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On some characterization of nearly Hall S-semiembedded subgroups of finite groups

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Abstract

Let K be a subgroup. Then K is known as partially Hall s-semiembedded subgruop in $\Game$ if for K$\mathcal{T}$ is s-semi permutable in $\Game$ and K $\cap\mathcal{T} \leq K_{\tilde{s}\Game}$ where $K_{\tilde{s}\Game}$ generated by all those subgroups of K which are Hall s-semiembedded in $\Game$, there exists a normal subgroup $\mathcal{T}$ of G. In this paper, we investigate the notion of partially Hall S-semi embedded subgroups on the structure of finite group $\Game$. We obtain some new criteria related to the p-nilpotency and super solubility of a finite group. Some earlier results about formations are also generalized by our results.

Keywords: Hall s-semiembedded subgroups, Partially S-embeded subgroup, p-nilpotent.

1. Introduction

We focus only finite groups and in this paper all groups will be saturated. Formation is denoted by F, Sylow subgroups of $\Game$ is denoted by Syl($\Game$) and maximal subgroups of P is denoted by max(P). All unexplained symbols and notations are standard. The readers can pass on [1] or [2] if compulsory.

In current years, many generalizations introduced sequentially by different authors in the field of normal subgroups. Ezquerro [3], introduced the concept of s-permutable in $\Game$. Another idea investigated extensively is the c-normal given by Guo *et al.,* [4]. Recently, Mao *et al.,* [5] introduced the idea s-semiembedded subgroups of finite groups.

In [6], the idea of Hall s-semiembedded subgroup was introduced by Guo and Li. Another generalizations is s-permutability, given by Petrillo [7] as: a subgroup K is s-semipermutable if $QK = KQ$ for each $Syl_q(\Game)$ with $(|Q|, |K|) = 1$.

Recently, Guo and Li [6] introduced an extension of s-semiembedded subgroups. In this paper we integrated these concepts and present the following notion:

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Definition 1.1. Let K be a subgroup. Then K is known as partially Hall s-semiembedded subgruop in ∂ if for K$\mathcal{T}$ is s-semi permutable in ∂ and K $\cap \mathcal{T} \leq K_{\tilde{s}\partial}$ where $K_{\tilde{s}\partial}$ generated by all those subgroups of K which are Hall s-semiembedded in ∂, there exists a normal subgroup $\mathcal{T}$ of G.

Example 1.2. Suppose $\partial = S_5$ be a group. Since $K = << 12 >>$ is known as partially Hall s- semiembedded subgroup in group ∂ if there exist a normal subgroup $\mathcal{T} = A_5$ such that K $\cap$ $\mathcal{T}$ is the subgroups of $\mathcal{H}$ which is Hall s-semiembedded in ∂.

2. Preliminaries

Now, we evaluate some partially Hall s- semiembedded subgroup on representation of ∂ which may help to proof proceeding theorem.

Lemma 2.1. [8] *Let K is s-permutable and $M \trianglelefteq \partial$. Then*

(a) If $\mathcal{T} \leq \partial$, then $K \cap \mathcal{T}$ is s-permutable in $\mathcal{T}$.
(b) KM and $K \cap M$ are s-permutable, $\dfrac{KM}{M}$ is s-permutable in $\dfrac{\partial}{M}$.
(c) G contained a subnormal K subgroup.
(d) If K is a q-group, then $\mathcal{N}_\partial \geqslant O^q(\partial)$, For some prime no. q.

Lemma 2.2. [3] *Let $K \leq slant\mathcal{T} \leq slant\, \partial$. Then*

(a) If $K \leq \partial$ is $s-semipermutable$, then $K \leq \mathcal{T}$ is $s-semipermutable$.
(b) Let $M \trianglelefteq \partial$, and K be q-group. If $K \leq \partial$ is S-semipermutable, then $\frac{KM}{M}$ is s-semipermutable in $\frac{\partial}{M}$.

Lemma 2.3. [6] *Suppose K is Hall ss-Embedded subgroup contained in ∂. Then*

(a) If $K \leq \mathcal{T} \leq \partial$, then K$\leq \mathcal{T}$ is Hall ss-Embedded.
(b) If K is a q-group, $M \trianglelefteq \partial$, so $KM/M \leq \partial/M$ is Hall ss-Embedded.

Lemma 2.4. [3] *Suppose K is subnormal q-subgroup. If $K \leq \partial$ is s-semipermutable, then $K \leq \partial$ is s-permutable.*

Lemma 2.5. *Consider $K \leq \mathcal{T} \leq \partial$ and $M \trianglelefteq \partial$. Then*

(1) If K is partially Hall s-semiembedded in ∂, then $\mathcal{H} \leq \mathcal{T}$ is partially Hall ss-Embedded.
(2) Consider K is a partially Hall ss-Embedded q-subgroup and $M \leq K$ or $(q, |M|) = 1$, then $\dfrac{KM}{M}$ is partially Hall ss-Embedded in $\dfrac{\partial}{M}$.
(3) If $\mathcal{T} \trianglelefteq \partial$ and K is partially Hall s-semiembedded in ∂, then $\exists$ some normal subgroup $M \leq \partial$ in $\mathcal{T}$ s.t $KM \leq \partial$ is s-permutable and $K \cap M \leqslant K_{s\partial}$.
(4) If K is partially Hall ss-Embedded and $K \leq F(\partial)$, then $K \leq \partial$ is S-embedded.

Proof. Suppose for few normal subgroup $\mathcal{H} \leq \partial$, we have $K\mathcal{H} \leq \partial$ is s-permutable and $K \cap \mathcal{H} \leq K_{s\partial}$. Then

(1) $\mathcal{T} \cap \mathcal{H} \trianglelefteq \mathcal{T}$, Lemmas 2.1 and 2.2 give $K(\mathcal{T} \cap \mathcal{H}) = \mathcal{T} \cap K\mathcal{H}$ is s-permutable in $\mathcal{T}$ and $K \cap (\mathcal{T} \cap \mathcal{H}) = K \cap \mathcal{H} \leq K_{s\partial} \leq K_{sk}$. Thus, K is partially Hall S-semiembedded in K.
(2) As $\mathcal{H}$M/M $\trianglelefteq\partial$/M and $(KM/M)(\mathcal{H}M/M) = K\mathcal{H}M/M$ is s-permutable in ∂/M. If $M \leqslant K$, then $K/M \cap \mathcal{H}M/M = (K \cap \mathcal{H})M/M \leq K_{s\partial}M/M$. If M is a q-group, then

$$|K \cap \mathcal{H}M| = \frac{|K|.|\mathcal{H}M|_q}{|K\mathcal{H}M|_q} = \frac{|K|.|\mathcal{H}|_q}{|K\mathcal{H}|_q} = |K \cap \mathcal{H}|$$

$\Rightarrow K \cap \mathcal{H}M = K \cap \mathcal{H}$, we see $(KM/M) \cap (\mathcal{H}M/M) = (KM \cap \mathcal{H}M)/M = (K \cap \mathcal{H}M)M/M = (K \cap \mathcal{H})M/M \leq K_{s\partial}M/M$. From Lemma 2.3, it is obviously $K_{s\partial}M/M$ is Hall s-semipermutable in ∂M. Hence, KM/M is partially Hall s-semiembedded in ∂/M.

(3) Let $M = \mathcal{T} \cap \mathcal{H}$, then $KM = K(\mathcal{T} \cap \mathcal{H}) = \mathcal{T} \cap K\mathcal{H}$ contained in G is s-permutable and $K \cap M = K \cap \mathcal{T} \cap \mathcal{H} = K \cap \mathcal{H} \leq K_{s\eth}$.

(4) By Lemma 2.4 and definitions, it follows easily.

□

Lemma 2.6. [9] *Suppose $\eth$ be a group and q a prime divided $|\eth|$ with $(|\eth|, q-1) = 1$. Then*

(1) If $M \trianglelefteq \eth$ and $|\ M\ | = q$ order, then M is inside the $Z(\eth)$.

(2) $\eth$ is q-nilpotent, if $\eth$ has cyclic Sylow q-subgroups.

Lemma 2.7. [10] *Let W, X and Y contained in $\eth$. Then*

(a) $W \cap XY = (W \cap X)(W \cap Y)$;

(b) $WX \cap WY = W(X \cap Y)$.

3. Main Results

In this section, we give our main results.

Theorem 3.1. *Suppose $|\eth|$ with $(|\eth|, p-1) = 1$ where p be prime divisor and P a $Syl_p(\eth)$. Consider that every max(Q) not a p-nilpotent is partially Hall S -semiembedded in $\eth$, so G is p-nilpotent.*

Proof. We take an example to prove the result. The example will be of minimal order which prove the result be false.

(1) P **is not cyclic.** Since $(|\eth|, p-1) = 1$. Clearly from Lemma 2.6(2), we consider P is not cyclic. Suppose for proper subgroup $\mathcal{H} \leq P$ contained a p-nilpotent supplemented. Now If G is not p-nilpotent, consider K is not a p-nilpotent containing P and every possible proper subgroup of K is p-nilpotent. Then by [[2], IV, Theorem 5.4], K is a minimal non-nilpotent such that
  (i) $K = [P]K_q$ and for normal P Syl_p-subgroup and K_q a non-normal cyclic $Syl_q(K)$.
  (ii) $P/\Phi(P)$ is minimal of $K/\Phi(P)$.

(2) **Every P proper subgroup has no p-nilpotent.** Take $G = HT$, '$K = K \cap HT = \mathcal{H}(K \cap \mathcal{T})$. The fact is $K \cap \mathcal{T} \leq \mathcal{T}$ is p-nilpotent but K is not p-nilpotent $\Rightarrow L = K \cap \mathcal{T} \leq K$, where L $\leq$ K. Thus L nilpotent and $L = L_p \times L_q$. Obviously, L_q is also a Syl_q(K). Since $P = HL_p, L_p \nleq \Phi = \Phi(P)$. Now suppose K/Φ. The fact $L_q \leq \mathcal{N}_K(L_p) \Rightarrow L_q\Phi/\Phi \leq \mathcal{N}_{K/\Phi}(L_p\Phi/\Phi)$. Also, $L_p\Phi/\Phi \trianglelefteq P/\Phi$ since P/Φ is an elementary abelian group. Hence $L_p\Phi/\Phi \trianglelefteq \langle L_q\Phi/\Phi, P/\Phi\rangle = K/\Phi$. Since $L_p\Phi/\Phi \neq 1^-$ and P/Φ is a chief factor of $K, L_p\Phi/\Phi = P/\Phi$. Following $L_p = P$. So $L = K$.

(3) $\eth$ **is simple group which is not abelian.** Let P_1 be a max(P), by hypothesis and (1) we know that p_1 is partially Hall S-semiembedded in $\eth$. Then there is $\mathcal{T} \trianglelefteq \eth$ such that $P_1\mathcal{T}$ is s-permutable and $P_1 \cap \mathcal{T} \leq (P_1)_{s\eth}$. If $\eth$ is non-abelian and simple, then $\mathcal{T} = \eth$ or *one*. If $\mathcal{T}$ is 1 thus $P_1 = P_1\mathcal{T}$ is s-permutable. So $P_1 \leq \eth$ (proper and subnormal), which contradicts the statement. Hence $\mathcal{T} = G$ and $P_1 = (P_1)_{s\eth}$, so we consider that any max(P) is s-semipermutable.
Let $K \leq P$ be non trivial, we consider $\mathcal{N}_G(K)$. Also $S_1 \in Syl_p(\mathcal{N}_\eth(K))$ and $Q_1 \in Syl_p(\mathcal{N}_\eth(K))$ for any $q \neq p$ where p and q are prime numbers. Let Q be a $Syl_q(G)$ containing Q_1, then every max(P) and Q are permutable. As P is not cyclic, $P = P_1P_2$ hold for some max(P_1) and P_2 of P. Thus $PQ = P_1P_2Q = QP_1P_2 = QP$ Hall subgroup (proper). Clearly PQ satisfied. Then the minimality $\Rightarrow PQ$ p-nilpotent. Hence $Q \trianglelefteq PQ$ and $Q_1 = Q \cap \mathcal{N}_{PQ}(K) \trianglelefteq \mathcal{N}_{PQ}(K)$. Obviously, $KQ_1 = K \times Q_1$ hold for every Sylow Q_1 of $\mathcal{N}_\eth(K)$ with prime order to p. Thus $\mathcal{N}_\eth(K)$ is p-nilpotent. From [[2], IV, Theorem 5.8] $\eth$ can not be a simple group which is non abelian.

(4) $\Phi(\eth) = 1$ **and** $\mathcal{N} \trianglelefteq \mathcal{N}\ \eth, \eth\ /\mathcal{N}$ **is** p**-nilpotent.** Clearly, $\mathcal{TN}/\mathcal{N}$ is a normal $\eth/\mathcal{N}$ subgroup and $P_1\mathcal{N}/\mathcal{N}.T\mathcal{N}/\mathcal{N} = P_1\mathcal{TN}/\mathcal{N}$ is s-permutable in $\eth/\mathcal{N}$. Moreover, since $P_1 \cap \mathcal{N}$ is a $Syl_p\ (\mathcal{N}), |(P_1 \cap \mathcal{N})(\mathcal{T} \cap \mathcal{N})|_p = |P_1 \cap \mathcal{N}| = |\mathcal{N}|_p = |\mathcal{N} \cap P_1\mathcal{T}|_p$. Since P_1 is a p-group,

$$|\mathcal{N} \cap P_1\mathcal{T}|_{p'}.\frac{|\mathcal{N}_{p'}.|P_1\mathcal{T}|_{p'}}{|NP_1\mathcal{T}|_{p'}} = \frac{|\mathcal{N}_{p'}.|\mathcal{T}|_{p'}}{|NT|_{p'}} = |\mathcal{N} \cap \mathcal{T}|_{P'}|(P_1 \cap \mathcal{N})(\mathcal{T} \cap \mathcal{N})|_{p'}.$$

The above equation implies $(\mathcal{N} \cap P_1)(\mathcal{N} \cap \mathcal{T}) = \mathcal{N} \cap P_1\mathcal{T}$. By Lemma 2.7, we have $P_1\mathcal{N} \cap TN = (P_1 \cap \mathcal{T})\mathcal{N}$. Thus $P_1\mathcal{N}/\mathcal{N} \cap TN/\mathcal{N} = (P_1 \cap \mathcal{T})\mathcal{N}/\mathcal{N} \leq (P_1)_{sG}\mathcal{NN}$, from Lemma 2.2 that $(P_1)_{sG}\mathcal{N}/\mathcal{N}$ is s-semipermutable in $\eth/\mathcal{N}$. Hence $M/\mathcal{N}$ is partially Hall s-semiembedded in $\eth/\mathcal{N}$. Therefore, $\eth/\mathcal{N}$ satisfied and so p-nilpotent by the minimality of $\eth$. As p-nilpotent groups class is saturated class, $\mathcal{N} \leq \eth$ where $\mathcal{N}$ is minimal and unique subgroup and $\Phi(\eth) = 1$.

(5) $O_{p'}(\eth) = 1$. If $O_{p'}(\eth) \neq 1$, then $\mathcal{N} \leq O_{p'}(\eth)$ and $\eth/O_{p'}(\eth)$ is p-nilpotent by (4). Thus $\eth$ is also p-nilpotent, contradict.

(6) $O_p(\eth) = 1$ **and** $\mathcal{N}$ **is not** p**-nilpotent.** If $O_p(\eth) \neq 1$, then $\mathcal{N} \leq O_p(\eth)$. As $\Phi(\eth) = 1$, $\eth$ has a max(M) such that $\eth = [\mathcal{N}]M$. Since $O_p(\eth) \leq F(\eth) \leq C_\eth(\mathcal{N})$ and $C_\eth(\mathcal{N}) \cap$ M $\trianglelefteq\eth$, the uniqueness of $\mathcal{N}$ yield $\mathcal{N} = O_P(\eth)$. Since $P = \mathcal{N}(P\cap$ M) and $\mathcal{N} \cap M = 1$,P$\cap$ M is a $Syl_p(M)$ and $\exists$ a max(P_1) $\leq$ P, such that P $\cap$ M $\leq$ P_1 and P = NP_1. By (3), we know that P_1 is Hall s-semiembedded in $\eth$. Then $\eth$ contained normal subgroup $\mathcal{T}$ such that $P_1\mathcal{T}$ is s-permutable in $\eth$ and $P_1 \cap \mathcal{T} \leq (P_1)_\eth$. If $\mathcal{T} = 1$, then $P_1 = P_1\mathcal{T}$ is s-permutable in $\eth$. It follows P_1 is subnormal in $\eth$ and so $P_1 \leq O_P(\eth) = \mathcal{N}$. Then we deduce $P = P_1\mathcal{N} = \mathcal{N}$ is a minimal normal subgroup of $\eth$. Since $\mathcal{N}_\eth(P_1) \geqslant O^p(\eth)$ by Lemma 2.1(d) and $P_1 \trianglelefteq$ P, P_1 is a proper normal subgroup of $\eth$ contained in $P = \mathcal{N}$, which contradicts. Thus we have $\mathcal{T} \neq 1$ and then $\mathcal{N} \leqslant \mathcal{T}$. In this case $\mathcal{N} \cap P_1 = \mathcal{N}\cap P_1 \cap\mathcal{T} = \mathcal{N} \cap (P_1 \cap\mathcal{T})Q \trianglelefteq (P_1 \cap\mathcal{T})Q$ for any $Syl_q(Q)$ of $\eth$ with q $\neq$ p. Hence Q $\leqslant \mathcal{N}_\eth(\mathcal{N} \cap P_1)$ hold for any $Syl_q(Q)$ of $\eth$ with q $\neq$ p. Since $\mathcal{N} \cap P_1 \trianglelefteq$ P, it is normal in $\eth$. Thus $\mathcal{N} \cap P_1 = 1$ and $|\ \mathcal{N}\ |$ = P. By Lemma 2.6(1), $\mathcal{N} \leq Z(\eth)$. Since $\eth/\mathcal{N}$ is nilpotent, $\eth$ is also p-nilpotent, a contradiction. $\mathcal{N}_p$ char $\mathcal{N}\trianglelefteq\ \eth$ as $\mathcal{N}$ p-nilpotent so $\mathcal{N}_p \leq O_p(\eth) = 1$ by (4). So there occurs a contradiction as $\mathcal{N}$ be p-group also $\mathcal{N} \leq O_p(\eth) = 1$.

(7) **The contradiction.** By Lemma 2.5, we known that PN satisfies. Therefore, PN is p-nilpotent if PN $< \eth$. Then obvious $\mathcal{N}$ is p-nilpotent, which contradict with (6). Hence $\eth$ = PN. Since $\mathcal{N}$ is not soluble, $\mathcal{N} = S_1 \times S_2 \times ... \times S_k$ is a simple group S_i which is non abelian. By (2), we know $\mathcal{N} < \eth$ and P $\cap\ \mathcal{N} <$ P. Let $S_p \in Syl_p(S_1)$, then by $S_p \leq P_1$ for some max(P_1) of P. By hypothesis and (1), we know $\exists\ \mathcal{T} \trianglelefteq \eth$ such that $P_1\mathcal{T}$ is s-permutable and $P_1 \cap\mathcal{T} \leq (P_1)_\eth$. If $\mathcal{T} = 1$, then P_1 is s-permutable and $O_p(\eth) \neq 1$, contradict with (6). Thus $\mathcal{T} \neq 1$ and so $\mathcal{N} \leq \mathcal{T}$. If $P_1 \cap\mathcal{T} = 1$, then $|\ \mathcal{T}\ | \leq$ p. Hence $\mathcal{T}$ is p-nilpotent by Lemma 2.6(2), $\mathcal{N}$ is also p-nilpotent. This contradiction shows $P_1 \cap\mathcal{T} \neq 1$. Let q is divisor of $|\ \eth\ |$ different from p and Q a $Syl_q(\eth)$.

$$|\ Q \cap P_1\mathcal{T}\ |=|\ Q\ |\ .\ |\ P_1\mathcal{T}\ |_q\ /\ |\ QP_1\mathcal{T}\ |_q=|\ Q\ |\ .\ |\ \mathcal{T}\ |_q\ /\ |\ QT\ |_q=|\ Q \cap \mathcal{T}\ |=|\ (Q \cap P_1)(Q \cap \mathcal{T})\ |\ .$$

The above statement implies $Q\cap P_1\mathcal{T} = (Q\cap P_1)(Q\cap\mathcal{T})$. Then $QP_1 \cap QT = Q(P_1 \cap\mathcal{T})$ by Lemma 2.7. So, $\mathcal{N}\cap P_1Q = \mathcal{N}\cap(P_1Q\cap TQ) = \mathcal{N}\cap(P_1 \cap\mathcal{T})Q$. Thus $S_1 \cap (P_1 \cap \mathcal{T}) = S_1 \cap P_1 = S_p$ is a Syl_p (S_1). Since $\mathcal{N}$ is non-abelian, p = 2. Then for q(q $\neq$ 2) where q is any prime divisor, |S|, simple and non-abelian S_1 has Hall {2,q}, contradiction with [[11], Lemma 2.6]. This completes proof.

□

Remark 3.2. Every supersolvable group $\eth$ for its smallest prime divisor p is p-nilpotent, finite group $\eth$ contains any subgroup $\mathcal{H}$ which has not a p-nilpotent supplement in $\eth$. Also there is no supersolvable supplement in $\eth$.

Corollary 3.3. *Suppose Q a* $Syl_q(\eth)$*, where* $q = min\pi(\eth)$*. If every max(P) without super solvable supplemented in* $\eth$ *is partially Hall s -semiembedded in* $\eth$*, then* $\eth$ *is nilpotent.*

Theorem 3.4. *Consider $\mathcal{F}$ and $\eth$ saturated formation and a finite group respectively consisting of solvable groups $\mathcal{U}$. Then $\eth \in \mathcal{F}$ iff $\eth \trianglelefteq E$ such that $\eth/E \in \mathcal{F}$ and for each non-cyclic $Syl_Q(E)$, each maximal of Q having no supersoluble supplemented in $\eth$ is partially Hall S -semiembedded in $\eth$.*

Proof. The necessary part of above statement is evident, so we prove only the second condition.

Consider the statement is not true and ($\eth$,E) is a counter example where | $\eth$ | | E | minimal.

(1) $\mathcal{N}$ **q-nilpotent for its minimal prime divisor q.** Let q = minπ(E) and Q a $Syl_q(E)$. E will be q-nilpotent if Q is cyclic from [Theorem 2.8, IV, [2]]. Consider Q is not cyclic. By Lemma 2.4, we know that each max(Q) contain no supersoluble supplement contained in E is Partially Hall S -semiembedded in E. Thus Corollary 3.3 $\Rightarrow$ E is q-nilpotent.

(2) **E = Q not cyclic.** Suppose that P < E and $\mathcal{N}$ be normal q-complement of E. Then $\mathcal{N} \trianglelefteq \eth$ and from Lemma 2.5(2), we the hypothesis still hold for $\eth/\mathcal{N}$ (w.r.t E/$\mathcal{N}$). So $\eth/\mathcal{N} \in \mathcal{F}$ by minimal of $\eth$. Then our deduced hypothesis still satisfied for ($\eth$,$\mathcal{N}$). Thus $\mathcal{N}$=1 by ($\eth$,E) and then Q = E. Since $\eth$/E $\in \mathcal{F}$, by [[12], Lemma 2.16] consider E is not cyclic.

(3) **For the minimal normal Q subgroup, $\eth^{\mathcal{F}}$ = Q.** Let $\mathcal{N}$ be min($\eth$) in Q, by Lemma 2.5, it is understood that the hypothesis hold for $\eth/\mathcal{H}$. Thus $\eth/\mathcal{H} \in \mathcal{F}$ by the minimal choice of $\eth$. Moreover , $\mathcal{H}$ is the only min($\eth$) such that $\eth = [\mathcal{H}]M$. Then Q = Q $\cap \mathcal{H}M = \mathcal{H}(Q\cap$ M). Since Q $\leq$ F($\eth$) $\leq C_{\eth}(\mathcal{H})$,(Q $\cap$ M) $\trianglelefteq \eth$ and Q $\cap$ M = 1. Following this Q = $\mathcal{N} = \eth^{\mathcal{F}}$ is a normal min($\eth$).

(4) **Final contradiction.** Consider P_1 be $max(P) = \mathcal{H}$. If P_1 consist of supersolvable supplement K $\leq$ $\eth$, then $PK = \eth$ and 1 $\neq$ P $\cap$ K $\trianglelefteq \eth$. Thus P $\cap K = P$ and so $\eth$ is K supersolvable, a contradiction. So, P_1 is Partially Hall s -semiembedded in $\eth$. There exist a $\mathcal{T} \leq \eth$ contained in P such that $P_1\mathcal{T}$ is s-permmutable also $P_1 \cap \mathcal{T} \leq (P_1)_{\eth}$. Since P is a min($\eth$), $\mathcal{T}$ is equal to 1 or P. If $\mathcal{T}$ = 1, $P_1 = P_1\mathcal{T}$ is s-permutable. If $\mathcal{T}$ = P, then $P_1 = P_1 \cap\mathcal{T} = (P_1)_{\eth} \cap\mathcal{T}$ is Hall s-semipermutable. From Lemma 2.4, we can also conclude that P_1 s-permutable. By Lemma 2.1(d), we have $O^p(\eth) \leq \mathcal{N}_{\eth}(P_1)$. So for any max($P_1$) of P, | $\eth$: $\mathcal{N}_{\eth}$(P_1)| = p^a, a is any integer. Let M_1, M_2,..., M_t set of all max(P), then p|t contradiction [[2],III, 8.5(d)]. The required result.

□

4. Conclusion

In this paper, we studied some properties of nearly Hall s-semiembedded subgroups of a finite group. We proved, for a prime divisor ρ of $|G|$ with $(|G|, \rho - 1) = 1$ and having a $Syl_\rho(G)$ subgroup W of order ρ that every max(V) not a ρ-nilpotent is N_{HsseG}, which leads to to show that G is ρ-nilpotent. Further, we proved a result related to formation $\mathcal{F}$. The idea of nearly Hall s-semiembedded subgroups can be extended further by involving τ-embedded subgroups, which is our next aim. By imposing some restriction on supplemented subgroups of Hall subgroups which are ss-embedded can also be extended to this idea. In this way our results can also be extended.

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