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Isomorphism Theorems in Generalized $d-$algebras

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Abstract

We introduce the generalized d-algebras, generalized d-ideals (d^*-ideals, $d^{\#}$-ideals, $d^{\$}$-ideals) and other related notions. We also prove some properties about d-ideal, $d^{\#}$-ideal and results related to quotient generalized d-algebra. Through these constructions, we prove the first, second and the third isomorphism theorems for the generalized d-algebras. These developments contribute to the theory of the BCI/BCK/BCH and the generalized BCH-algebras.

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1. Introduction

The notions of the BCK-algebras and the BCI-algebras were given in [12] and [13] among which the prior is the proper subclass of the latter. Several mathematicians studied multiple aspects of these algebras, for example BCI-algebras [25], BCK-algebras and ideals in BCK-algebras [19, 20], ideals, relevant theory and filters in BCH-algebras [8, 24].

The notion of a BCH-algebras was characterized by Hu et. al. in 1983 ([11]). The notion of BCH-algebras is a generalized notion of BCK-algebras and BCI-algebras. Chaudhry [9], Dudek et. al. [10] and many other researchers worked on this class.

Neggers et. al. [22] gave the idea of a d-algebra. The class of $d-$algebras is a generalized class of BCK-algebras. The authors in [22], worked on the relations between BCK-algebras and d-algebras. Several notions/aspepcts of d-algebras such as, ideal theory based on N-structures [2], fuzzy ideals [17], d-fuzzy ideals [21], d-algebra ideals [23], deformation in d/BCK-algebras [26] and BCK-neighborhood systems in d-algebras [27] have been studied extensively. Moreover, some other types related to d-algebras such as

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companion d-algebras [4], L-up and mirror d-algebras [5] and d-algebras of d-transitive d^*-algebra [18] have also been introduced and investigated. The theory of ideal of d-algebras was given by Neggers, et. al. ([23]). They gave the notions of a $d-$subalgebra, a d^*-ideal, a $d-$ideal and a $d^{\#}-$ideal and studied their connection.
In 2013, Abdullah et. al. [1] gave the idea of a semi $d-$ideal of a $d-$algebra, and also many relations between semi $d-$ideals and $d-$ideals in d-algebras are examined in [3].

The notion of a generalized $d-$algebra was firstly introduced by Chaudhry et. al. [7] and some elementary aspects of generalized $d-$algebras were discussed. The properties of these algebras have been not discovered extensively, yet.

The main objective of this paper is to inspect some ideals in generalized $d-$algebras and $d^{\#}-$ideals and prove some isomorphism theorems in these algebras.

2. Preliminaries

Throughout this article $X = (X, *, 0)$ will be a non-empty set with a binary operation "*" and a distinguished element $0 \in X$. K. Iseki and Y.Imai gave the notion of a BCK-algebras in 1966.

Definition 2.1. [13, 15] X is called a BCK-algebra if

1. $((\varrho * \upsilon) * (\varrho * n)) * (n * \upsilon) = 0$,
2. $(\varrho * (\varrho * \upsilon)) * \upsilon = 0$,
3. $\varrho * \varrho = 0$,
4. $0 * \varrho = 0$,
5. $\varrho * \upsilon = 0$ and $\upsilon * \varrho = 0$ imply $\varrho = \upsilon$, for all $\varrho, \upsilon, n \in X$.

In a BCK-algebra X, the following hold [23]:

I $(\varrho * \upsilon) * \varrho = 0$,

II $((\varrho * n) * (\upsilon * n)) * (\varrho * \upsilon) = 0$, for all $\varrho, \upsilon, n \in X$.

BCI-algebra was proposed in 1966, by K. Iseki, a generalization of a BCK-algebra [12].

Definition 2.2. [16] X is called a BCI-algebra if it satisfies $1, 2, 3, 5$ in definition 2.1 and

(6) $\varrho * 0 = 0$ implies $\varrho = 0$.

Every BCK-algebra is a BCI-algebra but the converse is not true [16]. In 1983, the concept of a BCH-algebra was presented by Q. P. Hu and X. Li [11]. They proved that BCI-algebras's class is a proper subclass of the BCH-algebras's class. Some basic properties of the BCH-algebras can be seen in [9] and their decomposition has been presented in [10].

Definition 2.3. [9] X is called a BCH-algebra if $\forall \varrho, \upsilon, n \in X$, it satisfies

(3) $\varrho * \varrho = 0$,

(5) $\varrho * \upsilon = 0$ and $\upsilon * \varrho = 0$ imply $\varrho = \upsilon$,

(7) $(\varrho * \upsilon) * n = (\varrho * n) * \upsilon$.

Definition 2.4. [11] Let X be a BCK/BCI/BCH-algebra. A relation $\leq$ is defined on X by: $\varrho \leq \upsilon$ if and only if $\varrho * \upsilon = 0$.

Every BCI/BCK-algebra, with respect to this relation $\leq$ is partially ordered (see [15]).

Definition 2.5. [14] Let X be a BCK/BCI/BCH-algebra. $\phi \neq I \subseteq X$ is a BCK/BCI/BCH-ideal of X if

(I) $0 \in I$
(II) $\varrho \in I$ and $\upsilon * \varrho \in I$, implies $\upsilon \in X$.

Definition 2.6. [14] Let X be a BCK/BCI/BCH-algebra and $\phi \neq I \subseteq X$. I is a closed ideal of X if
(I) $0 * \varrho \in I$, for all $\varrho \in I$,
(II) $\varrho \in I$ and $\upsilon * \varrho \in I$ imply $\upsilon \in I$, $\upsilon, \varrho \in X$.

Lemma 2.7. *[14] Let I be a BCK/BCI/BCH-ideal of a BCK-algebra X. If $\varrho \in I$ and $\upsilon * \varrho = 0$ then $\upsilon \in I$.*

2.1. $d-$algebra Notions

In 1999, Neggers et al. gave the notion of a $d-$algebra [22].

Definition 2.8. [22] X is called a $d-$algebra if $\forall \upsilon, \varrho \in X$, it satisfies
(I) $\varrho * \varrho = 0$,
(II) $0 * \varrho = 0$,
(III) $\varrho * \upsilon = 0$ and $\upsilon * \varrho = 0$ imply $\varrho = \upsilon$.

Every BCK-algebra is a $d-$algebra. But converse is not true. (see [22]).

Let X be a $d-$algebra. We specify the relation $\leq$ on X by: $\varrho \leq \upsilon$ if and only if $\varrho * \upsilon = 0$.

Definition 2.9. [22] A $d-$algebra X is $d-$transitive if $\varrho * n = 0$ & $n * \upsilon = 0$ imply $\varrho * \upsilon = 0$, for all $\varrho, \upsilon, n \in X$.

2.2. Some special Subsets and Ideals in $d-$algebra

Definition 2.10. [6] Let X be a $d-$algebra. $\phi \neq S \subseteq X$ is called a sub algebra of X if $\forall \varrho, \upsilon \in S, \varrho * \upsilon \in S$.

Definition 2.11. [23] Let X be a $d-$algebra and $\phi \neq I \subseteq X$ is a $d-$ideal of X if
(D_1) $\varrho * \upsilon \in I$ and $\upsilon \in I$ imply $\varrho \in I$.
(D_2) $\varrho \in I$ and $\upsilon \in X \Rightarrow \varrho * \upsilon \in I$, i.e., $I * X \subseteq I$.

It is not necessary that $d-$subalgebra is also $d-$ideal, see [23].

Lemma 2.12. *Suppose I is a $d-$ideal in a $d-$algebra X, then $0 \in I$.*

Proof. Since I is a non-empty set, so $\exists\ \varrho \in I$. Also $I \subseteq X$, so $\varrho \in X$. Thus $0 = \varrho * \varrho \in I$ by using (D_2). □

It is known that every $d-$ideal of a $d-$algebra is a BCK-ideal, but converse is not true.

Proposition 2.13. *Suppose I is a $d-$ideal of a $d-$algebra X. If $\varrho \in I$ and $\upsilon * \varrho = 0$, then $\upsilon \in I$.*

Proof. Assume that $\varrho \in I$ and $\upsilon * \varrho = 0$. By Lemma 2.12 and (D_1), we have $\upsilon \in I$. □

Definition 2.14. A $d-$ideal of a $d-$algebra X is called a d^*-ideal of X, if for arbitrary $\varrho, \upsilon, n \in X$,
(D_3) $\varrho * n \in I$, whenever $\varrho * \upsilon \in I$ and $\upsilon * n \in I$.

We note that every d^*-ideal is a $d-$ideal, but converse is not true. (see [23]).

Definition 2.15. A $d-$algebra X is called a d^*-algebra if it satisfies
$(\varrho * \upsilon) * \varrho = 0\ \forall \varrho, \upsilon \in X$.

Definition 2.16. A subset $I \neq \phi$ of a $d-$algebra X is called a $d^{\#}-$ideal of X if it is a d^*-ideal and it satisfies
(D_4) $\varrho * \upsilon \in I$ and $\upsilon * \varrho \in I$ imply $(\varrho * n) * (\upsilon * n) \in I$ and $(n * \varrho) * (n * \upsilon) \in I\ \forall\ \varrho, \upsilon, n \in X$.

Every $d^{\#}-$ideal in a $d-$algebra is a d^*-ideal, but converse is not true, (see [23]).

Definition 2.17. ([22]) A mapping $f : X \to Y$ is called a $d-$homomorphism if
$f(\varrho * \upsilon) = f(\varrho) * f(\upsilon),\ \forall\ \varrho, \upsilon \in X$.

Note that $f(0) = f(0 * 0) = f(0) * f(0) = 0$.

3. Generalized $d-$algebras Notions

In this section, few notions of generalized $d-$algebras are given. The concepts and some results about the generalized $d-$subalgebras and generalized $d-$ideal are proved to set the pathway towards the concept of quotient generalized $d-$algebras.

Definition 3.1. (Generalized $d-$algebra)
X is called generalized $d-$algebra if it satisfying these axioms:

(I) $\varrho * \varrho = 0$
(II) $(0 * \varrho) * \varrho = 0$
(III) $\varrho * \upsilon = 0$ and $\upsilon * \varrho = 0$ implies $\varrho = \upsilon$ for all $\varrho, \upsilon \in X$.

Remark 3.2. Every BCK/$d-$algebra is a generalized $d-$algebra because the condition $0 * \varrho = 0$ implies $(0 * \varrho) * \varrho = 0 * \varrho = 0$. Moreover. In general the converse is not true.

Example 3.3. Let $X = \{0, \varrho, \upsilon\}$ with binary operation "*" defined by:

$*$	0	ϱ	υ
0	0	0	υ
ϱ	ϱ	0	ϱ
υ	υ	υ	0

Then $(X, *, 0)$ is a generalized $d-$algebra, but it is not a BCK-algebra, as

$$(\varrho * (\varrho * \upsilon)) * \upsilon = (\varrho * \varrho) * \upsilon = 0 * \upsilon = \upsilon \neq 0.$$

Further, this is not a $d-$algebra because $0 * \upsilon = \upsilon \neq 0$.

We note that the class of $d-$algebras and the class of BCK-algebras contained in the class of generalized $d-$algebras. Thus, our results of this paper are valid for $d-$algebras as well as for BCK-algebras.
We remark that the relation $\leq$, definitions of transitive generalized $d-$algebras and generalized $d-$subalgebra are the same as the definitions of corresponding notions in $d-$algebras.

Remark 3.4. If X is a generalized $d-$algebra, then we define a relation $\leq$ on X as $\varrho \leq \upsilon$ if and only if $\varrho * \upsilon = 0, \forall \varrho, \upsilon \in X$.

Definition 3.5. (Transitive generalized $d-$algebra)
A generalized $d-$algebra X is a transitive generalized $d-$algebra if
$\varrho * n = 0$ and $n * \upsilon = 0$ imply $\varrho * \upsilon = 0$.

Definition 3.6. (Generalized $d-$subalgebra)
Let X be a generalized $d-$algebra. $\phi \neq S \subseteq X$ is called a generalized $d-$subalgebra of X if for all $\varrho, \upsilon \in S$, we have $\varrho * \upsilon \in S$.

Definition 3.7. (Generalized $d-$ideal)
Let X be a generalized $d-$algebra and $\phi \neq I \subseteq X$. I is said to be a generalized $d-$ideal of X if it satisfies:

(GD_1) $\varrho * \upsilon \in I$ and $\upsilon \in I \Rightarrow \varrho \in I$.
(GD_2) $\varrho \in I$ and $\upsilon \in X$ imply $(\varrho * \upsilon) * \upsilon \in I$.

Every generalized $d-$ideal is a generalized $d-$subalgebra. Suppose $\varrho, \upsilon \in I \subseteq X$, then $\upsilon \in X$. Thus by (GD_2), $(\varrho * \upsilon) * \upsilon \in I$ and by (GD_1), $\varrho * \upsilon \in I$.

Example 3.8. Let $X = \{0, a, b, c\}$ be a generalized $d-$algebra and "*" is a binary operation defined on X as:

*	0	a	b	c
0	0	0	0	c
a	a	0	c	a
b	b	c	0	a
c	c	c	c	0

Then $I = \{0, c\}$ is a generalized $d-$ideal of X.

It can be noted that every generalized $d-$subalgebra is not necessary a generalized $d-$ideal.

Theorem 3.9. *Every generalized $d-$ideal of a generalized $d-$algebra X is a BCK-ideal of X.*

Proof. Let X is a generalized $d-$algebra. Let I a generalized $d-$ideal of X.
Since $I \neq \phi$. So there exists an $\varrho \in I$. Further, $\varrho \in X$ so by (GD_2), $(\varrho * \varrho) * \varrho = 0 * \varrho \in I$. Now (GD_1) gives $0 \in I$.
Now suppose $\varrho * \upsilon, \upsilon \in I$ then by (GD_1), we have $\varrho \in I$. Thus, I is a BCK-ideal. □

Proposition 3.10. *Suppose I is a generalized $d-$ideal of a generalized $d-$algebra X. Then $0 * \varrho \in I$ for all $\varrho \in I$.*

Proof. Since $I \neq \phi$, there exists an $\varrho \in I$. Since $\varrho \in X$, so by (GD_2), $(\varrho * \varrho) * \varrho = 0 * \varrho \in I$. □

Definition 3.11. A generalized $d-$ideal I of a generalized $d-$algebra X is called a d^*-ideal of X, if for arbitrary $\varrho, \upsilon, n \in X$.

(GD_3) $\varrho * n \in I$ whenever $\varrho * \upsilon \in I$ and $\upsilon * n \in I$.

In generalized $d-$algebra, it is not necessary that generalized $d-$ideal is also generalized d^*-ideal.

Definition 3.12. A generalized $d-$algebra X is called a d^*-algebra if
$((\varrho * \upsilon) * \varrho) * \varrho = 0$ for all $\varrho, \upsilon \in X$.

Definition 3.13. If a d^*-ideal I of a generalized $d-$algebra X satisfies

(GD_4) $\varrho * \upsilon \in I$ and $\upsilon * \varrho \in I$ imply $(\varrho * n) * (\upsilon * n) \in I$ and $(n * \varrho) * (n * \upsilon) \in I$ for all $\varrho, \upsilon, n \in X$, then I is to be a $d^{\#}-$ideal in X.

In generalized $d^{\#}-$algebra, every d^*-ideal is $d^{\#}-$ideal but converse is not true in general.

4. Quotient Generalized $d-$algebras

The concept of a quotient generalized $d-$algebra is introduced in this section and also obtain some results.

Definition 4.1. **($d-$morphism)**
Let X and Y be generalized $d-$algebras. A mapping $f : X \to Y$ is called a $d-$morphism if
$f(\varrho * \upsilon) = f(\varrho) * f(\upsilon)$ $\forall$ $\varrho, \upsilon \in X$.

Let $f : X \to Y$ be a $d-$morphism then $f(0) = f(\varrho * \varrho) = f(\varrho) * f(\varrho)$. So $f(0_X) = 0_Y$.
Let I be a $d^{\#}-$ideal in generalized $d-$algebra $(X, *, 0_X)$. For any $\varrho, \upsilon \in X$, we define $\varrho \sim \upsilon$ if and only if $\varrho * \upsilon \in I$ and $\upsilon * \varrho \in I$. We claim that $\sim$ is an equivalence relation on X. Since $0 \in I$, we have $\varrho * \varrho = 0 \in I$, implies $\varrho \sim \varrho$ for any $\varrho \in X$. That is, $\sim$ is reflexive.
Let $\varrho \sim \upsilon$ and $\upsilon \sim n$. Then $\varrho * \upsilon, \upsilon * \varrho \in I$ and $\upsilon * n, n * \upsilon \in I$. By (GD_3), $\varrho * n, n * \varrho \in I$ and hence $\varrho \sim n$. Thus $\sim$ is transitive. The symmetry of $\sim$ is obvious. Thus $\sim$ is equivalence relation on X.
To show $\sim$ is a congruence, we suppose $\varrho, \upsilon, p, q \in X$ and let $\varrho \sim \upsilon$ and $p \sim q$. Then $\varrho * \upsilon, \upsilon * \varrho, p * q, q * p \in I$. Since I is a $d^{\#}-$ideal, so $(\varrho * p) * (\upsilon * p) \in I$ and $(\upsilon * p) * (\upsilon * q) \in I$. Hence $(\varrho * p) * (\upsilon * q) \in I$. Similarly $(\upsilon * q) * (\varrho * q) \in I$ and $(\varrho * q) * (\varrho * p) \in I$, imply $(\upsilon * q) * (\varrho * p) \in I$. Hence $\varrho * p \sim \upsilon * q$. So $\sim$ is a congruence relation on X.
The congruence class containing ϱ is denoted by $[\varrho]_I$ or C^I_ϱ. That is $[\varrho]_I = \{\upsilon \in X : \varrho \sim \upsilon\}$. We know that $\varrho \sim \upsilon$ if and only if $[\varrho]_I = [\upsilon]_I$. Collection of all equivalence classes of X is denoted by X/I, that is, $X/I = \{[\varrho]_I : \varrho \in X\}$ or $\{C^I_\varrho : \varrho \in I\}$.

Definition 4.2. A transitive generalized $d-$algebra X is called a generalized $d^{\$}-$algebra if it satisfies $x * 0 = x$.

Definition 4.3. Let X and X' be two generalized $d-$algebras. Let $f : X \to X'$ be a $d-$morphism. The set $ker(f) = \{\varrho : \varrho \in X \ and \ f(\varrho) = 0\}$ is the Kernal of the $d-$morphism f. Also the set $Im(f) = \{\upsilon : \upsilon \in X' \ and \ y = f(\varrho) \ for \ some \ \varrho \in X\}$ is called image of f.

Theorem 4.4. *Let X and X' be generalized $d-$algebra and generalized $d^{\$}-$algebra respectively. Let $f : X \to X'$ be a generalized $d-$morphism, then $ker(f)$ is a generalized $d-$ideal in X.*

Proof. Since $f(0) = 0$, so $0 \in ker(f)$. Hence $ker(f)$ is non-empty.
Let $\varrho * \upsilon, \upsilon \in ker(f)$, so $f(\varrho * \upsilon) = 0 = f(\upsilon)$. This implies

$$0 = f(\varrho * \upsilon) = f(\varrho) * f(\upsilon) = f(\varrho) * 0 = f(\varrho)$$

Thus $f(\varrho) = 0$, so $\varrho \in ker(f)$. So clearly $\varrho * \upsilon, \upsilon \in ker(f) \Rightarrow \varrho \in ker(f)$. Thus (GD_1) is satisfied. Now, let $\varrho \in ker(f)$ and $\upsilon \in X$. Now

$$\begin{aligned} f((\varrho * \upsilon) * \upsilon) &= f(\varrho * \upsilon) * f(\upsilon) \\ &= (f(\varrho) * f(\upsilon)) * f(\upsilon) \\ &= (0 * f(\upsilon)) * f(\upsilon) \\ &= 0 \quad (Since\ X'\ is\ a\ generalized\ d-algebra) \end{aligned}$$

This implies $(\varrho * \upsilon) * \upsilon \in ker(f)$.
Clearly $\varrho \in ker(f)$ and $\upsilon \in X$ imply that $(\varrho * \upsilon) * \upsilon \in ker(f)$. Hence (GD_2) is satisfied, so $ker(f)$ is an generalized ideal of X. □

Proposition 4.5. *Suppose $f : X \to Y$ is a $d-$morphism from a generalized $d-$algebra X into a generalized $d^{\$}-$algebra Y. Then $ker(f)$ is a $d^{\#}-$ideal of X.*

Proof. Let $\varrho * \upsilon, \upsilon \in ker(f)$. Then by Theorem 4.4, $ker(f)$ satisfies (GD_1) and (GD_2).
Now, if $\varrho * \upsilon, \upsilon * n \in ker(f)$, then

$$\begin{aligned} f(\varrho) * f(\upsilon) &= f(\upsilon) * f(n) \\ &= 0 \end{aligned}$$

Since Y is transitive generalized $d-$algebra, we obtain. $f(\varrho) * f(n) = 0$ and hence $\varrho * n \in ker(f)$, which proves (GD_3). Let $\varrho * \upsilon, \upsilon * \varrho \in ker(f)$, then

$$\begin{aligned} f(\varrho) * f(\upsilon) &= f(\upsilon) * f(\varrho) \\ &= 0. \end{aligned}$$

Since Y is a generalized $d^{\$}-$algebra, so we obtain $f(\varrho) = f(\upsilon)$.
$\Rightarrow$

$$\begin{aligned} f((\varrho * n) * (\upsilon * n)) &= f(\varrho * n) * f(\upsilon * n) \\ &= (f(\varrho) * f(n)) * (f(\upsilon) * f(n)) \\ &= (f(\varrho) * f(n)) * (f(\varrho) * f(n)) \\ &= 0 \end{aligned}$$

Hence $(\varrho * n) * (\upsilon * n) \in ker(f)$. Similarly $(n * \varrho) * (n * \upsilon) \in ker(f)$, which proves (GD_4). Hence $ker(f)$ is a $d^{\#}-$ideal. □

Lemma 4.6. *Suppose I is an $d^{\#}-$ideal of a generalized $d-$algebra X. Then $[0]_I = I$*

Proof. Suppose $\varrho \in I$. By Proposition 3.10, $0 * \varrho \in I$. Also $0 \in I$. For $\varrho, 0 \in I$ implies $\varrho * 0 \in I$ (Since every generalized $d^{\#}-$ideal is subalgebra). So $\varrho \in [0]_I$, that is $I \subseteq [0]_I$. Let $\upsilon \in [0]_I$. So $\upsilon \sim 0$. Thus $\upsilon * 0 \in I$. Since $0 \in I$, so $\upsilon \in I$. Hence $[0]_I \subseteq I$. Thus $[0]_I = I$

□

Theorem 4.7. *Suppose X is a generalized $d-$algebra and I be d^*-ideal in X. If $[\varrho]_I * [\upsilon]_I = [\varrho * \upsilon]_I (\varrho, \upsilon \in X)$, then $(X/I, *, [0]_I)$ is a generalized $d-$algebra, namely quotient generalized $d-$algebra.*

Proof. As $\sim$ is a congruence relation on X, $\varrho * \upsilon \sim \varrho' * \upsilon'$ for any $\varrho \sim \varrho', \upsilon \sim \upsilon'$. Hence $[\varrho]_I * [\upsilon]_I = [\varrho * \upsilon]_I$ is well defined. Let $[\varrho]_I, [\upsilon]_I \in X/I$. Then

(i) $[\varrho]_I * [\varrho]_I = [\varrho * \varrho]_I = [0]_I$.
(ii) Let $[\varrho * \upsilon]_I = [0]_I = [\upsilon * \varrho]_I$. Then $\varrho * \upsilon, \upsilon * \varrho \in I$. Thus $\varrho \sim \upsilon$ and hence $[\varrho]_I = [\upsilon]_I$.
(iii) $([0]_I * [\varrho]_I) * [\varrho]_I = ([0 * \varrho]_I * [\varrho]_I) = [(0 * \varrho) * \varrho]_I = [0]_I$.

Hence $(X/I, *, [0]_I)$ is a generalized $d-$algebra. □

5. Isomorphism Theorem in Generalized $d-$algebras

Three isomorphism theorems for generalized $d-$algebras are proved in this section.

Theorem 5.1. ***(First Isomorphism Theorem of generalized $d-$algebra)***
Suppose $f : X \to Y$ is a $d-$morphism from a generalized $d-$algebra X onto a generalized $d^{\$}-$algebra Y, then $X/ker(f) \cong Y$.

Proof. We define $\mu : X/ker(f) \to Y$ by $\mu([\varrho]_{ker(f)}) = f(\varrho)$.
We now show that (1) μ is well defined and (2) μ an isomorphism.
(1) If $[\varrho]_{ker(f)} = [\upsilon]_{ker(f)}$ then $\varrho * \upsilon, \upsilon * \varrho \in ker(f)$, and so $f(\varrho) * f(\upsilon) = 0 = f(\upsilon) * f(\varrho)$. Since Y is a $d^{\$}-$algebra, we have $f(\varrho) = f(\upsilon)$. That is, $\mu([\varrho]_{ker(f)}) = \mu([\upsilon]_{ker(f)})$. This shows that μ is well defined.
(2) To prove that μ is an isomorphism, we will show (i) μ is a $d-$morphism, (ii) is onto, (iii) μ is one-one.

(i) Let $[\varrho]_{ker(f)}, [\upsilon]_{ker(f)} \in X/ker(f)$. Then $\mu([\varrho]_{ker(f)} * [\upsilon]_{ker(f)}) = \mu([\varrho * \upsilon]_{ker(f)}) = f(\varrho * \upsilon) = f(\varrho) * f(\upsilon) = \mu([\varrho]_{ker(f)}) * \mu([\varrho]_{ker(f)})$. Thus μ is a $d-$morphism.
(ii) For any $\upsilon \in Y$, there is an $\varrho \in X$ such that $\upsilon = f(\varrho)$ because f is onto. Hence $\mu([\varrho]_{ker(f)}) = f(\varrho) = \upsilon$, which means that μ is onto.
(iii) If $\mu([\varrho]_{ker(f)}) = \mu([\upsilon]_{ker(f)})$, So $f(\varrho) = f(\upsilon)$, which gives $f(\varrho * \upsilon) = f(\varrho) * f(\upsilon) = 0$ and $f(\upsilon * \varrho) = f(\upsilon) * f(\varrho) = 0$. Hence $\varrho * \upsilon \in ker(f)$ and $(\upsilon * \varrho) \in ker(f)$. So $\varrho \sim \upsilon$. Hence $[\varrho]_{ker(f)} = [\upsilon]_{ker(f)}$. Hence μ is one-one.

Thus we have $X/ker(f) \cong Y$. □

Theorem 5.2. ***(Second Isomorphism Theorem of generalized*** $d-$***algebra)***
Let H, K be $d^{\#}-$ideals of a generalized $d^{\$}-$algebra X. Let $Y = \cup_{k\in K} C_k^H$. Then Y is a subalgebra of X containing H and K, $H \cap K$ is a d^*-ideal in K and $Y/H \cong K/(H \cap K)$.

Proof. First we show that Y is a subalgebra of X. Let $y_1, y_2 \in Y$. Then $\exists\ k_1, k_2 \in K$ such that $y_1 \in C_{k_1}^H$ and $y_2 \in C_{k_2}^H$. Hence $y_1 \sim k_1$ and $y_2 \sim k_2$ since $\sim$ is a congruence, so $y_1 * y_2 \sim k_1 * k_2$. Thus $y_1 * y_2 \in C_{k_1 * k_2}^H \subseteq \cup_{k\in K} C_k^H$. Hence Y is a subalgebra of X and obviously is a generalized $d^{\$}-$algebr. Now let $k \in K$. Since $k \sim k$, so $k \in C_k^H \subseteq Y$. Thus $K \subseteq Y$. Since $0 \in K$, so $C_0^H = H$. Thus $H \subseteq \cup_{k\in K} C_k^H = Y$.
We now show that $H \cap K$ is a $d^{\#}-$ideal in K. Obviously $0 \in K$ and $0 \in H$, so $0 \in H \cap K$, let $\sigma \in H \cap K$. Since H and K are d^*-ideals, so $0 * \sigma \in H, 0 * \sigma \in K$. Hence $0 * \sigma \in H \cap K$. Let $\varsigma \in K, \varsigma * \sigma \in H \cap K, \sigma \in H \cap K$. Since H is a d^*-ideal, so $\varsigma \in H$. Thus $\varsigma \in H \cap K$.
Let$\sigma * \varsigma, \varsigma * z \in H \cap K$, so $\sigma * \varsigma, \varsigma * z \in H$ and $\sigma * \varsigma, \varsigma * z \in K$. Since H and K are d^*-ideals, so using $\sigma * \varsigma \in H$ and $\varsigma * z \in K$, we have $\sigma * z \in H \cap K$.
Now let $\sigma * \varsigma, \varsigma * \sigma \in H \cap K$, so $\sigma * \varsigma, \varsigma * \sigma \in H$ and $\sigma * \varsigma, \varsigma * \sigma \in K$. Since H and K are d^*-ideals so $(\sigma * z) * (\varsigma * z), (z * \sigma) * (z * \varsigma) \in H$ and $(\sigma * z) * (\varsigma * z), (z * \sigma) * (z * \varsigma) \in K$, which gives $H \cap K$ is a d^*-ideal of K. So $K/(H \cap K)$ is well defined generalized $d^{\$}-$algebra.
Further H is a d^*-ideal of X, so obviously H is d^*-ideal of Y. Hence Y/H is well defined. We now define a mapping $\phi : K \to Y/H$ by $\phi(k) = C_k^H \in \cup_{k\in K} C_k^H = Y\ \forall k \in K$. We first show that ϕ is well-defined. Let $k_1, k_2 \in K$ and $k_1 = k_2$. Then $k_1 * k_2 = 0$ and $k_2 * k_1 = 0$. That is $k_1 * k_2 \in H$, $k_2 * k_1 \in H$. So $k_1 \sim k_2$. Thus $C_{k_1}^H = C_{k_2}^H$. So $\phi(k_1) = \phi(k_2)$. Hence ϕ is well-defined.
Let $k_1, k_2 \in K$. So

$$\phi(k_1 + k_2) = C_{k_1+k_2}^H = C_{k_1}^H * C_{k_2}^H = \phi(k_1) * \phi(k_2).$$

Thus ϕ is a generalized $d-$algebra morphism. Let $C_\varsigma^H \in Y/H = \cup_{k\in K} C_k^H$. So there exists $k \in K$ such that $\varsigma \in C_k^H$. Thus $C_\varsigma^H = C_k^H$. Now $\phi(k) = C_k^H = C_\varsigma^H$. Hence ϕ is onto.
We now show that $ker(\phi) = H \cap K$. Let $\sigma \in H \cap K$. So $\sigma \in H$ and $\sigma \in K$. Now $\phi(\sigma) = C_\sigma^H = H = C_0^H$. So $\sigma \in ker(\phi)$. Thus $H \cap K \subseteq ker(\phi)$. Let $\sigma \in ker(\phi) \subseteq K$. So $\sigma \in K$ and $\phi(\sigma) = C_0^H$. Also $\phi(\sigma) = C_\sigma^H$. Thus $C_0^H = C_\sigma^H$, which gives $\sigma \in H$. So $\sigma \in H \cap K$. Thus $ker(\phi) \subseteq H \cap K$. Thus we get $\ker(\phi) = H \cap K$. So by Theorem 5.1, we get that $K/ker(\phi) \cong Y/H$, that is, $Y/H \cong K/(H \cap K)$. □

Theorem 5.3. ***(Third Isomorphism Theorem of generalized*** $d-$***algebras)***
Let X be a generalized $d^{\$}-$algebra. Let H and K be $d^{\#}-$ideals of X such that $H \subseteq K$. Then $X/H \Big/ K/H \cong X/K$.

Proof. Since H and K are $d^{\#}-$ideals of X, so X/H and X/K are well defined $d^{\$}-$algebras. Let $C_k^H \in K/H$. So $k \in K \subseteq X$. Thus $k \in X$. Hence $C_k^H \in X/H$. So $K/H \subseteq X/H$.
Now we show that K/H is a $d^{\#}-$ideal of X/H. Since K is a d^*-ideal of X, so $0 \in K$. Thus $C_0^H = H \in K/H$. Let $C_k^H \in K/H$, so $k \in K$. Since K is an d^*-ideal of X, so $0 * k \in K$. Hence $C_{0*k}^H \in K/H$. Thus $C_0^H * C_k^H \in K/H$.
Let $C_\varsigma^H * C_\sigma^H \in K/H, C_\sigma^H \in K/H$. So $C_{\varsigma*\sigma}^H \in K/H, C_\sigma^H \in K/H$, so $\varsigma * \sigma \in K, \sigma \in K$. Since K is a d^*-ideal, so $\varsigma \in K$. Hence $C_\varsigma^H \in K/H$.
Let $C_\sigma^H * C_\varsigma^H \in K/H$ and $C_\varsigma^H * C_z^H \in K/H$. That is $C_{\sigma*\varsigma}^H \in K/H$ and $C_{\varsigma*z}^H \in K/H$, so $\sigma * \varsigma \in K$ and $\varsigma * z \in K$. Since K is a d^*-ideal, so $\sigma * z \in K$. Thus $C_{\sigma*z}^H \in K/H$. Hence $C_\sigma^H * C_z^H \in K/H$.

Now let $C_\sigma^H * C_\varsigma^H \in K/H$ and $C_\varsigma^H * C_\sigma^H \in K/H$. That is, $C_{\sigma*\varsigma}^H \in K/H$ and $C_{\varsigma*\sigma}^H \in K/H$. Thus $\sigma * \varsigma \in K$ and $\varsigma * \sigma \in K$. Since K is a $d^{\#}-$ideal of X, so $(\sigma * z) * (\varsigma * z) \in K$ and $(z * \sigma) * (z * \varsigma) \in K$ for $\sigma, \varsigma, z \in X$. Hence $C_{(\sigma*z)*(\varsigma*z)}^H \in K/H$ and $C_{(z*\sigma)*(z*\varsigma)}^H \in K/H$, which gives $C_{\sigma*z}^H * C_{\varsigma*z}^H \in K/H$ and $C_{z*\sigma}^H * C_{z*\varsigma}^H \in K/H$. Thus $(C_\sigma^H * C_z^H) * (C_\varsigma^H * C_z^H) \in K/H$ and $(C_z^H * C_\sigma^H) * (C_z^H * C_\varsigma^H) \in K/H$ for all $C_\sigma^H, C_\varsigma^H, C_z^H \in X/H$. Thus K/H is a $d^{\#}-$ideal of X/H. Hence $X/H \Big/ K/H$ is a well defined generalized $d^{\#}-$algebra.

Now we define a mapping $\phi : X/H \to X/K$ by $\phi(C_\sigma^H) = C_\sigma^K$ for all $\sigma \in X$. Let $C_\sigma^H = C_{\sigma'}^H$. Then $\sigma \sim \sigma'$ in H. Thus $\sigma * \sigma', \sigma' * \sigma \in H$. Since $H \subseteq K$, so $\sigma * \sigma', \sigma' * \sigma \in K$. Thus $\sigma \sim \sigma'$ in K. Hence $C_\sigma^K = C_{\sigma'}^K$.

That is, $\phi(C^H_\sigma) = \phi(C^H_{\sigma'})$. Hence ϕ is well defined. Now

$$\begin{aligned}\phi(C^H_\sigma * C^H_\varsigma) &= \phi(C^H_{\sigma*\varsigma}) \\ &= C^K_{\sigma*\varsigma} \\ &= C^K_\sigma * C^K_\varsigma \\ &= \phi(C^H_\sigma) = \phi(C^H_\varsigma).\end{aligned}$$

Let $C^K_\sigma \in X/K$, so $\sigma \in X$. Thus $C^H_\sigma \in X/H$ and $\phi(C^H_\sigma) = C^K_\sigma$. So ϕ is onto.
Now, we show that $K/H = ker(\phi)$. Let $C^H_\sigma \in ker(\phi)$. So $\phi(C^H_\sigma) = C^K_0$, that is, $C^K_\sigma = C^K_0$. Thus $\sigma \sim 0$ in K. So $0 * \sigma, \sigma * 0 \in K$. Hence $\sigma \in K$. Thus $C^H_\sigma \in K/H$. So $ker(\phi) \subseteq K/H$.
Let $C^H_k \in K/H$. So $k \in K$. Further

$$\begin{aligned}\phi(C^H_k) &= C^K_k \\ &= K \qquad (because\ \ k \in K) \\ &= C^K_0\end{aligned}$$

Hence $C^H_k \in ker(\phi)$, so $K/H \subseteq ker(\phi)$. Thus $ker(\phi) = K/H$.
So by Theorem 5.1 we get
$X/H \Big/ ker(\phi) \cong X/K$. That is, $X/H \Big/ K/H \cong X/K$. □

6. Conclusion

The concept of $d-$algebras is one of the very interesting topic among the study of algebraic structures, which has attracted many mathematicians. In this article, we have studied the structures of Generalized $d-$algebra, Transitive Generalized $d-$algebra, Generalized $d-$subalgebra, Generalized $d-$ideal and Quotient Generalized $d-$algebra. Some algebraic properties of these concepts are proved. In the last section, by defining $d-$morphism, first, second and third isomorphism theorems for Generalized $d-$algebra are proved.

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