject to (2).

The MFTSM is one of the efficient methods which use to solve the nonlinear fractional differential equations without using linearization and any other restriction, which gives the solution in the form of an infinite series which converges rapidly to the exact solution.

The rest of the article has been organized as follows. In Section 2, we describe the basic definitions of fractional calculus used throughout the article. In Section 3, we solve the fractional SIS epidemic model (1) subject to (2) by the MFTSM and establish its convergence. In Section 4, we suggest a numerical example which demonstrate the effectiveness of our proposed technique. In Section 5, we discuss our obtained results represented by figures and tables. At last, conclusion is given.

2. Basic definitions

In this section, we provide some definitions and properties of the fractional calculus which are used further in this article. For more details (see, [17–19]).

Definition 1. *The Riemann-Liouville fractional integral of order $\alpha \geq 0$ for a function $f(t)$ is defined by*

$$I^{\alpha}f(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\int\limits_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau, \alpha>0,\\ f(t), \qquad \alpha=0,\end{cases} \tag{3}$$

where $\Gamma(.)$ *denotes Euler's gamma function.*

Definition 2. *The Caputo fractional derivative of order* $\alpha\geq 0$ *for a function* $f(t)$ *is defined by*

$$D^{\alpha}f(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\int\limits_0^t(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau,\ n-1<\alpha<n,\\ f^{(n)}(t), \qquad \alpha=n,\end{cases} \tag{4}$$

where $n=[\alpha]+1$ *with* $[\alpha]$ *being the integer part of* α.

For this definition we have the following properties

1)

$$D^{\alpha}(\lambda)=0,\ \text{where } \lambda \text{ is a constant.}$$

2)

$$D^{\alpha}t^{\beta}=\begin{cases}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, \beta>n-1,\\ 0, \qquad \beta\leq n-1.\end{cases}$$

3)

$$D^{\alpha}(f(t)g(t))=f(t)D^{\alpha}g(t)+g(t)D^{\alpha}f(t).$$

3. Analysis of MFTSM for the fractional SIS epidemic model

Theorem 1. *Let consider the fractional SIS epidemic model (1) subject to (2). Then, by MFTSM, the solutions of (1) and (2) can be expressed by an infinite series as follow*

$$\begin{cases}S(t)=\sum\limits_{i=0}^{\infty}S_i\dfrac{t^{i\alpha}}{\Gamma(i\alpha+1)}\\ I(t)=\sum\limits_{i=0}^{\infty}I_i\dfrac{t^{i\alpha}}{\Gamma(i\alpha+1)}\end{cases}, 0<\alpha\leq 1, 0<t<R, \tag{5}$$

where S_i *and* I_i *are the coefficients of the series (5) and* R *is the radius of convergence.*

Proof. Consider the fractional SIS epidemic model (1) subject to (2).

Suppose the solutions takes the form of an infinite series as follows

$$S(t)=\sum_{i=0}^{\infty}S_i\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}, \tag{6}$$

$$I(t)=\sum_{i=0}^{\infty}I_i\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}. \tag{7}$$

Therefore, the $n-$order approximate solutions of (1), can be written as

$$S_n(t) = \sum_{i=0}^{n} S_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} = S_0 + \sum_{i=1}^{n} S_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}, \quad (8)$$

$$I_n(t) = \sum_{i=0}^{n} I_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} = I_0 + \sum_{i=1}^{n} I_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}. \quad (9)$$

Applying the operator D^α on (8) and (9), we get the following formula

$$D^\alpha S_n(t) = \sum_{i=0}^{n-1} S_{i+1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}, \quad (10)$$

$$D^\alpha I_n(t) = \sum_{i=0}^{n-1} I_{i+1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}. \quad (11)$$

After that, we substitute (8), (9), (10), and (11) in (1). So, we have the following recurrence relation

$$0 = \sum_{i=0}^{n-1} S_{i+1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} + r\left(S_0 + \sum_{i=1}^{n} S_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right)\left(I_0 + \sum_{i=1}^{n} I_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right) -\mu\left(I_0 + \sum_{i=1}^{n} I_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right),$$

and

$$0 = \sum_{i=0}^{n-1} I_{i+1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} - r\left(S_0 + \sum_{i=1}^{n} S_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right)\left(I_0 + \sum_{i=1}^{n} I_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right) +\mu\left(I_0 + \sum_{i=1}^{n} I_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right).$$

We follow the same analog used to obtain the Taylor series coefficients. In particular, to obtain the coefficient S_n and I_n $n = 1, 2, 3, ..$, we must solve the following

$$D^{(n-1)\alpha}\{F_1(t,\alpha,n)\} \downarrow_{t=0} = 0,$$
$$D^{(n-1)\alpha}\{F_2(t,\alpha,n)\} \downarrow_{t=0} = 0,$$

where

$$F_1(t,\alpha,n) = \sum_{i=0}^{n-1} S_{i+1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} + r\left(S_0 + \sum_{i=1}^{n} S_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right)\left(I_0 + \sum_{i=1}^{n} I_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right) -\mu\left(I_0 + \sum_{i=1}^{n} I_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right),$$

and

$$F_2(t,\alpha,n) = \sum_{i=0}^{n-1} I_{i+1}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)} - r\left(S_0+\sum_{i=1}^{n} S_i\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right)\left(I_0+\sum_{i=1}^{n} I_i\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right) +\mu\left(I_0+\sum_{i=1}^{n} I_i\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right).$$

Now, we obtain the first terms of the sequence $\{S_n\}_1^N$ and $\{I_n\}_1^N$.
For $n=1$ we have

$$F_1(t,\alpha,1) = S_1 + r\left(S_0+S_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\left(I_0+I_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right) - \mu\left(I_0+I_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right),$$

and

$$F_2(t,\alpha,1) = I_1 - r\left(S_0+S_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\left(I_0+I_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right) + \mu\left(I_0+I_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right).$$

Solving $F_1(t,\alpha,1)=0$ and $F_2(t,\alpha,1)=0$, gives

$$S_1 = -rS_0I_0+\mu I_0,$$
$$I_1 = rS_0I_0-\mu I_0.$$

To obtain S_2 and I_2, we consider

$$F_1(t,\alpha,2) = S_1+S_2\frac{t^{\alpha}}{\Gamma(\alpha+1)} + r\left(S_0+S_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}+S_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right) \times\left(I_0+I_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}+I_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right) -\mu\left(I_0+I_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}+I_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right),$$

and

$$F_2(t,\alpha,2) = I_1+I_2\frac{t^{\alpha}}{\Gamma(\alpha+1)} - r\left(S_0+S_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}+S_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right) \times\left(I_0+I_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}+I_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right) +\mu\left(I_0+I_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}+I_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right).$$

Then, we solve $D^{\alpha}\{F_1(t,\alpha,2)\}\downarrow_{t=0}=0$ and $D^{\alpha}\{F_2(t,\alpha,2)\}\downarrow_{t=0}=0$, to get

$$S_2 = -r(S_1I_0+S_0I_1)+\mu I_1,$$
$$I_2 = r(S_1I_0+S_0I_1)-\mu I_1.$$

To obtain S_3 and I_3, we consider $F_1(t,\alpha,3)$ and $F_2(t,\alpha,3)$ and we solve $D^{2\alpha}\{F_1(t,\alpha,3)\}\downarrow_{t=0}=0$ and $D^{2\alpha}\{F_2(t,\alpha,3)\}\downarrow_{t=0}=0$. Therefore

$$\begin{aligned} S_3 &= -r(S_2I_0+2S_1I_1+S_0I_2)+\mu I_2, \\ I_3 &= r(S_2I_0+2S_1I_1+S_0I_2)-\mu I_2. \end{aligned}$$

To obtain S_4 and I_4, we consider $F_1(t,\alpha,4)$ and $F_2(t,\alpha,4)$ and we solve $D^{3\alpha}\{F_1(t,\alpha,4)\}\downarrow_{t=0}=0$ and $D^{3\alpha}\{F_2(t,\alpha,4)\}\downarrow_{t=0}=0$. Therefore

$$\begin{aligned} S_4 &= -r(S_3I_0+3S_2I_1+3S_1I_2+S_0I_3)+\mu I_3, \\ I_4 &= r(S_3I_0+3S_2I_1+3S_1I_2+S_0I_3)-\mu I_3. \end{aligned}$$

In general, to obtain the other coefficient S_k and I_k, by solving $D^{(k-1)\alpha}\{F_1(t,\alpha,k)\}\downarrow_{t=0}=0$ and $D^{(k-1)\alpha}\{F_2(t,\alpha,k)\}\downarrow_{t=0}=0$.

Finally, the solutions of (1) subject to (2), can be expressed by

$$\begin{aligned} S(t) &= \lim_{n\longrightarrow\infty} S_n(t) = \lim_{n\longrightarrow\infty}\sum_{i=0}^{n} S_i\frac{t^{i\alpha}}{\Gamma(i\alpha+1)} = \sum_{i=0}^{\infty} S_i\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}, \\ I(t) &= \lim_{n\longrightarrow\infty} I_n(t) = \lim_{n\longrightarrow\infty}\sum_{i=0}^{n} I_i\frac{t^{i\alpha}}{\Gamma(i\alpha+1)} = \sum_{i=0}^{\infty} I_i\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}. \end{aligned}$$

□

Theorem 2. *If there exists a constant $0<\gamma,\delta<1$ such that*

$$\begin{cases} \|S_{n+1}(t)\| \le \gamma\,\|S_n(t)\|, \\ \|I_{n+1}(t)\| \le \delta\,\|I_n(t)\|, \end{cases}$$

where $n\in\mathbb{N}, 0<t<R$, then the sequence of approximate solutions (5) converges to the exact solutions.

Proof. For all $0<t<R$, we have

$$\begin{aligned} \|S(t)-S_n(t)\| &= \left\|\sum_{k=n+1}^{\infty} S_k(t)\right\| \le \sum_{k=n+1}^{\infty}\|S_k(t)\| \le \sum_{k=n+1}^{\infty}\gamma\,\|S_{k-1}(t)\| \\ &\le \sum_{k=n+1}^{\infty}\gamma^2\,\|S_{k-2}(t)\| \le \dots \le \|S_0\|\sum_{k=n+1}^{\infty}\gamma^k \\ &= \frac{\gamma^{n+1}}{1-\gamma}\,\|S_0\|. \end{aligned}$$

Since $0<\gamma<1$ and S_0 is bounded, then

$$\lim_{n\longrightarrow\infty}\|S(t)-S_n(t)\| = 0.$$

On another side

$$\begin{aligned}\|I(t)-I_n(t)\| &= \left\|\sum_{k=n+1}^{\infty} I_k(t)\right\| \leq \sum_{k=n+1}^{\infty}\|I_k(t)\| \leq \sum_{k=n+1}^{\infty}\delta\|I_{k-1}(t)\| \\ &\leq \sum_{k=n+1}^{\infty}\delta^2\|I_{k-2}(t)\| \leq ... \leq \|I_0\|\sum_{k=n+1}^{\infty}\delta^k \\ &= \frac{\delta^{n+1}}{1-\delta}\|b_0\|.\end{aligned}$$

Since $0<\delta<1$ and I_0 is bounded, then

$$\lim_{n\longrightarrow\infty}\|I(t)-I_n(t)\| = 0.$$

□

4. Application

In this section, we suggest a numerical example to demonstrate the efficiency and accuracy of the proposed approach.

Example 1. *Let consider the fractional SIS epidemic model with* $r=2$ *and* $\mu=1$*. Then (1) becomes*

$$\begin{cases} D^{\alpha}S(t) = -2S(t)I(t)+I(t), \\ D^{\alpha}I(t) = 2S(t)I(t)-I(t), \end{cases} \tag{12}$$

subject to

$$S(0)=0.45, I(0)=0.55, \tag{13}$$

where D^{α} *is the fractional derivative operator in the Caputo sense of order* $\alpha, 0<\alpha\leq 1$*.*

For $\alpha=1$*, the exact solutions of (12) subject to (13) is (See. [26])*

$$S(t) = 1-\frac{1}{2-\frac{2}{11}e^{-t}},$$
$$I(t) = \frac{1}{2-\frac{2}{11}e^{-t}}.$$

According the description of the MFTSM presented in Section 3, we have

$$S(t) = \sum_{i=0}^{\infty} S_i\frac{t^{i\alpha}}{\Gamma(i\alpha+1)},$$
$$I(t) = \sum_{i=0}^{\infty} I_i\frac{t^{i\alpha}}{\Gamma(i\alpha+1)},$$

and

$$\begin{aligned} S_0 &= 0.45, S_1 = 0.055, S_2 = -0.066, S_3 = 0.0913, S_4 = -0.15312... \\ I_0 &= 0.55, I_1 = -0.055, I_2 = 0.066, I_3 = -0.0913, I_4 = 0.15312... \end{aligned}$$

Therefore, the solutions of (12) subject to (13), is given by

$$S(t) = 0.45 + \frac{0.055t^{\alpha}}{\Gamma(\alpha+1)} - \frac{0.066t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{0.091\,3t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{0.153\,12t^{4\alpha}}{\Gamma(4\alpha+1)} + ...$$

$$I(t) = 0.55 - \frac{0.055t^{\alpha}}{\Gamma(\alpha+1)} + \frac{0.066t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{0.091\,3t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{0.153\,12t^{4\alpha}}{\Gamma(4\alpha+1)} - ...$$

The special case of $\alpha = 1$, will give the following result

$$S(t) = 0.45 + 0.055t - 0.033t^2 + 0.015217t^3 - 0.00638t^4 + ...$$

$$I(t) = 0.55 - 0.055t + 0.033t^2 - 0.015217t^3 + 0.00638t^4 - ...$$

which is the numerical solutions of the classical SIS epidemic model [3].

5. Numerical results and discussion

Figures 1 and 2 represents the graphical behaviour of the exact solutions $S(t)$ and $I(t)$ and approximate solutions $S_4(t)$ and $I_4(t)$ for different values of order of fractional derivative α. From Figures 1 and 2, we see that when the order of the fractional derivative α tends to 1, the approximate solutions obtained by MFTSM tends continuously to the exact solutions. Tables 1 and 2 represents the numerical values of the approximate solutions $S_4(t)$ and $I_4(t)$ and exact solutions $S(t)$ and $I(t)$. It can be observed from Tables 1 and 2 that exact solutions for (12) are in close agreement with 4−term approximate solutions using the MFTSM.

Table 1. The numerical values of the exact solution $S(t)$ and the approximate solutions $S_4(t)$ for different values of α.

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.95$	$\alpha = 1$	Exact solution	$\lvert S_{exact} - S_{MFTSM}\rvert$
0.1	0.46024	0.45831	0.45587	0.45518	0.45518	2.4978×10^{-8}
0.2	0.46509	0.46331	0.46064	0.45979	0.45979	7.7622×10^{-7}
0.3	0.46841	0.46710	0.46472	0.46389	0.46389	5.6872×10^{-6}
0.4	0.47070	0.47006	0.46825	0.46753	0.46755	2.3128×10^{-5}
0.5	0.47213	0.47234	0.47130	0.47075	0.47082	6.8175×10^{-5}

Remark 1. *In this paper, only 4−term approximate solution is used to calculate the numerical solution and the MFTSM can provide a more precise solution with less absolute error by calculating a higher order approximation.*

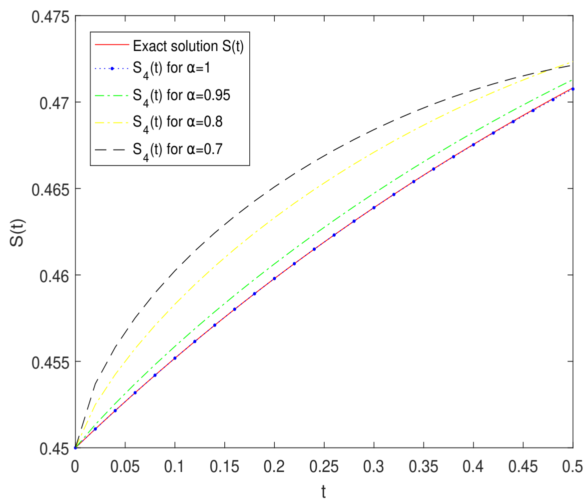

Figure 1. The behavior of the exact solution $S(t)$ and the approximate solutions $S_4(t)$ by MFTSM for different values of α.

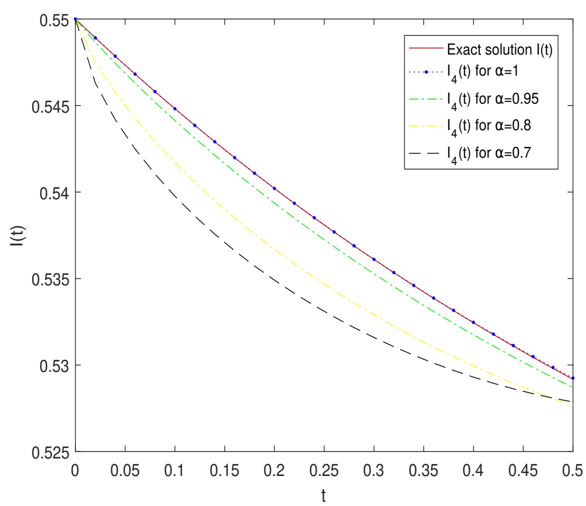

Figure 2. The behavior of the exact solution $I(t)$ and the approximate solutions $I_4(t)$ by MFTSM for different values of α.

TABLE 2. The numerical values of the exact solution $I(t)$ and the approximate solutions $I_4(t)$ for different values of α.

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.95$	$\alpha = 1$	Exact solution	$\lvert I_{exact} - I_{MFTSM}\rvert$
0.1	0.53976	0.54169	0.54413	0.54482	0.54482	2.4978×10^{-8}
0.2	0.53491	0.53669	0.53936	0.54021	0.54021	7.7622×10^{-7}
0.3	0.53159	0.53290	0.53528	0.53611	0.53611	5.6872×10^{-6}
0.4	0.52930	0.52994	0.53175	0.53247	0.53245	2.3128×10^{-5}
0.5	0.52787	0.52766	0.52870	0.52925	0.52918	6.8175×10^{-5}

6. Conclusion

In this article, a new technique has been introduced called modified fractional Taylor series method (MFTSM) to find numerical solutions for the fractional SIS epidemic model. In order to clarify the accuracy and effectiveness of the proposed method, it is applied to a numerical example. Numerical results demonstrate the ease and accuracy of the MFTSM to solve this model.In addition, this technique gives solutions in the form of convergent series with components that can be calculated directly without using linearization, perturbation or restrictive assumptions. Finally, based on the efficiency and simplicity of the proposed technique, we conclude that it is a powerful and effective mathematical tool for many systems of fractional differential equations.

In the future studies, we will extend this approach to study the solutions of a wide range of systems of nonlinear fractional differential equations involving a higher-order fractional derivatives α, where $n-1 < \alpha \leq n$ and $n \geq 1$.

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