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Fixed Point Approximations of a Family of α-nonexpansive Mappings in CAT(0) Spaces

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Abstract

In this article, the results deal with the strong convergence of Halpern iteration in CAT(0) spaces. The study revolves around finding a fixed point for nonexpansive mappings, which are also the metric projection points in CA(0) space.Moreover, the strong convergence of Halpern iteration for α-nonexpansive mapping sequence is also given.Our results extend some known results which appeared in the literature.

Keywords: CAT(0) space, Common Fixed Points, Halpern iteration, Semigroups.
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1. Introduction

The literature of CAT(0) space began to expand when M.Gromov stated an inequality based on Rimennian geometry which helped to study the hyperbolic spaces using comparision tirangles. The idea of geodesic metric spaces however was pioneered by H.Busemann and A.D.Alexandrov. Hence the name CAT(κ) stands for Cartan, Alexandrov and Toponogov respectively for their contributions in this field and κ stands for the curvature of the space to be studies. Our work here is for the spaces of zero curvature. For through study of CAT(0) spaces and of fundamental role they play in geometry, we refer the reader to [1]. For more rigorous study of CAT(0) spaces, we propose to study [2].

Consider Σ to be a CAT(0) space and S is any non-empty closed convex subset of Σ. Let $\boldsymbol{\tau} := \{\tau(\varphi) : \varphi > 0\} : S \to S$ be a family of mappings.They are called a one-parameter continuous semigroup of nonexpansive mappings if the following conditions are hold:

1. for each $\varphi > 0$, $\tau(\varphi)$ is a nonexpansive mapping on S, i.e.,

$$d\left(\tau(\varphi)u, \tau(\varphi)y\right) \leq d(u, y)\ , \quad \forall u, y \in S;$$

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2. $\tau(\nu+\varphi)=\tau(\varphi)\circ\tau(\nu)$ for all $\varphi,\nu>0$;
3. for each $u\in\Sigma$, the mapping $\tau(.)u$ from $R_{>0}$ into S is continuous.

A one-parameter α-nonexpansive semigroup is a family $\boldsymbol{\tau}=\{\tau(\varphi):\varphi>0\}$ of mapping with domain $D(\boldsymbol{\tau})=\bigcap\limits_{\varphi>o}D(\tau(\varphi))$ and range $R(\boldsymbol{\tau})$ such that:

1. for each $\varphi>0$, $\tau(\varphi)$ is a α-nonexpansive, i.e, for some $\alpha<1$
$$d(\tau(\varphi)u,\tau(\varphi)y)\leq \alpha d(\tau(\varphi)u,y)+\alpha d(\tau(\varphi)y,u)+(1-2\alpha)d(u,y)\ ,\quad \forall u,y\in S;$$
2. for each $u\in D(\boldsymbol{\tau})$, $\tau(0)u=u$;
3. $\tau(\nu+\varphi)u=\tau(\varphi)\circ\tau(\nu)u$ for all $\varphi,\nu>0$ and $u\in D(\boldsymbol{\tau})$.

The class of a one-parameter α-nonexpansive semigroup contains the classical nonexpansive semigroup hence is a special case. In fact, a mapping $\tau:\Sigma\to\Sigma$ is nonexpansive if and only if it is 0-nonexpansive.

For an anchor point $q\in\Sigma$ and an initial point $u_1\in\Sigma$ as a fixed point of a nonexpansive mapping, Halpern [3] was the one who first introduced the following explicit iterative scheme in Hilbert space, which is known as Halpern's iteration.
$$u_{n+1}=\alpha_n q+(1-\alpha_n)\tau u_n \tag{1.1}$$
for each $n\geq 1$, where α_n is a sequence in $(0,1)$. Various studies use the modified iteration results of Kopecka and Reich [4], Reich [5], Song and Wang [6], Zhou [7], Zhang and Su [8] and references therein. Moreover, for the search of a common fixed point of classical nonexpansive semigroup, in 1998, Xu [9] studied the following implicit iteration for the nonexpansive semigroup in a Hilbert space Σ,
$$u_n=\alpha_n q+(1-\alpha_n)\sigma_{\varphi_n}(u_n) \tag{1.2}$$
for each $n\geq 1$, where $\sigma_\varphi(u)$ is the average given by
$$\sigma_\varphi(u)=\frac{1}{\varphi}\int_0^\varphi \tau(\nu)x d\nu$$
for any $\varphi>0$. In 2012, Chen and Song [10] introduced the following explicit viscosity iteration processes to nonexpansive semigroup case,
$$u_{n+1}=\alpha_n f(u_n)+(1-\alpha_n)\sigma_{\varphi_n}(u_n) \tag{1.3}$$
for any $\varphi>0$. In 2002, Benavides et al. [11] and Xu [12], respectively, showed the strong convergence of the following Halpern iteration under the assumption that the semigroup τ satisfies the following asymptotic regularity condition:
$$u_{n+1}=\alpha_n q+(1-\alpha_n)\tau(\varphi_n)u_n \tag{1.4}$$
for any $\varphi>0$.Fixed point results for monotone nonexpansive mappings were introduced by Song et. al. [13] in 2016. The results were studied in an ordered Banach space Σ. Song et. al. [14] also proved some fixed point theorems of a monotone α-nonexpansive mapping in a uniformly convex Banach space with the partial order " $\leq$ ". There are ample studies for the convergence of Halpern's iterations of semigroups(of various nonlinear mappings) or under various spaces and conditions: for example, Cholamjiak and Suantai [15] for the nonexpansive semigroup with the gauge function, Cho and Kang [16] for the pseudo-contraction semigroup, Wangkeeree et al. [17] for the asymptotically nonexpansive semigroup, Tang and Chang [18] for the nonexpansive semigroup in CAT(0)-spaces, Chen et al. [19], Chen and He [20], Song and Xu [21], Shi and Chen [22], Wangkeeree and Preechasilp [23] and Yao et al. [24] for the nonexpansive semigroup under various conditions. For a family $\{\tau_n\}$ of nonexpansive mappings, Jung [25] and O'Hara et al. [26, 27] considered the following Halpern iteration:
$$u_{n+1}=\alpha_n q+(1-\alpha_n)\tau u_n \tag{1.5}$$

for each $n = 1$ and they, respectively, proved that this iteration converges strongly to a common fixed point of $\{\tau_n\}$ under different conditions (for more details, see Song and Chen [28] and others). In [29] authors consider Halpern's iteration (1.4) for an α-nonexpansive semigroup $\boldsymbol{\tau}$ and prove some strong convergence of this iteration to the metric projection point $P_{F(\tau)}q$ under the following conditions: for any sequence α_n in $(0, 1)$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=2}^{\infty} \alpha_n = \infty$$

Under the same conditions, we prove some strong convergence theorems of Halpern's iteration defined by (1.5) for a family $\{\tau_n\}$ of α-nonexpansive mappings in ordered Banach spaces.

Motivated by the above results, in this paper, we consider Halpern's iteration (1.4) for an α-nonexpansive semigroup $\boldsymbol{\tau}$ in the framework of CAT(0) spaces. We prove some strong convergence theorems of Halpern's iteration for a family $\{\tau_n\}$ of α-nonexpansive mappings.

2. Preliminaries

Let(Σ, d) be a geodesic metric space. A geodesic triangle $\triangle(u_1, u_2, u_3)$ in (Σ, d) consists of three points u_1, u_2,and u_3 in Σ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(u_1, u_2, u_3)$ in (Σ, d) is a triangle $\overline{\triangle}(u_1, u_2, u_3) := \triangle(\overline{u_1}, \overline{u_2}, \overline{u_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(u_i, u_j) = d(u_i, u_j)$for $i, j = 1, 2, 3$.
A geodesic metric space (Σ, d) is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom for all $u, y \in \triangle$ and all comparison points $\overline{u}, \overline{y} \in \overline{\triangle}$,

$$d(u, y) = d_{\mathbb{E}^2}(\overline{u}, \overline{y}) \tag{2.1}$$

Let $u, y \in \Sigma$ and by the Lemma 2.1(iv) of [30] for each $\varphi \in [0, 1]$, there exist a unique point $z \in [u, y]$ such that

$$d(u, z) = \varphi d(u, y), \quad d(y, z) = (1 - \varphi)d(u, y). \tag{2.2}$$

From now on, we will use the notation $(1 - \varphi)u \oplus \varphi y$ for the unique fixed point z satisfying the above equation.
We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 2.1. *[30] Let Σ be a CAT(0) spaces.*

- *For any $u, y, z \in \Sigma$ and $\varphi \in [0, 1]$,*

$$d((1 - \varphi)u \oplus \varphi y, z) \leq (1 - \varphi)d(u, z) + \varphi d(y, z) \tag{2.3}$$

- *For any $u, y, z \in \Sigma$ and $\varphi \in [0, 1]$,*

$$d^2((1 - \varphi)u \oplus \varphi y, z) \leq (1 - \varphi)d^2(u, z) + \varphi d^2(y, z) - \varphi(1 - \varphi)d^2(u, y) \tag{2.4}$$

Complete CAT(0) spaces are often called Hadamard spaces (see [2]). If u, y_1, y_2 are points of a CAT(0) spaces and y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denoted by $\frac{(y_1 \oplus y_2)}{2}$, then the CAT(0) inequality implies

$$d^2\left(u, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(u, y_1) + \frac{1}{2}d^2(u, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{2.5}$$

This inequality is the (CN) inequality of Bruhat and Tits [31]. In fact, a geodesic space is a CAT(0) space if and only if it satisfes the (CN) inequality (cf. [2], page 163).

Definition 2.2. Let Σ be a CAT(0) space and $\tau : \Sigma \to \Sigma$ be a mapping. Then τ is called nonexpansive if

$$d(\tau(u), \tau(y)) \leq d(u, y), \quad u, y \in S$$

Definition 2.3. Let Σ be a CAT(0) space and $\tau : \Sigma \to \Sigma$ be a mapping. Then τ is called contraction if

$$d(\tau(u), \tau(y)) \leq \theta d(u, y), \quad u, y \in S \;\; \theta \in [0, 1)$$

Berg and Nikolaev [32] introduce the concept of quasilinearization as follow: Let us denote the pair $(a, b) \in \Sigma \times \Sigma$ by the $\overrightarrow{ab}$ and call it a vector. Then, quasilinearization is defined as a map

$$\langle ., . \rangle : (\Sigma \times \Sigma) \times (\Sigma \times \Sigma) \longrightarrow \mathbb{R}$$

defined as

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)) \tag{2.6}$$

it is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d \in \Sigma$. We say that Σ satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(d, c)$$

for all $a, b, c, d \in \Sigma$. It is well-known [32] that a geodesically connected metric space is a CAT(0) space if and only if it satisfy the Cauchy-Schwarz inequality.
Let $\boldsymbol{\tau} = \{\tau(\varphi) : \varphi > 0\}$ stand for one-parameter α-nonexpansive semigroup with domain $D(\boldsymbol{\tau}) = \bigcap\limits_{\varphi > o} D(\tau(\varphi))$ and range $R(\boldsymbol{\tau})$ and $F(\tau) = \{u \in \Sigma : \tau u = u\}$ denote the set of all fixed points of a mapping τ.

Definition 2.4. Let S denote the non-empty closed convex subset of a CAT(0) space Σ. Then for each $q \in \Sigma$, the metric projection $P_S : \Sigma \to S$ is defined by

$$P_S q = z, \;\; d(z, q) = \inf_{u \in S} d(u, q).$$

For any $q \in \Sigma$ and $z \in S$, the following property of metric projection P_S is well known:

$$z = P_S \Longleftrightarrow \langle \overrightarrow{uz}, \overrightarrow{xz} \rangle \leq 0, \quad \forall u \in S. \tag{2.7}$$

Definition 2.5. A mapping τ with domain $D(\boldsymbol{\tau}) \in \Sigma$ and range $R(\boldsymbol{\tau})$ is said to be α-nonexpansive if there exists $\alpha < 1$ such that:

$$d(\tau(\varphi)u, \tau(\varphi)y) \leq \alpha d(\tau(\varphi)u, y) + \alpha d(\tau(\varphi)y, u) + (1 - 2\alpha)d(u, y) \;, \quad \forall u, y \in D(\boldsymbol{\tau});$$

For showing our main results, the following lemma is needed;

Lemma 2.6. *Let S be a closed convex subset of a complete real CAT(0) space Σ and $\tau : S \to S$ be an α-nonexpansive mapping with $F(\tau) \neq \emptyset$. Then $F(\tau)$ is closed convex and*

$$d(\tau u, p) \leq d(u, p)$$

for all $u \in S$ and $p \in F(\tau)$.

Proof. It follows form the definition of an α-nonexpansive mapping that, for all $u \in S$ and $p \in F(\tau)$,

$$\begin{aligned} d(\tau u, p) &= d(u, p) \\ &\leq \alpha d(\tau(\varphi)u, y) + \alpha d(\tau(\varphi)y, u) + (1 - 2\alpha)d(u, y) \\ &= \alpha d(\tau(\varphi)u, y) + \alpha d(\tau(\varphi)y, u) + (1 - 2\alpha)d(u, y) \end{aligned}$$

and hence

$$d(\tau u, p) \leq d(u, p). \tag{2.8}$$

Let $p, q \in F(\tau)$, $0 \leq \varphi \leq 1$ and $z = \varphi p \oplus (1-\varphi)q$. Then, using the well-known parallelogram Law, we have

$$d^2\left(\frac{z-p}{2} + \frac{\tau z - p}{2}\right) + \frac{1}{4}d^2(z, \tau z) = \frac{1}{2}d^2(z, p) + \frac{1}{2}d^2(\tau z, p)$$

$$d^2\left(\frac{z-q}{2} + \frac{\tau z - q}{2}\right) + \frac{1}{4}d^2(z, \tau z) = \frac{1}{2}d^2(z, q) + \frac{1}{2}d^2(\tau z, q)$$

which together with (2.8) imply that

$$\begin{aligned} d^2\left(\frac{z+\tau z}{2}, p\right) &= d^2\left(\frac{z-p}{2} + \frac{\tau z - p}{2}\right) \\ &\leq \frac{1}{2}d^2(z, p) + \frac{1}{4}d^2(\tau z, p) \\ &= (1-\varphi)^2 d^2(p, q) + \frac{1}{4}d^2(\tau z, p) \end{aligned}$$

$$\begin{aligned} d^2\left(\frac{z+\tau z}{2}, q\right) &= d^2\left(\frac{z-q}{2} + \frac{\tau z - q}{2}\right) \\ &\leq \frac{1}{2}d^2(z, q) + \frac{1}{4}d^2(\tau z, q) \\ &= (1-\varphi)^2 d^2(p, q) + \frac{1}{4}d^2(\tau z, q). \end{aligned}$$

Suppose that $z \neq \tau z$. Then we have

$$d^2\left(\frac{z+\tau z}{2}, p\right) \leq (1-\varphi)^2 d^2(p, q) \quad , \quad d^2\left(\frac{z+\tau z}{2}, p\right) \leq (1-\varphi)^2 d^2(p, q)$$

So, we obtain

$$\begin{aligned} d(p, q) &\leq d\left(p, \frac{z+\tau z}{2}\right) + d\left(\frac{z+\tau z}{2}, q\right) \\ &< (1-\varphi)d(p, q) + \varphi d(p, q) \\ &= d(p, q). \end{aligned}$$

which is a contradiction and so $z = \tau z$. Thus $F(\tau)$ is convex. Now, we show $F(\tau)$ is closed. In fact, for a sequence $\{p_n\} \in F(\tau)$ with $\lim_{n\to\infty} p_n = p$, it follows from (2.8) that $d^2(p, q) + \frac{1}{2}d^2(\tau z, p)$ as $n \to \infty$ and hence $\lim_{n\to\infty} p_n = p$. Thus $F(\tau)$ is closed. This completes the proof. □

It follows from Lemma 2.6 that for a α-nonexpansive mapping τ, $F(\tau)$ is closed convex and so is $F(\tau) = \bigcap_{\varphi>0} F(\tau(\varphi))$. Thus the metric projection $P_{F(\boldsymbol{\tau})} : \Sigma \to F(\boldsymbol{\tau})$ is well defined if $F(\boldsymbol{\tau}) \neq \emptyset$. Now, we present the concept of the uniformly asymptotically regular semigroup (see [11] for nonexpansive semigroup).

Definition 2.7. An α-nonexpansive semigroup $\boldsymbol{\tau} = \{\tau(\varphi) : \varphi > 0\}$ is said to be uniformly asymptotically regular (in short, u.a.r.) if, for any $\nu = 0$ and any bounded subset B of $D(\boldsymbol{\tau})$,

$$\lim_{n\to\infty} \sup_{u\in B} d(\tau(\nu)\tau(\varphi)u, \tau(\varphi)u) = 0.$$

Similarly, the uniformly asymptotically regular α-nonexpansive mappings sequence may be defined as follows (see [21] for u.a.r. nonexpansive mappings sequence).

Definition 2.8. A family $\{\tau_n\}$ of α-nonexpansive mappings is said to be uniformly asymptotically regular (in short, u.a.r.) if, for each positive integer m and any bounded subset B of $\bigcap_n D(\tau_n)$,

$$\lim_{n\to\infty} \sup_{u\in B} d(\tau_m(\tau_n u), \tau_n u) = 0.$$

Moreover, the following result gives the conditions for the convergence of a nonnegative real sequences.

Lemma 2.9. *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\beta_n)a_n + \delta_n, \forall n \geq 0$, where $\{\beta_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence with*

1. $\sum_{n=0}^{\infty} \beta_n = \infty$.
2. $\lim_{n\to\infty} \sup \frac{\delta_n}{\beta_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $\lim\limits_{n\to\infty} a_n \to 0$.

3. Main results

Following is the demi-closedness principle for an α-nonexpansive mapping, which plays key role in the proof of the main results in this paper

Proposition 3.1. *Let S be a closed convex subset of a complete real CAT(0) space Σ and $\tau : S \to S$ be an α-nonexpansive mapping. If a sequence $\{u_n\}$ in S converges weakly to $u \in S$ and*

$$\lim_{n\to\infty} d(u_n, \tau u_n) = 0,$$

then $u = \tau u$.

Proof. Since $\{u_n\}$ is weakly convergent, $\{u_n\}$ is bounded. Using the fact that

$$d(\tau u_n, 0) \leq d(\tau u_n, u_n) + d(u_n, 0),$$

it follows that $\{\tau u_n\}$ is bounded too. If $1 > \alpha \geq 0$, then we have

$$\begin{aligned} d(\tau u_n, \tau u) &\leq \alpha d(\tau u_n, u) + \alpha d(\tau u, u_n) + (1-2\alpha)d(u_n, u) \\ &\leq \alpha(d(u_n, \tau u_n) + d(u_n, u)) + \alpha(d(\tau u, \tau u_n) + d(\tau u_n, u_n)) \\ &\quad +(1-2\alpha)d(u_n, u) \\ &= 2\alpha d(u_n, \tau u_n) + (1-\alpha)d(u_n, u) + \alpha_n d(\tau u, \tau u_n) \end{aligned}$$

and

$$d(\tau u_n, \tau u) \leq d(u_n, u) + \frac{2\alpha}{1-\alpha} d(\tau u_n, u_n).$$

If $\alpha < 0$, then, we have

$$\begin{aligned} d(\tau u_n, \tau u) &\leq \alpha d(\tau u_n, u) + \alpha d(\tau u, u_n) + (1-2\alpha)d(u_n, u) \\ &\leq \alpha(d(u_n, u) - d(u_n, \tau u_n)) + \alpha(d(\tau u, \tau u_n) - d(\tau u_n, u_n)) \\ &\quad +(1-2\alpha)d(u_n, u) \\ &= -2\alpha d(u_n, \tau u_n) + (1-\alpha)d(u_n, u) + \alpha d(\tau u, \tau u_n) \end{aligned}$$

and so

$$d(\tau u_n, \tau u) \leq d(u_n, u) + \frac{-2\alpha}{1-\alpha} d(\tau u_n, u_n).$$

Thus, for all $\alpha \leq 1$, we have

$$d(\tau u_n, \tau u) \leq d(u_n, u) + \frac{2|\alpha|}{1-\alpha} d(\tau u_n, u_n).$$

which implies that

$$\limsup_{n\to\infty} d(\tau u_n, \tau u) \leq \limsup_{n\to\infty} d(u_n, u) \tag{3.1}$$

Thus, by the prosperities of a complete CAT(0) space Σ, we have

$$\begin{aligned} d^2(u_n, u) &= d^2(u_n, \tau u) + d^2(\tau u, u) + 2\langle \overrightarrow{u_n \tau u}, \overrightarrow{(\tau u) u} \rangle \\ &\leq (d(u_n, \tau u_n) + d(\tau u_n, \tau u))^2 + d^2(\tau u, u) + 2\langle \overrightarrow{u_n \tau u}, \overrightarrow{(\tau u) u} \rangle \end{aligned}$$

Since $\{u_n\}$ weakly converges to $u \in S$, it follows that from (3.1)

$$\begin{aligned} &\limsup_{n\to\infty} d^2(u_n, u) \\ = &\limsup_{n\to\infty} d^2(u_n, \tau u) + d^2(\tau u, u) + 2\limsup_{n\to\infty} \langle \overrightarrow{u_n \tau u}, \overrightarrow{(\tau u) u} \rangle \\ \leq &\limsup_{n\to\infty} d^2(u_n, u) + d^2(\tau u, u) + 2\langle \overrightarrow{u_n \tau u}, \overrightarrow{(\tau u) u} \rangle \\ = &\limsup_{n\to\infty} d^2(u_n, u) + d^2(\tau u, u). \end{aligned}$$

respectively, and hence $d^2(\tau u, u) \leq 0$. This complete the proof. □

Form the proof of the Proposition 3.1, we can show the lemma:

Lemma 3.2. *Let S be a nonempty closed convex subset of a complete real CAT(0) space Σ and $\tau : S \to S$ be an α-nonexpansive mapping . Then*

$$d(\tau u, \tau y) \leq d(u, y) + \frac{2|\alpha|}{1-\alpha} d(\tau u, u) \quad u, y \in S. \tag{3.2}$$

for all $u \in S$.

Theorem 3.3. *Let S be a nonempty closed convex subset of a real CAT(0) space Σ and $\tau = \{\tau(\varphi) : \varphi > 0\}$ be the u.a.r. semigroup of α-nonexpansive mappings from S into itself with $F(\tau) \neq \emptyset$. For any $q, u_0 \in S$, define the sequence $\{u_n\}$ by*

$$u_{n+1} = \beta_n q \oplus (1-\beta_n)\tau(\varphi_n) u_n \tag{3.3}$$

for each $n \geq 1$, where $\beta_n \in (0, 1)$ and $\varphi_n > 0$ satisfy the following conditions:

$$\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty, \quad \lim_{n\to\infty} \varphi_n = +\infty. \tag{3.4}$$

Then the sequence $\{u_n\}$ converges strongly to $z = P_{F(\tau)} q$.

Proof. Let $p \in F(\tau)$. Then $d(\tau(\varphi)u, p) \leq d(u, p)$ for all $u \in S$ and $\varphi > 0$ by Lemma 2.6 and so

$$\begin{aligned} d(u_{n+1}, p) &= d(\beta_n q \oplus (1-\beta_n)\tau(\varphi_n) u_n, p) \\ &\leq \beta_n d(q, p) + (1-\beta_n) d(\tau(\varphi_n) u_n, p) \\ &\leq \beta_n d(q, p) + (1-\beta_n) d(u_n, p) \\ &\leq \max\{d(u_n, p), d(q, p)\} \\ &\quad \ldots \\ &\leq \max\{d(u_0, p), d(q, p)\} \end{aligned}$$

Thus $\{u_n\}$ is bounded and hence so is $\{\tau(\varphi_n)u_n\}$. Since $d(\tau(\varphi_n)u_n, p) \leq d(u_n, p)$. Therefore, by the condition (3.4) we have

$$\lim_{n\to\infty} d(u_{n+1}, \tau(\varphi_n) u_n) = \lim_{n\to\infty} \beta_n d(q, \tau(\varphi_n) u_n) = 0 \tag{3.5}$$

Since $\{\tau(\varphi)\}$ is the u.a.r. α-nonexpansive semigroup, by the condition (3.4) , that is $\lim_{n\to\infty}\varphi_n = +\infty$ for any $\nu > 0$, we have

$$\begin{aligned}&\lim_{n\to\infty} d(\tau(\nu)(\tau(\varphi_n)u_n), \tau(\varphi_n)u_n)\\ &\leq \lim_{n\to} \sup_{u\in B} d(\tau(\nu)(\tau(\varphi_n)u), \tau(\varphi_n)u)\\ &= 0.\end{aligned} \tag{3.6}$$

where B is any bounded subset of S containing $\{u_n\}$ and $\{\tau(\varphi_n)u_n\}$. It follows from the definition of an α-nonexpansive mapping and Lemma 3.2 that

$$\begin{aligned}&d(\tau(\nu)(\tau(\varphi_n)u_n), \tau(\varphi_n)u_{n+1})\\ &\leq d((\tau(\varphi_n)u_n), u_{n+1})\\ &+\frac{2|\alpha|}{1-\alpha} d(\tau(\varphi_n)u_n, \tau(\nu)(\tau(\varphi_n)u_n).\end{aligned}$$

Thus it follows that, for all $\nu > 0$.

$$\begin{aligned}&d(u_{n+1}, \tau(\nu)u_{n+1})\\ \leq\ & d(u_{n+1}, \tau(\varphi_n)u_n) + d(\tau(\varphi_n)u_n, \tau(\nu)(\tau(\varphi_n))u_n)\\ &+d(\tau(\nu)(\tau(\varphi_n))u_n, \tau(\nu)u_{n+1})\\ \leq\ & d(u_{n+1}, \tau(\varphi_n)u_n) + d(\tau(\varphi_n)u_n, \tau(\nu)(\tau(\varphi_n))u_n)\\ &+d(u_{n+1}, \tau(\varphi_n)u_n) + \frac{2|\alpha|}{1-\alpha}, d(\tau(\varphi_n u_n, \tau(\nu)(\tau(\varphi_n)u_n))\\ =\ & 2d(u_{n+1}, \tau(\varphi_n)u_n) + \left(1+\frac{2|\alpha|}{1-\alpha}\right) d(\tau(\varphi_n u_n, \tau(\nu)(\tau(\varphi_n)u_n))\end{aligned}$$

that is,

$$\begin{aligned}&d(u_{n+1}, \tau(\nu)u_{n+1})\\ \leq\ & 2d(u_{n+1}, \tau(\varphi_n)u_n) + \left(1+\frac{2|\alpha|}{1-\alpha}\right) d(\tau(\varphi_n u_n, \tau(\nu)(\tau(\varphi_n)u_n))\end{aligned}$$

Thus, from (3.5) and (3.6), it follows that, for all $\nu > 0$,

$$\lim_{n\to\infty} d(u_{n+1}, \tau(\nu)u_{n+1}) = 0 \tag{3.7}$$

Since $\{u_n\}$ is a bounded sequence in S, there exists a subsequence $\{u_{n_k+1}\}$ of $\{u_{n+1}\}$ such that $\{u_{n_k+1}\}$ converges weakly to some point $u^* \in S$. It is obvious that, for all $\nu > 0$,

$$\lim_{n\to\infty} d(u_{n_k+1}, \tau(\nu)u_{n_k+1}) = 0 \tag{3.8}$$

as the result of (3.7). Then, by Proposition 3.1, we have $u^* \in F(\tau(\nu))$. Since ν is arbitrary, $u^* \in F(\boldsymbol{\tau})$. It follows from Lemma 2.6 that $F(\boldsymbol{\tau})$ is nonempty closed and convex. Then the metric projection $P_{F(\boldsymbol{\tau})} : \Sigma \to F(\boldsymbol{\tau})$ is well defined. Let $z = P_{F(\boldsymbol{\tau})}q$. Without loss of generality, we may assume

$$\lim_{n\to\infty} \langle \overrightarrow{uz}, \overrightarrow{u_{n+1}z}\rangle = \lim_{n\to\infty} \langle \overrightarrow{uz}, \overrightarrow{u_{n+1}z}\rangle \tag{3.9}$$

By the proposition of the metric projection, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \langle \overrightarrow{uz}, \overrightarrow{u_{n+1}z}\rangle &= \limsup_{k\to\infty} \langle \overrightarrow{uz}, \overrightarrow{u_{n_{k+1}}z}\rangle\\ &= \langle \overrightarrow{uP_{F(\boldsymbol{\tau})}q}, \overrightarrow{u^*P_{F(\boldsymbol{\tau})}q}\\ &\leq 0.\end{aligned} \tag{3.10}$$

Now, we prove that $u_n \to z$. Infact, we have

$$
\begin{aligned}
& d^2(u_{n+1}, z) \\
= \ & \langle \overrightarrow{\beta_n q \oplus (1-\beta_n)\tau(\varphi_n)u_{n+1} z}, \overrightarrow{u_{n+1} z} \rangle \\
\leq \ & \beta_n \langle \overrightarrow{uz}, \overrightarrow{u_{n+1} z} \rangle + (1-\beta_n) d(\tau(\varphi_n)u_n, z) d(u_{n+1}, z) \\
\leq \ & \beta_n \langle \overrightarrow{uz}, \overrightarrow{u_{n+1} z} \rangle + (1-\beta_n) d(u_n), z) d(u_{n+1}, z) \\
\leq \ & \beta_n \langle \overrightarrow{uz}, \overrightarrow{u_{n+1} z} \rangle + (1-\beta_n) \frac{d^2(u_n, z) + d^2(u_{n+1}, z)}{2}
\end{aligned}
$$

and hence

$$d^2(u_{n+1}, z) \leq (1-\beta_n) d^2(u_n, z) + 2\beta_n \langle \overrightarrow{uz}, \overrightarrow{u_{n+1} z} \rangle. \tag{3.11}$$

So, applying the Lemma 2.9 to (3.11) with (3.10), we obtain that

$$\lim_{n\to\infty} d(u_n, z) = 0.$$

This completes the proof. □

Using the similar proof technique of Theorem 3.3, we can prove easily the following theorem. We only need to replace the terms $\tau(\varphi_n)$ and $\tau(\nu)$ with the terms τ_n and τ_m, and so we omit the proof.

Theorem 3.4. *Let S be a nonempty closed convex subset of a real CAT(0) space Σ and $\{\tau_n\}$ be the u.a.r. α-nonexpansive mappings from S into itself with $F = \bigcap_n F(\tau_n) \neq \emptyset$. For any $q, u_0 \in S$, define the sequence $\{u_n\}$ by*

$$u_{n+1} = \beta_n q \oplus (1-\beta_n)\tau_n u_n \tag{3.12}$$

for each $n \geq 1$, where $\beta_n \in (0,1)$ satisfy the following conditions:

$$\lim_{n\to\infty} \beta_n = 0, \ \sum_{n=1}^{\infty} \beta_n = \infty. \tag{3.13}$$

Then the sequence $\{u_n\}$ converges strongly to $z = P_{F(\tau)} q$.

Corollary 3.5. *Let S be a nonempty closed convex subset of a complete real CAT(0) space Σ. Assume that $\{\tau_n\}$ be the family of u.a.r. mappings from S into itself defined by*

$$2d(\tau_n u, \tau_n y) \leq d(\tau_n u, y) + d(\tau_n y, u) \tag{3.14}$$

for all $u, y \in S$ and $n =\geq 1$. If $F = \bigcap\limits_n F(\tau n) \neq \emptyset$, then the sequence $\{u_n\}$ defined by (3.12) converges strongly to some fixed point $z = P_F q$.

Corollary 3.6. *Let S be a nonempty closed convex subset of a complete real CAT(0) space Σ. Assume that $\{\tau_n\}$ be the family of u.a.r. mappings from S into itself defined by*

$$3d(\tau_n u, \tau_n y) \leq d(\tau_n u, y) + d(\tau_n y, u) + d(u, y) \tag{3.15}$$

for all $u, y \in S$ and $n \geq 1$. If $F = \bigcap\limits_n F(\tau n) \neq \emptyset$, then the sequence $\{u_n\}$ defined by (3.12) converges strongly to some fixed point $z = P_F q$.

Proof. It is clear that τ is an α-nonexpansive mapping with $\alpha = \frac{1}{3}$. From Theorem (3.3), the desired conclusion is not difficult to be showed. □

Remark 3.7. For any semigroup $\boldsymbol{\tau} = \{\tau(\varphi) : \varphi > 0\}$ of nonlinear mappings with the conditions of Corollaries (3.10) and (3.11), we have the same conclusions similar to ones of Theorem (3.3).

Corollary 3.8. *Let S be a nonempty closed convex subset of a Hilbert space H and $\tau = \{\tau(\varphi) : \varphi > 0\}$ be the u.a.r. semigroup of α-nonexpansive mappings from S into itself with $F(\tau) \neq \emptyset$. For any $q, u_0 \in S$, define the sequence $\{u_n\}$ by*

$$u_{n+1} = \beta_n q + (1 - \beta_n)\tau(\varphi_n)u_n \tag{3.16}$$

for each $n = 1$, where $\beta_n \in (0, 1)$ and $\varphi_n > 0$ satisfy the following conditions:

$$\lim_{n\to\infty} \beta_n = 0, \ \sum_{n=1}^{\infty} \beta_n = \infty, \ \lim_{n\to\infty} \varphi_n = +\infty. \tag{3.17}$$

Then the sequence $\{u_n\}$ converges strongly to $z = P_{F(\tau)}q$.

Corollary 3.9. *Let S be a nonempty closed convex subset of a real Hilbert space H and $\{\tau_n\}$ be the u.a.r. α-nonexpansive mappings from S into itself with $F = \bigcap_n F(\tau_n) \neq \emptyset$. For any $q, u_0 \in S$, define the sequence $\{u_n\}$ by*

$$u_{n+1} = \beta_n q + (1 - \beta_n)\tau_n u_n \tag{3.18}$$

for each $n \geq 1$, where $\beta_n \in (0, 1)$ satisfy the following conditions:

$$\lim_{n\to\infty} \beta_n = 0, \ \sum_{n=1}^{\infty} \beta_n = \infty. \tag{3.19}$$

Then the sequence $\{u_n\}$ converges strongly to $z = P_{F(\tau)}q$.

Corollary 3.10. *Let S be a nonempty closed convex subset of a complete real Hilbert space H. Assume that $\{\tau_n\}$ be the family of u.a.r. mappings from S into itself defined by*

$$2\|\tau_n u - \tau_n y\| \leq \|\tau_n u - y\| + \|\tau_n y - u\| \tag{3.20}$$

for all $u, y \in S$ and $n =\geq 1$. If $F = \bigcap_n F(\tau n) \neq \emptyset$, then the sequence $\{u_n\}$ defined by (3.18) converges strongly to some fixed point $z = P_F q$.

Corollary 3.11. *Let S be a nonempty closed convex subset of a complete real Hilbert space H. Assume that $\{\tau_n\}$ be the family of u.a.r. mappings from S into itself defined by*

$$3\|\tau_n u - \tau_n y\| \leq \|\tau_n u - y\| + \|\tau_n y - u\| + \|u - y\| \tag{3.21}$$

for all $u, y \in S$ and $n =\geq 1$. If $F = \bigcap_n F(\tau n) \neq \emptyset$, then the sequence $\{u_n\}$ defined by (3.18) converges strongly to some fixed point $z = P_F q$.

Remark 3.12. Our Theorem 3.3 is an analog of Theorem 3.3 of S. Yisheng *et al*, [29] in Hilbert spaces.

Remark 3.13. Our results immediately apply to any CAT(κ) space with $\kappa = 0$ since any CAT(κ') space is a CAT(κ) space for any $\kappa' > \kappa$ (see [32]).

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