



# Partition dimension of generalized Peterson and Harary graphs

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## Abstract

The distance of a connected, simple graph  $\mathbb{P}$  is denoted by  $d(\alpha_1, \alpha_2)$ , which is the length of a shortest path between the vertices  $\alpha_1, \alpha_2 \in V(\mathbb{P})$ , where  $V(\mathbb{P})$  is the vertex set of  $\mathbb{P}$ . The  $l$ -ordered partition of  $V(\mathbb{P})$  is  $K = \{K_1, K_2, \dots, K_l\}$ . A vertex  $\alpha \in V(\mathbb{P})$ , and  $r(\alpha|K) = \{d(\alpha, K_1), d(\alpha, K_2), \dots, d(\alpha, K_l)\}$  be a  $l$ -tuple distances, where  $r(\alpha|K)$  is the representation of a vertex  $\alpha$  with respect to set  $K$ . If  $r(\alpha|K)$  of  $\alpha$  is unique, for every pair of vertices, then  $K$  is the resolving partition set of  $V(\mathbb{P})$ . The minimum number  $l$  in the resolving partition set  $K$  is known as partition dimension ( $pd(\mathbb{P})$ ). In this paper, we studied the generalized families of Peterson graph,  $P_{\lambda, \chi}$  and proved that these families have bounded partition dimension.

**Keywords:** Generalized Peterson graph, Harary Graph, partition dimension, partition resolving set, sharp bounds of partition dimension.

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## 1. Introduction

Let  $\mathbb{G} = (\mathbb{P}_k)_{k \geq 1}$  be a family with connected graphs  $\mathbb{P}_k$  depending on the number of graphs in the family which is denoted by  $k$  as follows:  $|V(\mathbb{P}_k)| = O(k)$  is the order with condition  $\lim_{k \rightarrow \infty} O(k) = \infty$ . If  $\exists$  a constant number  $\alpha \geq 0$  such that  $pd(\mathbb{P}_k) \leq \alpha$  for each possible value  $k \geq 1$ , then we can deduce the result that  $\mathbb{G}$  has bounded partition dimension and the upper bound is the constant number  $\alpha$ , otherwise the  $\mathbb{G}$  does

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not have bounded partition dimension. If every graph in the family  $\mathbb{G}$  have the equal partition dimension (which mean that the partition dimension do not depends on the number  $k$ ),  $\mathbb{G}$  is said to be a family with constant partition dimension [16].

In 1975 the idea delivered by Slater had a background in networking, usually referred to as locating set or beacons set. The entire network or a graph is controlled by specifically chosen vertices from the vertex set in this concept. These vertices have to choose with a specific condition that each vertex of a graph has a unique position in terms of representations, we refer to the Definition 1.2 for this concept. Later Melter and Harary rename this concept as resolving set [12]. In the graph's theoretical study, this concept is called a metric basis or basis set of a graph. The count of vertices in a resolving set or metric basis is referred as the metric dimension of a graph [27]. Instead of choosing particular nodes into a subset with the defined condition, it is possible to arrange the entire vertex set into subsets keeping the defined condition of  $r(\gamma|F)$ , which is actually came from the idea of the unique position of each vertex in a graph. This concept is called as the partition resolving set, and the least number of subsets is called the partition dimension, introduced by Chartrand et al. in 2000 [5]. To better understand this concept, we refer to the mathematical Definition 1.3 and 1.4 and for the latest ideas related to this concept, see [10].

Representing a graph with each of its vertex has unique position is falling in different real-world applications, such as for the strategies, coding, and decoding of mastermind games brief in [9], the popular relation which is named as Djokovic-Winkler linked to this concept [4], the piloting or the guidance of a robot also associated with this unique idea [17], the procedure of verifying and discovering a network related to this concept [3]. There are many applications to explore those, we refer to see [12, 21]. Finding of a resolving set is NP-hard problem [13, 18] and the partition resolving set is the generalization of resolving set it also falls in the category of NP-hard [5].

The concept of resolving partition set and partition dimension extensively appeared in the literature. For example, the graph with partition dimension  $|V| - 3$  discussed [2], the graph obtained by few graph operations and its corresponding partition dimension studied in [28], bounds on the partition dimension for convex polytopes in [6, 8, 14], bounds of partition on the circulant and multipartite discussed in [11, 19], chemical structure partitioning discussed in [20], on the bounded partition dimension of the Cartesian product of graphs are studied in [30], [1] gave bounds for the subdivision of different graphs, [25] provide the bounds on tree graph, [26] discussed bounds of unicyclic graphs in the form of subgraphs. For the resolving set and metric dimension of Peterson and generalized Peterson graph, we refer to the articles [23, 15]. For more recent literature and results, we refer to see [25, 22, 24].

Following are basic mathematical definitions of the concepts used in this research work.

**Definition 1.1.** Suppose  $P$  be an undirected, simple graph with the set of vertices named as  $V(P)$  and edge set  $E(P)$ , the distance which also known as geodesics, between  $\alpha_1, \alpha_2 \in V(P)$  two vertices is the count of minimum edges between  $\alpha_1 - \alpha_2$  path. It is denoted by  $d(\alpha_1, \alpha_2)$ .

**Definition 1.2.** Suppose an ordered set of vertices from  $V(P)$  labeled as  $R = \{\alpha_1, \alpha_2, \dots, \alpha_s\}$  and  $\alpha \in V(P)$ . The representations  $r(\alpha|R)$  of  $\alpha$ -vertex with respect to an ordered subset  $R$  is the  $s$ -tuple distances  $(d(\alpha, \alpha_1), d(\alpha, \alpha_2), \dots, d(\alpha, \alpha_s))$ . If each vertex from  $V(P)$  have unique representations according to  $R$ , then  $R$  is called a resolving set of graph  $P$ , and minimum count of the elements in  $R$  is called the metric dimension of graph  $P$  and it is represented by  $dim(P)$ .

**Definition 1.3.** Let  $\chi$  is the  $s$ -ordered partition set and  $r(\alpha|\chi) = \{d(\alpha, \chi_1), d(\alpha, \chi_2), \dots, d(\alpha, \chi_s)\}$ , is the  $s$ -tuple distance representations of a vertex  $\alpha$  with respect to  $\chi$ . If the representations of  $\alpha$  with respect to  $\chi$  are unique, then  $B$  is the partition resolving set of the vertex set of a graph  $P$ .

**Definition 1.4.** The minimum count of subsets in the partition resolving set of  $V(P)$  is defined as the partition dimension ( $pd(P)$ ) of  $P$ .

Following theorems are very helpful in finding the partition dimension of a graph.

**Theorem 1.5.** [5] *Let  $\chi$  be a partition resolving set of  $V(P)$  and  $\alpha_1, \alpha_2 \in V(P)$ . If  $d(\alpha_1, \alpha) = d(\alpha_2, \alpha)$  for all vertices  $\alpha \in V(P) \setminus \{\alpha_1, \alpha_2\}$ , then  $\alpha_1, \alpha_2$  belongs to different subsets of  $\chi$ .*

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$
$a_0$	1	4	3	0	$a_1$	2	5	4	0
$a_2$	3	5	5	0	$a_3$	4	4	5	0
$a_4$	5	3	4	0	$a_5$	5	2	3	0
$a_6$	4	1	2	0	$a_7$	3	2	1	0
$a_9$	2	4	1	0	$a_{10}$	3	5	2	0
$a_{11}$	4	5	3	0	$a_{12}$	4	2	5	0
$a_{13}$	5	3	5	0	$a_{14}$	4	2	5	0
$a_{15}$	3	2	4	0	$a_{16}$	2	3	3	0

Table 1: Representations of outer vertices

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$
$b_1$	2	6	3	0	$b_2$	4	6	4	0
$b_3$	5	5	5	0	$b_4$	6	4	5	0
$b_5$	6	2	4	0	$b_7$	3	2	2	0
$b_8$	1	4	1	0	$b_9$	1	5	2	0
$b_{10}$	3	6	3	0	$b_{11}$	5	6	4	0
$b_{12}$	6	5	5	0	$b_{13}$	6	3	5	0
$b_{14}$	5	1	4	0	$b_{15}$	4	1	3	0
$b_{16}$	2	3	2	0					

Table 2: Representations of inner vertices

## 2. Generalized Petersen Graph $P_{\alpha,\beta}$

For each odd integer  $\alpha = 2\beta + 1 \geq 3$ , the generalized Petersen graph  $P_{\alpha,\beta}$  is a graph with vertex set  $a \cup b$  where  $\{a_\eta, b_\eta : \eta = 0, 1, \dots, \alpha - 1\}$  and edge set  $E = \{a_\eta a_{\eta+1}, b_\eta b_{\eta+\beta}, a_\eta b_\eta : \eta = 0, 1, \dots, \alpha - 1\}$ . For our purpose, we call the vertices  $a_0, a_1, \dots, a_{\alpha-1}$  outer vertices and  $b_0, b_1, \dots, b_{\alpha-1}$  inner vertices. Here and throughout the paper, the subscripts are to be taken as integers modulo  $\alpha$ . Following are some bounds on the partition dimension of generalized Petersen graph  $P_{\alpha,\beta}$ .

**Theorem 2.1.** *Let  $P_{\alpha,\beta}$  be generalized Petersen graphs with  $\alpha = 2\beta + 1$  and  $\beta \equiv 0 \pmod{4}$ , then  $pd(P_{\alpha,\beta}) \leq 4$ .*

*Proof.* For  $\beta = 4$ , it is easy to see that  $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$  where  $\chi_1 = \{b_0\}$ ,  $\chi_2 = \{b_4\}$ ,  $\chi_3 = \{a_5\}$ ,  $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_0, b_4, a_5\}$  is a resolving partitioning for  $V(P_{9,4})$ .

For  $\beta \geq 8$ ,  $\beta \equiv 0 \pmod{4}$  and for the chosen index  $\eta$  such that  $0 \leq \eta \leq \alpha - 1$ , we shall show that  $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$  where  $\chi_1 = \{b_\eta\}$ ,  $\chi_2 = \{b_{\eta+2\xi+2}\}$ ,  $\chi_3 = \{a_{\eta+\beta}\}$ ,  $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_\eta, b_{\eta+2\xi+2}, a_{\eta+\beta}\}$  is a resolving partitioning for  $V(P_{\alpha,\beta})$ , where  $\xi = \frac{\beta}{4}$ .

For  $\beta = 8$ , the representations of the vertices of  $V(P_{17,8})$ , are in the Tables 1 and 2.

It can be seen that all the vertices in  $V(P_{17,8})$ , have distinct representations with respect to  $\chi$ . Now for  $\beta \geq 8$  the representations of the vertices of  $V(P_{\alpha,\beta})$ , are:  $r(b_{\eta+1}|\chi) = (2, 2\xi + 2, 3, 0)$ ,  $r(b_{\eta+2\beta}|\chi) = (2, 2\xi, 2, 0)$ , and remaining in Tables 3 and 4.

From these tables, one can see that all the vertices of  $P_{\alpha,\beta}$  lying in column 1 of Table 3 and Table 4 have distinct representations with respect to  $\chi$ . Thus  $\chi$  is resolving partitioning for  $P_{\alpha,\beta}$ . Hence, for all  $\alpha = 2\beta + 1$ ,  $\beta \geq 8$  and  $\beta \equiv 0 \pmod{4}$ ,  $pd(P_{\alpha,\beta}) \leq 4$ . □

**Theorem 2.2.** *Let  $P_{\alpha,\beta}$  be generalized Petersen graphs with  $\alpha = 2\beta + 1 \geq 5$  and  $\beta \equiv 1 \pmod{4}$ , then  $pd(P_{\alpha,\beta}) \leq 4$ .*

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	
$a_{\eta+\zeta}$	$\zeta + 1$	$2\xi + \zeta$	$\zeta + 3$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\zeta+2}$	$\zeta + 3$	$2\xi - \zeta + 1$	$\zeta + 5$	0	$0 \leq \zeta \leq 2\xi - 4$
$a_{\eta+2\xi+\zeta-1}$	$2\xi + \zeta$	$4 - \zeta$	$2\xi - \zeta + 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+2\xi+\zeta+1}$	$2\xi - \zeta + 1$	$2 - \zeta$	$2\xi - \zeta - 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\beta-\zeta-1}$	$\zeta + 3$	$2\xi - \zeta - 2$	$\zeta + 1$	0	$0 \leq \zeta \leq 2\xi - 4$
$a_{\eta+\beta+\zeta+1}$	$\zeta + 2$	$2\xi + \zeta$	$\zeta + 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\beta+\zeta+3}$	$\zeta + 4$	$2\xi - \zeta + 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 3$
$a_{\eta+2\xi+\zeta+\beta+1}$	$2\xi - \zeta + 1$	$3 - \zeta$	$2\xi + 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+2\beta-\zeta}$	$\zeta + 2$	$2\xi - \zeta - 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 3$

Table 3: Representations of outer vertices

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	
$b_{\eta+\zeta+2}$	$\zeta + 4$	$2\xi - \zeta + 2$	$\zeta + 4$	0	$0 \leq \zeta \leq 2\xi - 3$
$b_{\eta+\zeta+2\xi}$	$2\xi + 2$	$4 - 2\zeta$	$2\xi - \zeta + 1$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\xi+\zeta+3}$	$2\xi - \zeta$	$2\zeta + 2$	$2\xi - \zeta - 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\xi+\zeta+5}$	$2\xi - \zeta - 2$	$\zeta + 5$	$2\xi - \zeta - 4$	0	$0 \leq \zeta \leq 2\xi - 7$
$b_{\eta+\beta-\zeta}$	$2\zeta + 1$	$2\xi + \zeta + 1$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+1}$	$2\zeta + 1$	$2\xi + \zeta + 1$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+3}$	$\zeta + 5$	$2\xi - \zeta + 2$	$\zeta + 4$	0	$0 \leq \zeta \leq 2\xi - 3$
$b_{\eta+2\xi+\zeta+\beta+1}$	$2\xi - \zeta + 2$	$3 - 2\zeta$	$2\xi - \zeta + 1$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\xi+\zeta+\beta+3}$	$2\xi - \zeta$	$2\zeta + 1$	$2\xi - \zeta - 1$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\beta-\zeta-1}$	$\zeta + 4$	$2\xi - \zeta - 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 6$

Table 4: Representations of inner vertices

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	
$a_{\eta+\zeta}$	$\zeta + 3$	$2\xi + \zeta + 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\zeta+2}$	$\zeta + 1$	$2\xi - \zeta + 1$	$\zeta + 5$	0	$0 \leq \zeta \leq 2\xi - 3$
$a_{\eta+2\xi+\zeta}$	$2\xi + \zeta - 1$	$3 - \zeta$	$2\xi - \zeta + 1$	0	$0 \leq \zeta \leq 2$
$a_{\eta+\beta-\zeta-1}$	$\zeta + 5$	$2\xi - \zeta - 1$	$\zeta + 1$	0	$0 \leq \zeta \leq 2\xi - 3$
$a_{\eta+\beta+\zeta+1}$	$3 - \zeta$	$2\xi + \zeta + 1$	$\zeta + 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\beta+\zeta+3}$	$\zeta + 2$	$2\xi - \zeta + 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 1$
$a_{\eta+2\beta-\zeta}$	$\zeta + 4$	$2\xi - \zeta$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 2$

Table 5: Representations of outer vertices

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	
$b_{\eta+\zeta}$	$4 - 2\zeta$	$2\xi + \zeta + 2$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\zeta+4}$	$\zeta + 4$	$2\xi - \zeta$	$\zeta + 6$	0	$0 \leq \zeta \leq 2\xi - 4$
$b_{\eta-\beta+\zeta}$	$\zeta + 5$	$2\xi - \zeta + 1$	$\zeta + 1$	0	$0 \leq \zeta \leq 2\xi - 3$
$b_{\eta+\beta+\zeta+1}$	$3 - 2\zeta$	$2\xi + \zeta + 2$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+3}$	$2\zeta + 1$	$2\xi - \zeta + 2$	$\zeta + 4$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+5}$	$\zeta + 5$	$2\xi - \zeta$	$\zeta + 6$	0	$0 \leq \zeta \leq 2\xi - 5$
$b_{\eta+\beta+2\xi+\zeta+1}$	$2\xi + \zeta + 1$	$3 - 2\zeta$	$2\xi - \zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+2\xi+\zeta+3}$	$2\xi - \zeta + 3$	$2\zeta + 1$	$2\xi - \zeta$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\beta-\zeta}$	$\zeta + 5$	$2\xi + \zeta + 1$	$\zeta + 2$	0	$0 \leq \zeta \leq 2\xi - 4$

Table 6: Representations of inner vertices

*Proof.* For  $\beta = 5$ , it is easy to see that  $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$  where  $\chi_1 = \{b_2\}$ ,  $\chi_2 = \{b_4\}$ ,  $\chi_3 = \{a_5\}$ ,  $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_2, b_4, a_5\}$  is a resolving partitioning for  $V(P_{11,5})$ .

For  $\beta \geq 5$ ,  $\beta \equiv 1(mod 4)$  and for the chosen index  $\eta$  such that  $0 \leq \eta \leq \alpha - 1$ , we shall show that  $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ , where  $\chi_1 = \{b_{\eta+2}\}$ ,  $\chi_2 = \{b_{\eta+2\xi+2}\}$ ,  $\chi_3 = \{a_{\eta+\beta}\}$ ,  $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_{\eta+2}, b_{\eta+2\xi+2}, a_{\eta+\beta}\}$  is a resolving partitioning for  $V(P_{\alpha,\beta})$ .

The representations of  $V(P_{\alpha,\beta})$ , are:  $r(b_{\eta+3}|\chi) = (2, 2\xi + 1, 5, 0)$ ,  $r(b_{\eta+2\xi+1}|\chi) = (2\xi + 1, 2, 2\xi + 1, 0)$ ,  $r(b_{\eta+2\xi+3}|\chi) = (2\xi + 3, 2, 2\xi - 1, 0)$  and in the Tables 5 and 6.

From these tables, one can see that all the vertices of  $P_{\alpha,\beta}$  lying in column 1 of Table 5 and Table 6 have distinct representations with respect to  $\chi$ . Thus  $\chi$  is resolving partitioning for  $P_{\alpha,\beta}$ . Hence, for all  $\alpha = 2\beta + 1$ ,  $\beta \geq 5$  and  $\beta \equiv 1(mod 4)$ ,  $pd(P_{\alpha,\beta}) \leq 4$ . □

**Theorem 2.3.** *Let  $P_{\alpha,\beta}$  be generalized Petersen graphs with  $\alpha = 2\beta + 1 \geq 2$  and  $\beta \equiv 2(mod 4)$ , then  $pd(P_{\alpha,\beta}) \leq 4$ .*

*Proof.* For  $\beta = 6$ , it is easy to see from the Tables 7 and 7 that  $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$  where  $\chi_1 = \{b_3\}$ ,  $\chi_2 = \{b_4\}$ ,  $\chi_3 = \{a_7\}$ ,  $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_3, b_4, a_7\}$  is a resolving partitioning for  $V(P_{13,6})$ .

For  $\beta = 10$ , it is easy to see from the Tables 9 and 10 that  $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$  where  $\chi_1 = \{b_3\}$ ,  $\chi_2 = \{b_6\}$ ,  $\chi_3 = \{a_{11}\}$ ,  $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_3, b_6, a_{11}\}$  is a resolving partitioning for  $V(P_{21,10})$ .

For  $\beta \geq 10$ ,  $\beta \equiv 2(mod 4)$  and for the chosen index  $\eta$  such that  $0 \leq \eta \leq \alpha - 1$ , we shall show that  $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$  where  $\chi_1 = \{b_{\eta+3}\}$ ,  $\chi_2 = \{b_{\eta+2\xi+2}\}$ ,  $\chi_3 = \{a_{\eta+\beta+1}\}$ ,  $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_{\eta+3}, b_{\eta+2\xi+2}, a_{\eta+\beta+1}\}$  is a resolving partitioning for  $V(P_{\alpha,\beta})$ , where  $\xi = \frac{\beta-2}{4}$ .

For  $\beta \geq 11$ ,  $\beta \equiv 2(mod 4)$  the representations of the vertices of  $V(P_{\alpha,\beta})$ , are:  $r(a_{\eta}|\chi) = (4, 2\xi + 2, 3, 0)$ ,  $r(b_{\eta}|\chi) = (5, 2\xi + 3, 2, 0)$ ,  $r(b_{\eta+4}|\chi) = (2, 2\xi, 5, 0)$ ,  $r(b_{\eta+2\xi+1}|\chi) = (2\xi, 2, 2\xi + 2, 0)$ ,  $r(b_{\eta+2\xi+3}|\chi) = (2\xi + 2, 2, 2\xi + 1, 0)$  and in the Tables 11 and 12.

From these tables, one can see that all the vertices of  $P_{\alpha,\beta}$  lying in column 1 of Table 1 and Table 12 have distinct representations with respect to  $\chi$ . Thus  $\chi$  is resolving partitioning for  $P_{\alpha,\beta}$ . Hence, for all

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$
$a_0$	4	4	3	0	$a_1$	3	4	3	0	$a_2$	2	3	4	0
$a_3$	1	2	4	0	$a_4$	2	1	3	0	$a_5$	3	2	2	0
$a_6$	4	3	1	0	$a_8$	3	4	1	0	$a_9$	2	3	2	0
$a_{10}$	2	2	3	0	$a_{11}$	3	2	4	0	$a_{12}$	4	3	4	0

Table 7: Representations of outer vertices

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$
$b_0$	5	5	2	0	$b_1$	4	5	2	0	$b_2$	2	4	3	0
$b_5$	4	2	3	0	$b_6$	5	4	2	0	$b_7$	5	5	1	0
$b_8$	3	5	2	0	$b_9$	1	3	3	0	$b_{10}$	1	1	4	0
$b_{11}$	3	1	4	0	$b_{12}$	5	3	3	0					

Table 8: Representations of inner vertices

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$
$a_0$	4	6	3	0	$a_1$	3	6	3	0
$a_2$	2	5	4	0	$a_3$	1	4	5	0
$a_4$	2	3	6	0	$a_5$	3	2	6	0
$a_6$	4	1	5	0	$a_7$	5	2	4	0
$a_8$	6	3	3	0	$a_9$	6	4	2	0
$a_{10}$	5	5	1	0	$a_{12}$	3	6	1	0
$a_{13}$	2	5	2	0	$a_{14}$	2	4	3	0
$a_{15}$	3	3	4	0	$a_{16}$	4	2	5	0
$a_{17}$	5	2	6	0	$a_{18}$	6	3	6	0
$a_{19}$	6	4	5	0	$a_{20}$	5	5	4	0

Table 9: Representations of outer vertices

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$
$b_0$	5	7	2	0	$b_1$	4	7	2	0
$b_2$	2	6	3	0	$b_4$	2	4	5	0
$b_5$	4	2	6	0	$b_7$	6	2	5	0
$b_8$	7	4	4	0	$b_9$	7	5	3	0
$b_{10}$	6	6	2	0	$b_{11}$	5	7	1	0
$b_{12}$	3	7	2	0	$b_{13}$	1	6	3	0
$b_{14}$	1	5	4	0	$b_{15}$	3	3	5	0
$b_{16}$	5	1	6	0	$b_{17}$	6	1	6	0
$b_{18}$	7	3	5	0	$b_{19}$	7	5	4	0
$b_{20}$	6	6	3	0					

Table 10: Representations of inner vertices

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	
$a_{\eta+\zeta+1}$	$\zeta + 3$	$2\xi - \zeta + 2$	$\zeta + 3$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\zeta+3}$	$\zeta + 1$	$2\xi - \zeta$	$\zeta + 5$	0	$0 \leq \zeta \leq 2\xi - 3$
$a_{\eta+2\xi+\zeta+1}$	$2\xi + \zeta - 1$	$2 - \zeta$	$2\xi - \zeta + 2$	0	$0 \leq \zeta \leq 1$
$a_{\eta+2\xi+\zeta+3}$	$2\xi + \zeta + 1$	$\zeta + 2$	$2\xi - \zeta$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\beta-\zeta}$	$\zeta + 5$	$2\xi - \zeta + 1$	$\zeta + 1$	0	$0 \leq \zeta \leq 2\xi - 3$
$a_{\eta+\beta+\zeta+2}$	$3 - \zeta$	$2\xi - \zeta + 2$	$\zeta + 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\beta+\zeta+4}$	$\zeta + 2$	$2\xi - \zeta$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 2$
$a_{\eta+\beta+\zeta+2\xi+3}$	$2\xi + \zeta + 1$	$\zeta + 2$	$2\xi + 2$	0	$0 \leq \zeta \leq 1$
$a_{\eta+2\beta-\zeta}$	$\zeta + 5$	$2\xi - \zeta + 1$	$\zeta + 4$	0	$0 \leq \zeta \leq 2\xi - 3$

Table 11: Representations of outer vertices

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	
$b_{\eta+\zeta+1}$	$4 - 2\zeta$	$2\xi - \zeta + 3$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\zeta+5}$	$\zeta + 4$	$2\xi - \zeta - 1$	$\zeta + 6$	0	$0 \leq \zeta \leq 2\xi - 5$
$b_{\eta+2\xi+\zeta+4}$	$2\xi + 3$	$\zeta + 4$	$2\xi - \zeta$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta-\zeta+1}$	$\zeta + 5$	$2\xi - \zeta + 3$	$\zeta + 1$	0	$0 \leq \zeta \leq 2\xi - 3$
$b_{\eta+\beta+\zeta+2}$	$3 - 2\zeta$	$2\xi - \zeta + 3$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+4}$	$2\zeta + 15$	$2\xi - \zeta + 1$	$\zeta + 4$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+6}$	$\zeta + 5$	$2\xi - \zeta - 1$	$\zeta + 6$	0	$0 \leq \zeta \leq 2\xi - 6$
$b_{\eta+\beta+\zeta+1+2\xi}$	$2\xi + \zeta$	$3 - 2\zeta$	$2\xi + \zeta + 1$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+3+2\xi}$	$2\xi + \zeta + 2$	$2\zeta + 1$	$2\xi - \zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\beta-\zeta}$	$\zeta + 6$	$2\xi - \zeta + 2$	$2\xi - \zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 3$

Table 12: Representations of inner vertices

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$
$a_0$	2	5	4	0	$a_1$	3	4	5	0	$a_2$	4	3	4	0
$a_3$	5	2	3	0	$a_4$	4	1	2	0	$a_5$	3	2	1	0
$a_7$	2	4	1	0	$a_8$	3	5	2	0	$a_9$	4	1	3	0
$a_{10}$	5	3	4	0	$a_{11}$	4	2	5	0	$a_{12}$	3	2	4	0
$a_{13}$	2	3	3	0	$a_{14}$	1	4	3	0					

Table 13: Representations of outer vertices

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$
$b_0$	2	6	3	0	$b_1$	4	5	4	0	$b_2$	5	4	5	0
$b_3$	6	2	4	0	$b_5$	3	2	2	0	$b_6$	1	4	1	0
$b_7$	1	5	2	0	$b_8$	3	6	3	0	$b_9$	5	5	4	0
$b_{10}$	6	3	5	0	$b_{11}$	5	1	4	0	$b_{12}$	4	1	3	0
$b_{13}$	2	3	4	0										

Table 14: Representations of inner vertices

$\alpha = 2\beta + 1, \beta \geq 2$  and  $\beta \equiv 2(mod 4), pd(P_{\alpha,\beta}) \leq 4$ . □

**Theorem 2.4.** Let  $P_{\alpha,\beta}$  be generalized Petersen graphs with  $\alpha = 2\beta + 1 \geq 3$  and  $\beta \equiv 3(mod 4)$ , then  $pd(P_{\alpha,\beta}) \leq 4$ .

*Proof.* For  $\beta = 6$ , it is easy to see that  $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$  where  $\chi_1 = \{b_2\}, \chi_2 = \{b_6\}, \chi_3 = \{a_2\}, \chi_4 = V(P_{\alpha,\beta}) \setminus \{b_2, b_6, a_2\}$  is a resolving partitioning for  $V(P_{7,3})$ .

For  $\beta \geq 7, \beta \equiv 3(mod 4)$  and for the chosen index  $\eta$  such that  $0 \leq \eta \leq \alpha - 1$ , we shall show that  $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$  where  $\chi_1 = \{b_{\eta+2\beta}\}, \chi_2 = \{b_{\eta+2\xi+2}\}, \chi_3 = \{a_{\eta+\beta-1}\}, \chi_4 = V(P_{\alpha,\beta}) \setminus \{b_{\eta+2\beta}, b_{\eta+2\xi+2}, a_{\eta+\beta-1}\}$  is a resolving partitioning for  $V(P_{\alpha,\beta})$ , where  $\xi = \frac{\beta-3}{4}$ .

For  $\beta = 7$  the representations of the vertices of  $V(P_{15,7})$ , are in the Tables 13 and 14. For  $\beta > 7, \beta \equiv 3(mod 4)$  the representations of the vertices of  $V(P_{\alpha,\beta})$ , are:  $r(a_{\eta+\beta+2\xi+2}|\chi) = (2\xi + 2, 2, 2\xi + 3, 0), r(a_{\eta+2\beta}|\chi) = (1, 2\xi + 2, 3, 0), r(b_{\eta}|\chi) = (2, 2\xi + 4, 3, 0) r(b_{\eta+2\xi+1}|\chi) = (2\xi + 4, 2, 2\xi + 2, 0) r(b_{\eta+2\xi+3}|\chi) = (2\xi + 2, 2, 2\xi, 0), r(b_{\eta+2\beta-1}|\chi) = (2, 2\xi + 2, 2, 0)$  and in the Tables 15 and 16.

From these tables, one can see that all the vertices of  $P_{\alpha,\beta}$  lying in column 1 of Table 15 and Table 16 have distinct representations with respect to  $\chi$ . Thus  $\chi$  is resolving partitioning for  $P_{\alpha,\beta}$ . Hence, for all  $\alpha = 2\beta + 1, \beta \geq 7$  and  $\beta \equiv 3(mod 4), pd(P_{\alpha,\beta}) \leq 4$ . □

$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	
$a_{\eta+\zeta}$	$\zeta + 2$	$2\xi - \zeta + 3$	$\zeta + 4$	0	$0 \leq \zeta \leq 2\xi - 1$
$a_{\eta+2\xi+\zeta}$	$2\xi + \zeta + 2$	$3 - \zeta$	$2\xi - \zeta + 4$	0	$0 \leq \zeta \leq 1$
$a_{\eta+2\xi+\zeta+2}$	$2\xi - \zeta + 2$	$\zeta + 1$	$2\xi - \zeta$	0	$0 \leq \zeta \leq 2\xi - 1$
$a_{\eta+\beta+\zeta}$	$\zeta + 2$	$2\xi + \zeta + 2$	$\zeta + 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\beta+\zeta+2}$	$\zeta + 4$	$2\xi - \zeta + 2$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 1$
$a_{\eta+2\beta-\zeta-1}$	$\zeta + 2$	$2\xi - \zeta + 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 1$

Table 15: Representations of outer vertices



$r(\cdot \chi)$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	
$b_{\eta+\zeta+1}$	$\zeta + 1$	$2\xi - \zeta + 3$	$\zeta + 4$	0	$0 \leq \zeta \leq 2\xi - 1$
$b_{\eta+2\xi+\zeta+4}$	$2\xi - \zeta + 1$	$\zeta + 4$	$2\xi - \zeta - 1$	0	$0 \leq \zeta \leq 2\xi - 4$
$b_{\eta+\beta-\zeta-1}$	$2\zeta + 1$	$2\xi - \zeta + 2$	$\zeta + 1$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta}$	$2\zeta + 1$	$2\xi + \zeta + 3$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+2}$	$\zeta + 5$	$2\xi - \zeta + 3$	$\zeta + 4$	0	$0 \leq \zeta \leq 2\xi - 2$
$b_{\eta+2\xi+\beta+\zeta+1}$	$2\xi - \zeta + 4$	$3 - 2\zeta$	$2\xi - \zeta + 3$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\xi+\beta+\zeta+3}$	$2\xi - \zeta + 2$	$2\zeta + 1$	$2\xi - \zeta + 1$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\beta-\zeta-2}$	$\zeta + 4$	$2\xi - \zeta + 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 4$

Table 16: Representations of inner vertices

### 3. Harary Graph $H_{\psi,\lambda}$

Harary  $H_{\psi,\lambda}$ , is an  $\psi$ -regular graph with order  $\lambda$  the vertex set  $V(H_{\psi,\lambda}) = \{\beta_\epsilon : \epsilon = 1, 2, \dots, \lambda\}$ , if  $\psi$  is even then  $\psi = 2\xi \leq \lambda - 1$  for some integer  $\xi \leq \frac{\lambda-1}{2}$ . For each  $\epsilon$  ( $1 \leq \epsilon \leq \lambda$ ), we join  $\beta_\epsilon$  to  $\beta_{\epsilon+1}, \beta_{\epsilon+2}, \dots, \beta_{\epsilon+\xi}$  and to  $\beta_{\epsilon-1}, \beta_{\epsilon-2}, \dots, \beta_{\epsilon-\xi}$ . If we think of arranging of the vertices  $\beta_1, \beta_2, \dots, \beta_\lambda$  cyclically then each vertex  $\beta_\epsilon$  is adjacent to the  $\xi$  vertices that immediately follow  $\beta_\epsilon$  and the  $\xi$  vertices that immediately precede  $\beta_\epsilon$ . Following are some bounds on the partition dimension of Harary graph  $H_{\psi,\lambda}$ .

**Theorem 3.1.** *Let  $H_{\psi,\lambda}$  is a Harary graph with  $\lambda \geq 5, \lambda \equiv 0, 2, 3(mod \psi)$  and  $\psi = 4$ . Then the partition dimension of  $H_{\psi,\lambda}$  is  $\leq 4$ .*

*Proof.* To prove  $pd(H_{\psi,\lambda}) \leq 4$  we split the proof into following cases;

**Case 1:**  $\lambda \equiv 0(mod 4), \lambda = 4\xi, \xi(\geq 2) \in \mathbb{Z}^+$ .

Assume the partition resolving set  $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$  where  $\chi_1 = \{\beta_1\}, \chi_2 = \{\beta_3\}, \chi_3 = \{\beta_{\lambda-2}\}, \chi_4 = V(H_{\psi,\lambda}) \setminus \{\beta_1, \beta_3, \beta_{\lambda-2}\}$  following are the representations of the entire vertex set of  $H_{\psi,\lambda}$  with respect to  $\chi$ .

$$r(\beta_{2\epsilon}|\chi) = \begin{cases} (\epsilon, \epsilon - 1, \epsilon - 1, 0) & \text{if } \epsilon = 2, 3, \dots, \xi; \\ (\xi, \xi, \xi, 0) & \text{if } \epsilon = \xi + 1; \\ (2\xi - \epsilon + 1, 2\xi - \epsilon + 1, 2\xi - \epsilon + 1, 0) & \text{if } \epsilon = \xi + 2, \dots, 2\xi. \end{cases}$$

$$r(\beta_{2\epsilon+1}|\chi) = \begin{cases} (\epsilon, \epsilon, \epsilon - 1, 0) & \text{if } \epsilon = 2, 3, \dots, \xi; \\ (2\xi - \epsilon, 2\xi - \epsilon + 1, 2\xi - \epsilon + 1, 0) & \text{if } \epsilon = \xi + 1, \dots, 2\xi - 1. \end{cases}$$

**Case 2:**  $\lambda \equiv 2(mod 4), \lambda = 4\xi + 2, \xi \in \mathbb{Z}^+$ .

Assume the partition resolving set  $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$  where  $\chi_1 = \{\beta_1\}, \chi_2 = \{\beta_2\}, \chi_3 = \{\beta_3\}, \chi_4 = V(H_{\psi,\lambda}) \setminus \{\beta_1, \beta_2, \beta_3\}$  following are the representations of the entire vertex set of  $H_{\psi,\lambda}$  with respect to  $\chi$ .

$$r(\beta_{2\epsilon}|\chi) = \begin{cases} (\epsilon, \epsilon - 1, \epsilon - 1, 0) & \text{if } \epsilon = 2, 3, \dots, \xi + 1; \\ (2\xi - \epsilon + 2, 2\xi - \epsilon + 2, 2\xi - \epsilon + 3, 0) & \text{if } \epsilon = \xi + 2, \dots, 2\xi + 1. \end{cases}$$

$$r(\beta_{2\epsilon+1}|\chi) = \begin{cases} (\epsilon, \epsilon, \epsilon - 1, 0) & \text{if } \epsilon = 2, 3, \dots, \xi; \\ (\xi, \xi + 1, \xi, 0) & \text{if } \epsilon = \xi + 1; \\ (2\xi - \epsilon + 1, 2\xi - \epsilon + 2, 2\xi - \epsilon + 2, 0) & \text{if } \epsilon = \xi + 2, \dots, 2\xi. \end{cases}$$

**Case 3:**  $\lambda \equiv 3(\text{mod } 4)$ ,  $\lambda = 4\xi + 3$ ,  $\xi \in \mathbb{Z}^+$ .

Assume the partition resolving set  $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$  where  $\chi_1 = \{\beta_1\}$ ,  $\chi_2 = \{\beta_2\}$ ,  $\chi_3 = \{\beta_3\}$ ,  $\chi_4 = V(H_{\psi, \lambda}) \setminus \{\beta_1, \beta_2, \beta_3\}$  following are the representations of the entire vertex set of  $H_{\psi, \lambda}$  with respect to  $\chi$ .

$$r(\beta_{2\epsilon}|\chi) = \begin{cases} (\epsilon, \epsilon - 1, \epsilon - 1, 0) & \text{if } \epsilon = 2, 3, \dots, \xi + 1; \\ (2\xi - \epsilon + 2, 2\xi - \epsilon + 3, 2\xi - \epsilon + 3, 0) & \text{if } \epsilon = \xi + 2, \dots, 2\xi + 1. \end{cases}$$

$$r(\beta_{2\epsilon+1}|\chi) = \begin{cases} (\epsilon, \epsilon, \epsilon - 1, 0) & \text{if } \epsilon = 2, 3, \dots, \xi + 1; \\ (2\xi - \epsilon + 2, 2\xi - \epsilon + 2, 2\xi - \epsilon + 3, 0) & \text{if } \epsilon = \xi + 2, \dots, 2\xi + 1. \end{cases}$$

The entire vertex set of  $H_{\psi, \lambda}$  w.r.t. to the partition resolving set  $\chi$  have distinct representations hence,  $pd(H_{\psi, \lambda}) \leq 4$ .  $\square$

#### 4. Conclusion

In this paper provide the sharp bounds of the partition dimension for the generalized Peterson graph  $P_{\alpha, \beta}$ , and we also studied the partition dimension of Harary graph  $H_{\psi, \lambda}$ , and we concluded that

$$pd(P_{\alpha, \beta}) = pd(H_{\psi, \lambda}) \leq 4.$$

Theorem 2.1 to 2.4 discussed the partition dimension of the generalized Peterson graph when  $\beta \equiv 0, 1, 2, 3(\text{mod } 4)$ , respectively and the partitioning of Harary graph studied in the Theorem 3.1.

#### References

- [1] Azmi, S., Soeprianto, H., Turmuzi, M., & Anwar, Y. S. (2019, November). The partition dimension of subdivision graph on the star. In *Journal of Physics: Conference Series* (Vol. 1280, No. 2, p. 022037). IOP Publishing.
- [2] Baskoro, E. T., & Haryeni, D. O. (2020). All graphs of order  $n = 11$  and diameter 2 with partition dimension  $n - 3$ . *Heliyon*, 6(4), e03694. 1
- [3] Beerliova, Z., Eberhard, F., Erlebach, T., Hall, A., Hoffmann, M., Mihal'ak, M., & Ram, L. S. (2006). Network discovery and verification. *IEEE Journal on Selected Areas in Communications*, 24(12), 2168-2181. 1
- [4] Cceres, J., Hernando, C., Mora, M., Pelayo, I. M., Puertas, M. L., Seara, C., & Wood, D. R. (2007). On the metric dimension of cartesian products of graphs. *SIAM Journal on Discrete Mathematics*, 21(2), 423-441. 1
- [5] Chartrand, G., Salehi, E., & Zhang, P. (2000). The partition dimension of a graph. *Aequationes Mathematicae*, 59(1), 45-54. 1
- [6] Chu, Y. M., Nadeem, M. F., Azeem, M., & Siddiqui, M. K. (2020). On sharp bounds on partition dimension of Convex polytopes. *IEEE Access*, 8, 224781-224790. 1, 1.5
- [7] Coxeter, H. S. (1950). Self-dual configurations and regular graphs. *Bulletin of the American Mathematical Society*, 56(5), 413-455. 1
- [8] Liu, J. B., Nadeem, M. F., & Azeem, M. (2020). Bounds on the partition dimension of convex polytopes. *Combinatorial chemistry & high throughput screening*, 10.2174/1386207323666201204144422.
- [9] Chvtal, V. (1983). Mastermind. *Combinatorica*, 3(3-4), 325-329. 1
- [10] Nadeem, M. F., Azeem, M., & Khalil, A. (2021). The locating number of hexagonal Mbius ladder network. *Journal of Applied Mathematics and Computing*, 66(1), 149-165. 1
- [11] Grigorious, C., Stephen, S., Rajan, B., & Miller, M. (2017). On the partition dimension of circulant graphs. *The Computer Journal*, 60(2), 180-184. 1
- [12] Harary, F., & Melter, R. A. (1976). On the metric dimension of a graph. *Ars Combinatoria*, 2, 191-195. 1
- [13] Hauptmann, M., Schmied, R., & Viehmann, C. (2012). Approximation complexity of metric dimension problem. *Journal of Discrete Algorithms*, 14, 214-222. 1
- [14] Hussain, Z., Kang, S. M., Rafique, M., Munir, M., Ali, U., Zahid, A., & Saleem, M. S. (2019). Bounds for partition dimension of M-wheels. *Open Physics*, 17(1), 340-344. 1
- [15] Imran, M., Siddiqui, M. K., & Naeem, R. (2018). On the metric dimension of generalized petersen multigraphs. *IEEE Access*, 6, 74328-74338. 1

- [16] Javaid, I., Rahim, M. T., & Ali, K. (2008). Families of regular graphs with constant metric dimension. *Utilitas Mathematica*, 75, 21-34. 1
- [17] Khuller, S., Raghavachari, B., & Rosenfeld, A. (1996). Landmarks in graphs. *Discrete Applied Mathematics*, 70(3), 217-229. 1
- [18] Hartmanis, J. (1982). Computers and intractability: a guide to the theory of np-completeness (Michael R. Garey and David S. Johnson). *Siam Review*, 24(1), 90. 1
- [19] Maritz, E. C., & Vetrk, T. (2018). The partition dimension of circulant graphs. *Quaestiones Mathematicae*, 41(1), 49-63. 1
- [20] Mehreen, N., Farooq, R., & Akhter, S. (2018). On partition dimension of fullerene graphs. *AIMS Mathematics*, 3(3), 343-352. 1
- [21] Melter, R. A., & Tomescu, I. (1984). Metric bases in digital geometry. *Computer Vision, Graphics, and Image Processing*, 25(1), 113-121. 1
- [22] Chris Monica, M., & Santhakumar, S. (2016). Partition dimension of honeycomb derived networks. *International Journal of Pure and Applied Mathematics*, 108(4), 809-818. 1
- [23] Naz, S., Salman, M., Ali, U., Javaid, I., & Bokhary, S. A. U. H. (2014). On the constant metric dimension of generalized Petersen graphs  $P(n, 4)$ . *Acta Mathematica Sinica, English Series*, 30(7), 1145-1160. 1
- [24] Rehman, T. U., & Mehreen, N. (2020). Partition dimension and strong metric dimension of chain cycle. *Jordan Journal of Mathematics and Statistics*, 13(2), 305-325. 1
- [25] Rodr`uez-Vel`azquez, J. A., Yero, I. G., & Lemanska, M. (2014). On the partition dimension of trees. *Discrete Applied Mathematics*, 166, 204-209 1
- [26] Fernau, H., Rodr`uez-Vel`azquez, J. A., & Yero, I. G. (2014). On the partition dimension of unicyclic graphs. *Bulletin mathmatique de la Soci`et`e des Sciences Math`ematiques de Roumanie*, 381-391. 1
- [27] Slater, P. J. (1975). Leaves of trees. *Congress Numerentium*, 14, 549-559. 1
- [28] Vertana, H., & Kusmayadi, T. A. (2016). On the partition dimension of  $C_m + P_n$  graph. *Journal of Physics: Conference Series*, 855. 1
- [29] Watkins, M. E. (1969). A theorem on Tait colorings with an application to the generalized Petersen graphs. *Journal of Combinatorial Theory*, 6(2), 152-164. 1
- [30] Yero, I. G., & Rodr`uez-Vel`azquez, J. A. (2010). A note on the partition dimension of Cartesian product graphs. *Applied Mathematics and Computation*, 217(7), 3571-3574.