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Journal of Prime Research in Mathematics, 17(1), (2021), 1–6

Journal of Prime Research in Mathematics

Quantum Painlevé II solution with approximated analytic solution in form of nearly Yukawa potential

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Abstract

In this article it has been shown that one dimensional non-stationary Schrödinger equation with a specific choice of potential reduces to the quantum Painlevé II equation and the solution of its riccati form appears as a dominant term of that potential. Further, we show that Painlevé II Riccati solution is an equivalent representation of centrifugal expression of radial Schrödinger potential. This expression is used to derive the approximated to the Yukawa potential of radial Schrödinger equation which can be solved by applying the Nikiforov-Uvarov method. Finally, we express the approximated form of Yukawa potential explicitly in terms of quantum Painlevé II solution.

Keywords: Quantum Painlevé II equation; Riccati equation; Yukawa potential.

1. Introduction

The Painlevé six equations (Painlevé I-VI) were discovered by Painlevé and his colleagues while classifying the nonlinear second-order ordinary differential equations with respect to their solutions [1]. The study of Painlevé equations is important due to the wide applications of these equations in various areas of mathematics and physics. For example, in hydrodynamics and plasma physics the Painlevé equations are usually obtained as reduced ODEs of some PDEs that describe the evolution of flows or convective flows with viscous dissipation [2]. In nonlinear optics, the nonlinear Schrödinger equations play an important role to explain the wave propagation in media, the ODE reduction of this equation is Painlevé IV equation [2]. It was shown that the description of two dimensional quantum gravity involves Painlevé I equation [3, 4, 5] and one of the applications of Painlevé II equation has been studied in [6] for the wave collapse in the three-dimensional nonlinear Schrödinger equation.

Further it has been shown that the exactly solvable models of statistical physics and the quantum field theory can be described in terms of Painlevé transcendents [7, 8, 9]. The classical Painlevé equations are regarded as completely integrable equations and obeyed the Painlevé test [10, 11, 12]. These equations admit some properties such as linear representations, hierarchies, they possess Darboux transformations(DTs) and Hamiltonian structure. These equations also arise as ordinay differential equations (ODEs) reduction of some integrable systems, i.e, the ODE reduction of the KdV equation is Painlevé II equation [13] and [14].

Received : Received: 24 September 2020*; Accepted:* 2 February 2021*; Published Online:* 10 March 2021.

It seems quite interesting to derive the quantum analogue of these equation and to explore their physical aspects in various areas of physics and mathematics.

In this paper, we construct a connection of quantum Painlevé II equation [15] and its Riccati solution to the quantum mechanical system that involves the approximated form of the Yukawa potential. We show that one dimensional non-stationary Schrödinger equation with a specific choice of potential reduces to the quantum Painlevé II equation and the dominant term of that potential can be obtained as the solution of quantum Pianlevé II Riccati equation. Further we observe that quantum Painlevé II solution coincides with centrifugal expression [16] of the radial schrödinger potential that expression was applied to derive the approximated form of Yukawa potential [17, 18, 19].

2. Quantum Painlevé II equation and its Riccati form

The quantum Painlevé II equation in the following form

$$\begin{cases} f^{''} = 2f^3 - 2[z, f]_+ + c \\ zf - fz = i\hbar f \end{cases} \tag{2.1}$$

can be derived from the compatibility condition of following linear system

$$\Psi_\lambda = A(z;\lambda)\Psi, \quad \Psi_z = B(z;\lambda)\Psi \tag{2.2}$$

and the Lax matrices A and B are given by

$$\begin{cases} A = (8i\lambda^2 + if^2 - 2iz)\sigma_3 + f^{'}\sigma_2 + (\frac{1}{4}c\lambda^{-1} - 4\lambda f)\sigma_1 + i\hbar\sigma_2 \\ B = -2i\lambda\sigma_3 + f\sigma_1 + fI \end{cases} \tag{2.3}$$

where $\hbar$ is Planck constant and σ_1, σ_2, σ_3 are Pauli spin matrices and $[z, f]_+$ is anti-commutator of z and f. The quantum Painlevé II equation (2.1) was derived in [15] by taking the quantum commutation relations [20] as a kind of quantization for Noumi-Yamada Painlevé II system [21]. This quatization can be taken aa a particular case of deformed space that involves the star product and the version of pure non-commutative Painlevé II under the star product has been derived by V. Retakh and V. Roubtsov [22]. By using the commutation relation $zf - fz = i\hbar f$ the first equation in system (2.1) can be written as

$$f^{''} = 2f^3 - 4zf - 2i\hbar f + c. \tag{2.4}$$

The quantum Painlevé II equation (2.4) reduces to the ordinary Painlevé II equation under the classical limit when $\hbar \to 0$. Substituting the eigenvector $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ in linear system (2.1) and setting $\mathit{\Delta} = \psi_1\psi_2^{-1}$, we obtain the quantum Painlevé II equation in Riccati form as follow

$$\mathit{\Delta}^{'} = -4i\mathit{\Delta} + f + [f, \mathit{\Delta}]_- - \mathit{\Delta} f \mathit{\Delta}. \tag{2.5}$$

In section 5 we will show that how the Riccati form (2.5) of the Painlevé II equation helpful to yield the approximated solution of the Yukawa potential.

3. Non-Stationary Shrödinger equation and Quantum Painlevé II equation

In this section we will observe that how the quantum Painlevé II equation is connected to the quantum system described by one dimensional non-stationary Schrödinger equation with a specific choice of potential $V(x, t)$ and eigenfunction $\psi(x, t)$ that has been explained in the following proposition.

Proposition 3.1. *The time dependent Shrödinger equation for a particle of mass m with potential $V(x,t)$*

$$-i\hbar\frac{\partial\psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+V(x,t)\psi(x,t) \tag{3.1}$$

reduces to quantum Painlevé II equation (2.4) at constant $c=0$ with the choice of potential $V(x,t)$ as follow

$$V(x,t)=\gamma x-2\psi^*(x,t)\psi(x,t). \tag{3.2}$$

where $\gamma=4i(\frac{2m}{\hbar})^{\frac{1}{2}}$ and the eigenfunction

$$\psi(x,t)=f(z,\lambda)e^{i\alpha t},\quad z=i(\frac{2m}{\hbar^2})^{\frac{1}{2}}x. \tag{3.3}$$

The function $f(z,\lambda)$ satisfies the quantum Painlevé II equation (2.4).

Proof. For the Schrödinger equation (3.1) we can easily evaluate $\frac{\partial\psi(x,t)}{\partial t}$ and $\frac{\partial^2\psi(x,t)}{\partial x^2}$ in terms of z variable as follow

$$\frac{\partial\psi(x,t)}{\partial t}=i\alpha f(z,\lambda)e^{i\alpha t} \tag{3.4}$$

and

$$\frac{\partial^2\psi(x,t)}{\partial x^2}=-\frac{2m}{\hbar^2}f^{''}(z,\lambda)e^{i\alpha t}. \tag{3.5}$$

Now substituting the potential from (3.2) and using the results from (3.4) and (3.5) in Schrödinger equation (3.1). After simplifications, we obtain the following expression

$$f^{''}=2f^3-4zf-\alpha\hbar f. \tag{3.6}$$

The expression (3.6) represents quantum Painlevé II equation (2.4) at constant $c=0$ and $\alpha=2i$. □

In the following section we construct the solutions to the Painlevé II equation and its Riccati form which are used to derive the approximated solution to the Yukawa potential in terms of quantum Painlevé II variable z and its parameter λ.

4. Solution to the Quantum Painlevé II Riccati equation

This section consists the derivation of solution for the quantum Painlevé II in Riccati form (2.5) that has been described in Proposition 4.1 and further, in next section we will observe that how this Riccati solution coincides to the centrifugal expression [16] and it is helpful to construct approximated analytic solution for the radial Schrödinger equation with the Yukawa potential [17, 18, 19].

Proposition 4.1. *We can show that the following choice of $f(z;\lambda)$ and $\varDelta$*

$$\begin{cases} f=\beta(1-e^{-8\lambda z})^{-1}e^{-4\lambda z}\\ \varDelta=e^{4\lambda z}\end{cases} \tag{4.1}$$

satisfies quantum Painlevé II Riccati equation (2.5) where β is a complex parameter.

Proof. By using system (4.1), we can show that

$$\varDelta f\varDelta=\beta e^{4\lambda z}(1-e^{-8\lambda z})^{-1},\ [f,\varDelta]_-=0. \tag{4.2}$$

Now after using the values of $f(z;\lambda)$ and $\varDelta$ from (4.1) in Riccati equation (2.5), we get

$$4\lambda e^{4\lambda z}-4\lambda e^{-4\lambda z}=-4ie^{4\lambda z}+4ie^{-4\lambda z}+\beta e^{-4\lambda z}-\beta e^{4\lambda z}$$

or

$$4(\lambda+i)e^{4\lambda z}-4(\lambda+i)e^{-4\lambda z}=\beta e^{-4\lambda z}-\beta e^{4\lambda z}. \tag{4.3}$$

We can show that system (4.1) satisfies the Riccati equation (2.5) by taking $\beta=-4(\lambda+i)$ in equation (4.3). □

5. Quantum Painlevé II solution and Yukawa potential approximation

In this section, we show that the quantum Painlevé II solution f coincides with the centrifugal expression of [16] and finally we express an approximated form of Yukawa potential [17, 18, 19] in terms of quantum Painlevé II solution f. The explicit expression of Yukawa Potential [23] in radial form can be expressed as follow

$$V(r) = -V_0 \frac{e^{-2ar}}{r}, \tag{5.1}$$

where $V_0 = \alpha Z$, $\alpha = (137.037)^{-1}$ is the fine-structure constant and Z is the atomic number of neutral atom. This potential can be applied to evaluate the normalized bound-state and the energy levels of neutral atoms. In following Proposition **??** we have described a procedure to express approximated form to the Yukawa potential in terms of Painlevé II solution f.

Proposition 5.1. **??** *The potential (3.2) can be expressed in terms of z and spectral parameter λ as follows*

$$V(z, \lambda) = 4z - 2f^2 = 4z - 2\beta^2(e^{-4\lambda z} - e^{4\lambda z})^{-2} \tag{5.2}$$

and a solution to the non-stationary Schrödinger equation (3.1) can written as

$$\psi(x, t) = \beta(1 - e^{-8\lambda z})^{-1} e^{-4\lambda z + i\alpha t}, \tag{5.3}$$

where $\beta^2 = \beta^* \beta$. The quantum Painlevé II solution f in the potential (5.2) is equivalent to the centrifugal term of the radial Schrödinger equation considered in [16] applied to derive the approximated form of the Yukawa potential [17, 18, 19]. Later can be sued to solve radial Schrödinger equation by applying Nikiforov-Uvarov method.

Proof. The dominant term f^2 in potential (5.2) represents the square of quantum Painlmevé II solution and explicitly can be written as and given by

$$f^2 = \beta^2(e^{-4\lambda z} - e^{4\lambda z})^{-1} \tag{5.4}$$

or

$$f^2 = \beta^2 \frac{e^{-8\lambda z}}{(1 - e^{-8\lambda z})^2}. \tag{5.5}$$

The expression (5.4) is equivalent to the following centrifugal term [16] of radial Schrödinger equation

$$\frac{1}{r^2} \approx 4a^2 \frac{e^{-2ar}}{(1 - e^{-2ar}})^2, \tag{5.6}$$

by setting the parameters β and a of (5.5) and (5.6) as follows

$$\begin{cases} \beta^2 = 4a^2 \\ a = 4\lambda \end{cases} \tag{5.7}$$

and for one dimensional case we can take r along $z - axis$. Now after comparing the equation (5.5) and equation (5.6), we can express $\frac{1}{r^2}$ in terms of quantum Painlevé II solution f

$$\frac{1}{r^2} = f^2 \tag{5.8}$$

and

$$\frac{1}{r} = \pm f. \tag{5.9}$$

We only consider the positive value of (5.9)

$$\frac{1}{r} = f \tag{5.10}$$

as the physical meaningful one. The result (5.10) holds for $ar \ll 1$ and holds also equivalently for $\lambda z \ll 1$. Finally we can express the approximated form [17, 18, 19] of Yukawa potential (5.1) in terms of quantum Painlevé II solution with the help of (5.10) in the following form

$$V(z) = -V_0|\beta|e^{-4\lambda z}\frac{e^{-4\lambda z}}{(1-e^{-8\lambda z})} \tag{5.11}$$

or

$$V(z) = -V_0|\beta|\frac{e^{-8\lambda z}}{(1-e^{-8\lambda z})}, \tag{5.12}$$

where $|\beta| = 4(\lambda^2+1)^{\frac{1}{2}}$. □

6. Conclusion

In this article, we derived a connection of one dimensional non-stationary Schrödinger equation to the quantum Painlevé II equation and constructed the solution of Schrödinger equation by using quantum Painleé II Riccati solution. We expressed the centrifugal term $\frac{1}{r^2}$ of the radial Schrödinger potential [16] in terms of quantum Painlevé II solution. We observed that the quantum Painlevé II solution $f(z;\lambda)$ coincides to the results of [16] for centrifugal expression by setting the parameters β of quantum Painlevé II solution and a screening parameter of Yukawa potential, as $\beta^2 = 4a^2$. The radial centrifugal expression has been applied to derive the approximated analytic solutions of radial Schrödinger equation by using Nikiforov-Uvarov method.

Acknowledgements:

I am very thankful to the Punjab University 54590 on providing me facilities. My special thanks to Science and technology commission of Shanghai, China recruited me to work on project, No .20590742900, as Belt and Road Yung Scientist from 2020 to 2012. This work is completed as one of the elements of that project.

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