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Some Integral Inequalities in the Framework of Conformable Fractional Integral

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Abstract

In this paper, we use new definition of left and right conformable fractional integral to obtain some new inequalities. The results obtained are refinements of existing results.

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1. Introduction and Preliminaries

Recently, a significant number of considerations have been given to fractional calculus due to its various applications in different field of sciences. In pure and applied mathematics, fractional calculus is the most developed areas of classical calculus. The development of several fractional operators is a noteworthy feature of this investigation (See [1, 2, 3, 6, 12, 15]). A more complete overview of the development of this area with its overlapping with the generalized local calculus can be found at [5], [8] and [9].

Integral inequalities have significant role as these are helpful for the study of different classes of differential and integral equations.

In this paper, we present new integral inequalities in the framework of conformable fractional integral of order α.

The class of functions that we will consider in our study is defined below (see [6] and [9]) as:

Definition 1.1. A function h is said to be in $L_{q,r}[0,+\infty)$ space if

$$L_{q,r}[0,+\infty) = \{h : \|h\|_{L_{q,r}[0,+\infty)} = \left(\int_{a_1}^{a_2} |h(s)|^q s^r ds\right)^{\frac{1}{q}},$$

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where $< +\infty, \quad 1 \leq q < +\infty, \quad r \geq 0\}$ and for $r = 0$, we have

$$\begin{aligned} L_q[0,+\infty) &= \{h : \|h\|_{L_q[0,+\infty)} \\ &= \left(\int_{a_1}^{a_2} |h(s)|^q ds\right)^{\frac{1}{q}} < +\infty, \quad 1 \leq q < +\infty\}. \end{aligned}$$

In [8], the following functional class is defined as:

Definition 1.2. Let $h \in L_1[0,+\infty)$ and Ψ be an increasing and positive monotone function on $[0,+\infty)$ and Ψ' is also continuous on $[0,+\infty)$ and $\Psi(0) = 0$. The space $X_\Psi^q(0,+\infty)$ where $1 \leq q < +\infty$ of those real-valued Lebesgue measurable functions h on $[0,+\infty)$ for which

$$\|h\|_{X_q^\Psi} = \left(\int_{a_1}^{a_2} |h(s)|^q \Psi'(s) ds\right)^{\frac{1}{q}} < +\infty, \quad 1 \leq q < +\infty.$$

Based on the previous definition, there is another functional class defined as:

Definition 1.3. Let $h \in L_1[0,+\infty)$ and F continuous and positive function on $[0,+\infty)$ with $F(0) = 0$. The space $X_F^q(0,+\infty)(1 \leq q < +\infty)$ of those real-valued Lebesgue measurable functions h on $[0,+\infty)$ for which

$$\|h\|_{X_F^q} = \left(\int_{a_1}^{a_2} |h(s)|^q F(s) ds\right)^{\frac{1}{q}} < +\infty, \quad 1 \leq q < +\infty,$$

and for the case $q = +\infty$

$$\|h\|_{X_F^\infty} = ess \sup_{0 \leq s < \infty} [F(s)h(s)].$$

Remark 1.4. If $F(t) = 1$, $1 \leq q < +\infty$ the space $X_F^q(0,+\infty)$ coincides with the $L_q[0,+\infty)$-space.

Remark 1.5. If $F(t) = \frac{1}{t}$ the space $X_F^q(0,+\infty)$ coincides with $L_{q,r}[1,+\infty)$-space (see [10, 16, 17]).

More and more researchers have dedicated themselves to this area [6, 7]. A more complete overview of the development of this area with its overlapping with the generalized local calculus can be found at [4] and [18].

To make the work easier to read, we begin by these definitions for fractional integrals and derivatives, here authors proved that (see [14]):

$$\int_0^1 F^{\delta+1}(s) ds \geq \int_0^1 s^\delta F(s) ds,$$

and

$$\int_0^1 F^{\delta+1}(s) ds \geq \int_0^1 s F^\delta(s) ds,$$

where $\delta > 0$ and f is a positive continuous function on $[0, 1]$ such that

$$\int_u^1 F(s) ds \geq \int_u^1 s ds, \qquad u \in [0, 1].$$

Then in [12] , W. J. Liu et al. established the following result:

$$\int_a^b F^{\alpha+\beta}(s) ds \geq \int_a^b (s-a)^\alpha F^\beta(s) ds.$$

Definition 1.6. [18](Left Conformable derivative). Let a function $F : [u_1, u_2] \to \mathbb{R}$, where $0 \leq u_1 < u_2$. Then left conformable fractional derivative of F of order α is defined as:

$$D_{u_1}^{\alpha}(G)(\top) = \lim_{\varepsilon \to 0} \frac{G(\top + \varepsilon \top^{-\alpha}(\top - u_1)) - G(\top)}{\varepsilon(1 - u_1 \top^{-\alpha})},$$

where $\top > u_1$, $\alpha \in (0, 1)$ and $\top^{\alpha} \neq u_1$.

Theorem 1.7. *Let $\top > u_1$, $\alpha \in (0, 1)$ and $\top^{\alpha} \neq u_1$, we have following results for left conformable fractional derivative of order α*

(i) $D_{u_1}^{\alpha}(aF \pm bG) = aD_{u_1}^{\alpha}(F) \pm bD_{u_1}^{\alpha}(G)$,
(ii) $D_{u_1}^{\alpha}(FG) = FD_{u_1}^{\alpha}(G) + GD_{u_1}^{\alpha}(F)$,
(iii) $D_{u_1}^{\alpha}(\frac{F}{G}) = \frac{GD_{u_1}^{\alpha}(F) - FD_{u_1}^{\alpha}(G)}{G^2}$,
(iv) $D_{u_1}^{\alpha}(c) = 0$,
(v) $D_{u_1}^{\alpha}(\top^n) = \frac{n\top^{n-1}(\top - a)}{(\top^{\alpha} - a)}$, where $n \in \mathbb{R}$,
(vi) $D_{u_1}^{\alpha}(F \circ G) = F'(G(\top))D_{u_1}^{\alpha}(G(\top))$,
(vii) $D_{u_1}^{\alpha}(G(\top)) = \left(\frac{\top - a}{\top^{\alpha} - a}\right) G'(\top)$.

Proof. *(vii)* We know

$$D_{u_1}^{\alpha}(G)(\top) = \lim_{\varepsilon \to 0} \frac{G(\top + \varepsilon \top^{-\alpha}(\top - u_1)) - G(\top)}{\varepsilon(1 - u_1 \top^{-\alpha})}.$$

Substitute $\varepsilon \top^{-\alpha}(\top - u_1) = \ell$, we have

$$D_{u_1}^{\alpha}(G)(\top) = \frac{\top - u_1}{\top^{\alpha} - u_1} \lim_{\ell \to 0} \frac{G(\top + \ell) - G(\top)}{\ell}.$$

□

Now, we will define right conformable derivative as:

Definition 1.8. (Right Conformable derivative). Let a function $F : [u_1, u_2] \to \mathbb{R}$, where $0 \leq u_1 < u_2$. Then right conformable fractional derivative of F of order α is defined as u

$$D_{u_2}^{\alpha}(G)(\top) = -\lim_{\varepsilon \to 0} \frac{G(\top + \varepsilon \top^{-\alpha}(u_2 - \top)) - G(\top)}{\varepsilon(1 - u_2 \top^{-\alpha})},$$

where $\top < u_2$, $\alpha \in (0, 1)$ and $\top^{\alpha} \neq u_2$.

Theorem 1.9. *Let $\top < u_2$, $\alpha \in (0, 1)$ and $\top^{\alpha} \neq u_2$, we have following results for right conformable fractional derivative of order α*

(i) $D_{u_2}^{\alpha}(aF \pm bG) = aD_{u_2}^{\alpha}(F) \pm bD_{u_2}^{\alpha}(G)$,
(ii) $D_{u_2}^{\alpha}(FG) = FD_{u_2}^{\alpha}(G) + GD_{u_2}^{\alpha}(F)$,
(iii) $D_{u_2}^{\alpha}(\frac{F}{G}) = \frac{GD_{u_2}^{\alpha}(F) - FD_{u_2}^{\alpha}(G)}{G^2}$,
(iv) $D_{u_2}^{\alpha}(c) = 0$,
(v) $D_{u_2}^{\alpha}(\top^n) = n\top^{n-1}\left(\frac{u_2 - \top}{u_2 - \top^{\alpha}}\right)$, where $n \in \mathbb{R}$,
(vi) $D_{u_2}^{\alpha}(F \circ G) = F'(G(\top))D_{u_2}^{\alpha}(G(\top))$,
(vii) $D_{u_2}^{\alpha}(G(\top)) = \left(\frac{u_2 - \top}{u_2 - \top^{\alpha}}\right) G'(\top)$.

Proof. (vii) We know

$$D^{\alpha}_{u_2}(G)(\top) = -\lim_{\varepsilon\to 0}\frac{G(\top+\varepsilon\top^{-\alpha}(u_2-\top))-G(\top)}{\varepsilon(1-u_2\top^{-\alpha})}.$$

Substitute $\varepsilon\top^{-\alpha}(u_2-\top)=\ell$, we have

$$D^{\alpha}_{u_2}(G)(\top) = \frac{u_2-\top}{u_2-\top^{\alpha}}\lim_{\ell\to 0}\frac{G(\top+\ell)-G(\top)}{\ell}.$$

□

Definition 1.10. Let a function $G:[u_1,u_2]\longrightarrow\mathbb{R}$, where $0\le u_1<u_2$. Then left conformable fractional integral of G of order α is defined as:

$$I^{\alpha}_{u_1}(G)(\top)=\int_{u_1}^{\top}G(u)\frac{(u^{\alpha}-u_1)}{(u-u_1)}du, \tag{1.1}$$

where $\top>u_1$, $\alpha\in(0,1]$.

Definition 1.11. Let a function $G:[u_1,u_2]\longrightarrow\mathbb{R}$,where $0\le u_1<u_2$. Then right conformable fractional integral of G of order α is defined as:

$$I^{\alpha}_{u_2}(G)(\top)=\int_{\top}^{u_2}G(u)\frac{(u_2-u^{\alpha})}{(u_2-u)}du, \tag{1.2}$$

where $\top<u_2$, $\alpha\in(0,1]$.

Remark 1.12. When $\alpha=1$ and $u_1=0$, we obtain from (1.1) :

$$I^{1}_{0}(G)(\top)=\int_{0}^{\top}G(u)du.$$

Remark 1.13. When $\alpha=1$ and $u_2=0$, we obtain from (1.2) :

$$I^{1}_{0}(G)(\top)=\int_{0}^{\top}G(u)du.$$

In [11] and [12], the following two theorems were proved.

Theorem 1.14. *Let F and G be continuous and positive functions defined on the interval $[a,b]$, such that $F\le G$ on $[a,b]$. Such that $\frac{F}{G}$ is decreasing and F is increasing. Assume that Φ is a convex function $\Phi:\Phi(0)=0$. Then the inequality*

$$\frac{\int_a^b F(s)ds}{\int_a^b G(s)ds}\ge\frac{\int_a^b\Phi(F(s))ds}{\int_a^b\Phi(G(s))ds},$$

holds.

Theorem 1.15. *Let F, G and H be three continuous and positive functions defined on the interval $[a,b]$, such that $F\le H$ on $[a,b]$. Such that $\frac{F}{H}$ is decreasing and F and G are increasing. Assume that Φ is a convex function $\Phi:\Phi(0)=0$. Then the inequality*

$$\frac{\int_a^b F(s)ds}{\int_a^b H(s)ds}\ge\frac{\int_a^b\Phi(F(s))G(s)ds}{\int_a^b\Phi(H(s))G(s)ds},$$

holds.

2. Main Results

In this section we obtain new integral inequalities within the framework of the generalized operators of the definition 1.10 and 1.11.

Theorem 2.1. *Let F and G be continuous and positive functions defined on the interval $[1,+\infty)$, such that $F \leq G$. Under the condition that $\frac{F}{G}$ is decreasing and F is increasing over $[1,+\infty)$, then for any convex function Ω that satisfies $\Omega(0)=0$, the following inequality holds:*

$$\frac{I_{u_1}^{\alpha}\left[F(\top)\right]}{I_{u_1}^{\alpha}\left[G(\top)\right]} \geq \frac{I_{u_1}^{\alpha}\left[\Omega(F(\top))\right]}{I_{u_1}^{\alpha}\left[\Omega(G(\top))\right]}, \tag{2.1}$$

where $\alpha \in (0,1]$.

Proof. Utilizing the convexity of Ω and from the fact that $\Omega(0)=0$, the function $\frac{\Omega(F(u))}{u}$ is increasing and F is also increasing, then we have the increasing function $\frac{\Omega(F(u))}{F(u)}$. Using the fact that $\frac{F}{G}$ is decreasing function,

$$\left(\frac{\Omega(F(\nu))}{F(\nu)} - \frac{\Omega(F(u))}{F(u)}\right)\left(\frac{F(u)}{G(u)} - \frac{F(\nu)}{G(\nu)}\right) \geq 0, \tag{2.2}$$

for all $u,\,\nu \in [1,+\infty)$. From the inequality (2.2), we have

$$\frac{\Omega(F(\nu))}{F(\nu)}\frac{F(u)}{G(u)} + \frac{\Omega(F(u))}{F(u)}\frac{F(\nu)}{G(\nu)} \geq \frac{\Omega(F(u))}{F(u)}\frac{F(u)}{G(u)} + \frac{\Omega(F(\nu))}{F(\nu)}\frac{F(\nu)}{G(\nu)}. \tag{2.3}$$

On multiplying inequality (2.3) by $G(\nu)G(u)$, we obtain

$$\frac{\Omega(F(\nu))}{F(\nu)}G(\nu)F(u) + \frac{\Omega(F(u))}{F(u)}G(u)F(\nu) \geq \Omega(F(u))G(\nu) + \Omega(F(\nu))G(u). \tag{2.4}$$

Multiplying the inequality (2.4) by $\frac{(u^{\alpha}-u_1)}{(u-u_1)}$ and integrating the above inequality over $(u_1,\top)$ w.r.t. u, we obtain

$$\begin{aligned} &\frac{\Omega(F(\nu))}{F(\nu)}G(\nu)I_{u_1}^{\alpha}(F\ )(\top) + I_{u_1}^{\alpha}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right]F(\nu) \\ \geq\ & I_{u_1}^{\alpha}\left[\Omega(F(\top))\right]G(\nu) + \Omega(F(\nu))I_{u_1}^{\alpha}\left[G(\top)\right]. \end{aligned} \tag{2.5}$$

Similarly, multiplying the inequality (2.5) by $\frac{(v^{\alpha}-v_1)}{(v-v_1)}$, integrating the resulting inequality over $(u_1,\top)$ with respect to ν, we obtain

$$\begin{aligned} &I_{u_1}^{\alpha}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right]I_{u_1}^{\alpha}\left[F(\top)\right] + I_{u_1}^{\alpha}\left[F(\top)\right]I_{u_1}^{\alpha}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] \\ \geq\ & I_{u_1}^{\alpha}\left[\Omega(F(\top))\right]I_{u_1}^{\alpha}\left[G(\top)\right] + I_{u_1}^{\alpha}\left[\Omega(F(\top))\right]I_{u_1}^{\alpha}\left[G(\top)\right], \end{aligned} \tag{2.6}$$

We have from (2.6)

$$\frac{I_{u_1}^{\alpha}\left[F(\top)\right]}{I_{u_1}^{\alpha}\left[G(\top)\right]} \geq \frac{I_{u_1}^{\alpha}\left[\Omega(F(\top))\right]}{I_{u_1}^{\alpha}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right]}.$$

Since, $F \leq G$ and the from property of Ω, it is easy to obtain that

$$\frac{\Omega(F(\top))}{F(\top)} \leq \frac{\Omega(G(\top))}{G(\top)}, \top \in [a_1,+\infty)$$

Then in the same way, we obtain

$$I_{u_1}^{\alpha}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] \leq I_{u_1}^{\alpha}\left[\Omega(G(\top))\right]. \tag{2.7}$$

By utilizing (2.7) in (2.6), we obtained the required inequality (2.1). □

Remark 2.2. If $u_1 = a$, $\top = b$ and $\alpha = 1$, then we obtain Theorem 1.14.

Theorem 2.3. *Let F and G be continuous and positive functions defined on the interval $[1, +\infty)$, such that $F \leq G$. Under the condition that $\frac{F}{G}$ is decreasing and F is increasing over $[1, +\infty)$, then for any convex function Ω that satisfies $\Omega(0) = 0$, the following inequality holds:*

$$\begin{aligned} & I_{u_1}^{\alpha}\left[\Omega(G(\top))\right] I_{u_1}^{\gamma}\left[F(\top)\right] + I_{u_1}^{\alpha}\left[F(\top)\right] I_{u_1}^{\gamma}\left[\Omega(G(\top))\right] \\ & \geq I_{u_1}^{\gamma}\left[\Omega(F(\top))\right] I_{u_1}^{\alpha}\left[G(\top)\right] + I_{u_1}^{\alpha}\left[\Omega(F(\top))\right] I_{u_1}^{\gamma}\left[G(\top)\right], \end{aligned} \tag{2.8}$$

with $\top \in (0, 1]$.

Proof. Utilizing the convexity of Ω and from the fact that $\Omega(0) = 0$, the function $\frac{\Omega(F(u))}{u}$ is increasing and F is also increasing, then we have the increasing function $\frac{\Omega(F(u))}{F(u)}$. Using the fact that $\frac{F}{G}$ is decreasing function,

$$\left(\frac{\Omega(F(\nu))}{F(\nu)} - \frac{\Omega(F(u))}{F(u)}\right)\left(\frac{F(u)}{G(u)} - \frac{F(\nu)}{G(\nu)}\right) \geq 0, \tag{2.9}$$

for all $u, \nu \in [1, +\infty)$. From the inequality (2.9), we have

$$\frac{\Omega(F(\nu))}{F(\nu)}\frac{F(u)}{G(u)} + \frac{\Omega(F(u))}{F(u)}\frac{F(\nu)}{G(\nu)} \geq \frac{\Omega(F(u))}{F(u)}\frac{F(u)}{G(u)} + \frac{\Omega(F(\nu))}{F(\nu)}\frac{F(\nu)}{G(\nu)}. \tag{2.10}$$

On multiplying inequality (2.10) by $G(\nu)G(u)$, we obtain

$$\frac{\Omega(F(\nu))}{F(\nu)}G(\nu)F(u) + \frac{\Omega(F(u))}{F(u)}G(u)F(\nu) \geq \Omega(F(u))G(\nu) + \Omega(F(\nu))G(u). \tag{2.11}$$

If we multiply the inequality (2.11) by $\frac{(u^{\gamma} - u_1)}{(u - u_1)}$ and integrating the resulting inequality over $(u_1, \top)$ with respect to u, we obtain

$$\begin{aligned} & \frac{\Omega(F(\nu))}{F(\nu)}G(\nu)I_{u_1}^{\gamma}\left[F(\top)\right] + I_{u_1}^{\gamma}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right]F(\nu) \\ \geq \; & I_{u_1}^{\gamma}\left[\Omega(F(\top))\right]G(\nu) + \Omega(F(\nu))I_{u_1}^{\gamma}\left[G(\top)\right]. \end{aligned} \tag{2.12}$$

Similarly, multiplying the inequality (2.12) by $\frac{(v^{\alpha} - v_1)}{(v - v_1)}$, integrating the resulting inequality over $(u_1, \top)$ with respect to ν, we obtain

$$\begin{aligned} & I_{u_1}^{\alpha}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] I_{u_1}^{\gamma}\left[F(\top)\right] + I_{u_1}^{\alpha}\left[F(\top)\right] I_{u_1}^{\gamma}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] \\ \geq \; & I_{u_1}^{\gamma}\left[\Omega(F(\top))\right] I_{u_1}^{\alpha}\left[G(\top)\right] + I_{u_1}^{\alpha}\left[\Omega(F(\top))\right] I_{u_1}^{\gamma}\left[G(\top)\right]. \end{aligned} \tag{2.13}$$

Since $F \leq G$ and using the property of Ω, it is easy to obtain that

$$\frac{\Omega(F(u))}{F(u)} \leq \frac{\Omega(G(u))}{G(u)}, u \in [a_1, +\infty)$$

If we multiply with both sides with $\frac{(u^{\alpha} - u_1)}{(u - u_1)}$ $G(u)$ and integrating the above inequality over $(u_1, \top)$ with respect to u, we obtain

$$I_{u_1}^{\alpha}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] \leq I_{u_1}^{\alpha}\left[\Omega(G(\top))\right]. \tag{2.14}$$

Similarly. we obtain

$$I_{u_1}^{\gamma}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] \leq I_{u_1}^{\gamma}\left[\Omega(G(\top))\right]. \tag{2.15}$$

By utilizing inequalities (2.14) and (2.15) in (2.13), we obtained the required inequality. □

Remark 2.4. If $\gamma = \alpha$ in 2.8, we obtain Theorem 2.1.

Remark 2.5. If $u_1 = a$, $\top = b$ and $\gamma = \alpha = 1$, then we obtain Theorem 1.14.

Theorem 2.6. *Let F and G be continuous and positive functions defined on the interval $[1, +\infty)$, such that $F \leq G$. Under the condition that $\frac{F}{G}$ is decreasing and F is increasing over $[1, +\infty)$, then for any convex function Ω that satisfies $\Omega(0) = 0$, the following inequality holds:*

$$\begin{aligned} & I^{\alpha}_{v_2}\left[\Omega(G(\top))\right] I^{\alpha}_{u_2}\left[F(\top)\right] + I^{\alpha}_{v_2}\left[F(\top)\right] I^{\alpha}_{u_2}\left[\Omega(G(\top))\right] \\ \geq\ & I^{\alpha}_{u_2}\left[\Omega(F(\top))\right] I^{\alpha}_{v_2}\left[G(\top)\right] + I^{\alpha}_{v_2}\left[\Omega(F(\top))\right] I^{\alpha}_{u_2}\left[G(\top)\right], \end{aligned} \tag{2.16}$$

where $\alpha \in (0, 1]$.

Proof. Utilizing the convexity of Ω and from the fact that $\Omega(0) = 0$, the function $\frac{\Omega(F(u))}{u}$ is increasing and F is also increasing, then we have the increasing function $\frac{\Omega(F(u))}{F(u)}$. Using the fact that $\frac{F}{G}$ is decreasing function,

$$\left(\frac{\Omega(F(\nu))}{F(\nu)} - \frac{\Omega(F(u))}{F(u)}\right)\left(\frac{F(u)}{G(u)} - \frac{F(\nu)}{G(\nu)}\right) \geq 0, \tag{2.17}$$

for all $u, \nu \in [1, +\infty)$. From the inequality (2.17), we have

$$\frac{\Omega(F(\nu))}{F(\nu)}\frac{F(u)}{G(u)} + \frac{\Omega(F(u))}{F(u)}\frac{F(\nu)}{G(\nu)} \geq \frac{\Omega(F(u))}{F(u)}\frac{F(u)}{G(u)} + \frac{\Omega(F(\nu))}{F(\nu)}\frac{F(\nu)}{G(\nu)}. \tag{2.18}$$

On multiplying inequality (2.18) by $G(\nu)G(u)$, we obtain

$$\frac{\Omega(F(\nu))}{F(\nu)}G(\nu)F(u) + \frac{\Omega(F(u))}{F(u)}G(u)F(\nu) \geq \Omega(F(u))G(\nu) + \Omega(F(\nu))G(u). \tag{2.19}$$

Multiplying both members of above inequality by $\frac{(u_2 - u^{\alpha})}{(u_2 - u)}$ and integrating the resulting inequality over $(\top, u_2)$ with respect to u, we obtain

$$\begin{aligned} & \frac{\Omega(F(\nu))}{F(\nu)}G(\nu)I^{\alpha}_{u_2}(F\ )(\top) + I^{\alpha}_{u_2}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right]F(\nu) \\ \geq\ & I^{\alpha}_{u_2}\left[\Omega(F(\top))\right]G(\nu) + \Omega(F(\nu))I^{\alpha}_{u_2}\left[G(\top)\right]. \end{aligned} \tag{2.20}$$

Similarly, multiplying the inequality (2.20) by $\frac{(v_2 - v^{\alpha})}{(v_2 - v)}$, integrating the resulting inequality over $(\top, u_2)$ with respect to v, we obtain

$$\begin{aligned} & I^{\alpha}_{v_2}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] I^{\alpha}_{u_2}\left[F(\top)\right] + I^{\alpha}_{v_2}\left[F(\top)\right] I^{\alpha}_{u_2}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] \\ \geq\ & I^{\alpha}_{u_2}\left[\Omega(F(\top))\right] I^{\alpha}_{v_2}\left[G(\top)\right] + I^{\alpha}_{v_2}\left[\Omega(F(\top))\right] I^{\alpha}_{u_2}\left[G(\top)\right]. \end{aligned} \tag{2.21}$$

Since, $F \leq G$ and the properties of Ω

$$\frac{\Omega(F(\top))}{F(\top)} \leq \frac{\Omega(G(\top))}{G(\top)}, \top \in [a_1, +\infty).$$

It is easy to obtain that

$$I^{\alpha}_{u_2}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] \leq I^{\alpha}_{u_2}\left[\Omega(G(\top))\right], \tag{2.22}$$

and

$$I^{\alpha}_{v_2}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] \leq I^{\alpha}_{v_2}\left[\Omega(G(\top))\right]. \tag{2.23}$$

By utilizing (2.22) and (2.23) in (2.21), we obtained the required inequality.

Remark 2.7. If $u_2 = v_2 = b$, $\top = a$ and $\alpha = 1$, then we obtain Theorem 1.14.

□

Theorem 2.8. *Let F and G be continuous and positive functions defined on the interval $[1, +\infty)$, such that $F \leq G$. Under the condition that $\frac{F}{G}$ is decreasing and F is increasing over $[1, +\infty)$, then for any convex function Ω that satisfies $\Omega(0) = 0$, the following inequality holds:*

$$\begin{aligned} & I_{v_1}^{\alpha}\left[\Omega(G(\top))\right] I_{u_2}^{\gamma}\left[F(\top)\right] + I_{v_1}^{\alpha}\left[F(\top)\right] I_{u_2}^{\gamma}\left[\Omega(G(\top))\right] \\ & \geq I_{u_2}^{\gamma}\left[\Omega(F(\top))\right] I_{v_1}^{\alpha}\left[G(\top)\right] + I_{v_1}^{\alpha}\left[\Omega(F(\top))\right] I_{u_2}^{\gamma}\left[G(\top)\right], \end{aligned} \tag{2.24}$$

where $\alpha \in (0, 1]$.

Proof. Utilizing the convexity of Ω and from the fact that $\Omega(0) = 0$, the function $\frac{\Omega(F(u))}{u}$ is increasing and F is also increasing, then we have the increasing function $\frac{\Omega(F(u))}{F(u)}$. Using the fact that $\frac{F}{G}$ is decreasing function,

$$\left(\frac{\Omega(F(\nu))}{F(\nu)} - \frac{\Omega(F(u))}{F(u)}\right)\left(\frac{F(u)}{G(u)} - \frac{F(\nu)}{G(\nu)}\right) \geq 0, \tag{2.25}$$

for all $u, \nu \in [1, +\infty)$. From the inequality (2.9), we have

$$\frac{\Omega(F(\nu))}{F(\nu)}\frac{F(u)}{G(u)} + \frac{\Omega(F(u))}{F(u)}\frac{F(\nu)}{G(\nu)} \geq \frac{\Omega(F(u))}{F(u)}\frac{F(u)}{G(u)} + \frac{\Omega(F(\nu))}{F(\nu)}\frac{F(\nu)}{G(\nu)}. \tag{2.26}$$

On multiplying inequality (2.10) by $G(\nu)G(u)$, we obtain

$$\frac{\Omega(F(\nu))}{F(\nu)}G(\nu)F(u) + \frac{\Omega(F(u))}{F(u)}G(u)F(\nu) \geq \Omega(F(u))G(\nu) + \Omega(F(\nu))G(u). \tag{2.27}$$

Multiplying both sides of above inequality by $\frac{(u_2 - u^{\gamma})}{(u_2 - u)}$, and integrating the resulting inequality over $(\top, u_2)$ with respect to u, we obtain

$$\begin{aligned} & \frac{\Omega(F(\nu))}{F(\nu)}G(\nu)I_{u_2}^{\gamma}\left[F(\top)\right] + I_{u_2}^{\gamma}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right]F(\nu) \\ \geq \; & I_{u_2}^{\gamma}\left[\Omega(F(\top))\right]G(\nu) + \Omega(F(\nu))I_{u_2}^{\gamma}\left[G(\top)\right]. \end{aligned} \tag{2.28}$$

Similarly, multiplying the inequality (2.28) by $\frac{(v^{\alpha} - v_1)}{(v - v_1)}$, integrating the resulting inequality over $(\top, v_1)$ with respect to v, we obtain

$$\begin{aligned} & I_{v_1}^{\alpha}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] I_{u_2}^{\gamma}\left[F(\top)\right] + I_{v_1}^{\alpha}\left[F(\top)\right] I_{u_2}^{\gamma}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] \\ & \geq I_{u_2}^{\gamma}\left[\Omega(F(\top))\right] I_{v_1}^{\alpha}\left[G(\top)\right] + I_{v_1}^{\alpha}\left[\Omega(F(\top))\right] I_{u_2}^{\gamma}\left[G(\top)\right]. \end{aligned} \tag{2.29}$$

Since, $F \leq G$ and from the properties of Ω, it is easy to obtain that

$$\frac{\Omega(F(u))}{F(u)} \leq \frac{\Omega(G(u))}{G(u)}, \; u \in [a_1, +\infty)$$

If we multiply both sides with $\frac{(u_2 - u^{\gamma})}{(u_2 - u)}G(u)$ and integrating the resulting inequality over $(\top, u_2)$ w.r.t. u, we obtain

$$I_{u_2}^{\gamma}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] \leq I_{u_2}^{\gamma}\left[\Omega(G(\top))\right]. \tag{2.30}$$

Similarly, we obtain

$$I_{v_1}^{\alpha}\left[\frac{\Omega(F(\top))}{F(\top)}G(\top)\right] \leq I_{v_1}^{\alpha}\left[\Omega(G(\top))\right]. \tag{2.31}$$

By utilizing (2.30) and (2.31) in (2.29), we obtained the required inequality. □

Remark 2.9. If $u_2 = v_2 = b$, $\top = a$ and $\alpha = \gamma = 1$, then we obtain Theorem 1.14.

3. Conclusion

We established some new inequality for new conformable left and right fractional Integrals. The existing inequalities can be established from new established inequalities as special cases.

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References

[1] B. Ahmad, A. Alsaedi, M. Kirane and B. Toberek, *Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals*, J. Comput. Appl. Math. **353** (2019), 120–129, doi:10.1016/j.cam.2018.12.030. 1

[2] D. Baleanu, *Comments on: "The failure of certain fractional calculus operators in two physical models" by M. Ortigueira, V. Martynyuk, M. Fedula and J.A.T. Machado*, Fract. Calc. Appl. Anal. **23:1** (2020), 292–297, doi:org/10.1515/fca-2020-0012. 1

[3] D. Baleanu, A. Fernandez, *On Fractional Operators and Their Classifications*, Mathematics **7(9), 830** (2019), doi:org/10.3390/math7090830. 1

[4] H. Budak, *On refinements of Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral operators*, IJOCTA **9:1** (2019), 41–48, doi.org/10.11121/ijocta.01.2019.00585. 1

[5] V. L. Chinchane and D. B. Pachpatte, *on fractional integral inequalities involving convex functions using Saigo fractional integral*, arXiv:1712.03414v2. 1

[6] Z. Dahmani, *A note on some new fractional results involving convex functions*, Acta Math. Univ. Comenianae. **LXXXI:2**, (2012), 241–246. 1, 1

[7] A. Carpinteri and F. Mainardi, *Fractals and fractional calculus in continuum mechanics*, Springer, Vienna, 1997. 1

[8] E. Kacar, Z. Kacar and H. Yildirim, *Integral inequalities for Riemann-Liouville fractional integrals of a function with respect to another function*, Iran. J. Math. Sci. Inform. **13:1** (2018), 1–13. 1, 1

[9] U. N. Katugampola, *New approach generalized fractional integral*, Appl. Math. Comp. **218** (2011), 860–865. 1

[10] A. A. Kilbas, O. I. Marichev and S. G. Samko, *Fractional Integrals and Derivative* Theory and Application, Gordon & Breach, Switzerland, 1993. 1.5

[11] W. J. Liu , Q. A. Ngo and V. N. Huy, *Several interesting integral inequalities*, J. Math. Inequal. **3:2** (2009), 201–212. 1.13

[12] J. Liu, G.S. Cheng and C.C. Li, *Further development of an open problem concerning an integral inequality*, J. Inequal. Pure Appl. Math. **9:1** (2008), 1–5. 1, 1, 1.13

[13] S. Mehmood, J. E. Nápoles Valdes, N. Fatima, B. Shahid *Some New inequalities using conformable fractional integral of order β* , Journal of Mathematical Extension, (2022), 1–22.

[14] Q. A. Ngo, D. D. Thang, T. T. Dat, and D. A. Tuan, *Notes on an integral inequality*, J. Inequal. Pure Appl. Math. **7:4** (2006), 1–5. 1

[15] T. F. Nonnenmacher and R. Metzler, *Applications of fractional calculus ideas to biology*, World Scientific, 1998. 1

[16] K. Oldham and J. Spanier, *Applications of differentiation and integration to arbitrary order*, Elsevier Science, 1974. 1.5

[17] I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications* (Mathematics in science and engineering; volume: 198). San Diego: Academic Press, 1999. 1.5

[18] M. Z. Sarikaya, H Budak, F. Usta, *On Generalized the conformable fractional calculus*, TWMS J. App. Eng. Math. **9:4** (2019), 792–799. 1, 1.6