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On A Mathematical Contradiction to Rethink Associativity and Commutativity for Infinite Series and Infinite Products

Md. Shafiqul Islam[a,*], Sumit Bhowmick[a]

[a] *Department of Materials and Metallurgical Engineering, Bangladesh University of Engineering and Technology, Dhaka-1000, Bangladesh.*

Abstract

This paper deals with two contradictory values of π, focusing on the invalidity of associative and commutative laws for infinite series. The argument shows that operating with some infinite products leads to some dangerous contradictions such as the π value turns out to be 4 or 8/3. The study and the findings embedded with the article's methodology points out that these classical operations like associativity and commutativity on infinite series or infinite products must be handled carefully.

Keywords: Infinite Series, Infinite Product, Commutative Law.

1. Introduction

Associativity and commutativity [1] are two operational properties used in algebra and number theory. The word associativity comes from "associate" which means "to group", i.e., we can always re-group numbers for pure addition or multiplication and the final answer will remain unchanged. The term commutativity comes from the root "commute" which means "to move around" which is to exchange or swap numbers while pure addition or multiplication and the final answer will not depend on the operation. Associative and commutative operations are used for addition and multiplication of rational or irrational numbers. Though the two properties are common laws used in algebra and number theories, there are some pieces of evidence where the application of these properties misleads to dangerous findings. For example, the Ramanujan Summation [2] can be mentioned, where the application of these principles misleads to a nonsense result, i.e., the sum of all the positive integers turns out to be a negative fraction, -1/12. Further, it is interesting that, $\sum_{n=1}^{\infty} n = -\frac{1}{12}$ has been used to derive the equations in 'string theory, quantum field theory and in some aspects of complex analytics. In this paper, we will present a novel pedagogical approach to

*Corresponding author
Email addresses: shafiqmme@gmail.com (Md. Shafiqul Islam), sumitbhowmick357@gmail.com (Sumit Bhowmick)

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study the application of associativity and commutativity principles for infinite series and infinite products. Anyone with an elementary knowledge of integral calculus can be a prospective reader of this article. Other prospective readers of this article might be the STEM folks who work on education and pedagogy as the authors undertook some pedagogical approaches in the current and their previous works [3].

The constant π, a transcendental number can be expressed as the ratio of a circle's circumference to its corresponding dia. The number is frequently used in all branches of mathematics and science and has been extensively known for several centuries. Several millenniums passed since it was attempted to calculate by the mathematicians and since then it was well known that π is less than 4 and greater than 3. As the circle of unity radius can be inscribed inside a square of side length 2, the early mathematicians confirmed that is less than 4. Consequently, like an equilateral do-decagon (an equilateral polygon having twelve edges) can be inscribed inside a circle of unity radius, touching all vertices with it, π was confirmed to be greater than 3.

In the following argument, we will show two contradictory values of π. We will start with the reduction formula of an integrand, motivated by Wallis' theorem [4], [5]. Upon applying commutative and associative principles for infinite products, contradictory results arise, which can be described by the well-known postulate [6] for an infinite product of infinite sum, which states that "associativity and commutativity hold for infinite sum or product if and only if the sum or product is convergent". The Ramanujan Summation is a relevant example of this, where the application of these associativity and commutativity principles for summation of all the positive integers turns out to be a negative value i.e., using an argument that looks valid but is not.

2. Arguments

Let's start from the reduction formula of the following integral:

$$I\left(n\right)=\int e^{ax}sin^{n}x\ dx.$$

$$\begin{aligned}
I\left(n\right) &= \frac{e^{ax}\sin^{n}x}{a}-\frac{n}{a}\int e^{ax}\sin^{n-1}x\cos x dx\\
&= \frac{e^{ax}\sin^{n}x}{a}-\frac{n}{a}\left[\frac{e^{ax}\sin^{n-1}x\cos x}{a}-\frac{1}{a}\int e^{ax}[(n-1)\sin^{n-2}x\cos^{2x}-\sin^{n}x]dx\right]\\
&= \frac{e^{ax}\sin^{nx}}{a}-\frac{n}{a}\left[\frac{e^{ax}\sin^{n-1}x\ \ \cos x}{a}-\frac{1}{a}\int e^{ax}[(n-1)\sin^{n-2}x-n\sin^{n}x]\ dx\right]\\
&= \frac{e^{ax}\sin^{n}x}{a}-\frac{ne^{ax}\sin^{n-1}x\cos x}{a^{2}}+\frac{n\left(n-1\right)}{a^{2}}I\left(n-2\right)-\frac{n^{2}}{a^{2}}I\left(n\right).
\end{aligned}$$

So, the reduction formula for $I\left(n\right)=\int e^{ax}\sin^{n}x\ dx$ becomes,

$$I\left(n\right)=\frac{ae^{ax}\sin^{n}x}{n^{2}+a^{2}}-\frac{ne^{ax}\sin^{n-1}x\ \ \cos x}{n^{2}+a^{2}}+\frac{n\left(n-1\right)}{n^{2}+a^{2}}I\left(n-2\right).$$

The reduction formula for $\int_0^{\pi} e^{ax}\sin^{2n}x\ dx$,

$$\begin{aligned} I\left(2n\right) &= \int_0^{\pi} e^{ax}\sin^{2n}x\ dx\ = \frac{2n(2n-1)}{\left(2n\right)^2+\left(a\right)^2}\ I\left(2n-2\right) \\ &= \frac{2n(2n-1)}{\left[\left(2n\right)^2+\left(a\right)^2\right]}\frac{\left(2n-2\right)\left(2n-3\right)}{\left[\left(2n-2\right)^2+\left(a\right)^2\right]}I\left(2n-4\right), \\ I\left(2n\right) &= \frac{2n(2n-1)}{\left[\left(2n\right)^2+\left(a\right)^2\right]}\frac{\left(2n-2\right)\left(2n-3\right)}{\left[\left(2n-2\right)^2+\left(a\right)^2\right]}\frac{\left(2n-4\right)\left(2n-5\right)}{\left[\left(2n-4\right)^2+\left(a\right)^2\right]}\cdots\frac{2\cdot 1}{\left[\left(2\right)^2+\left(a\right)^2\right]}\ I\left(0\right), \\ I\left(2n+1\right) &= \frac{\left(2n+1\right)\cdot\left(2n\right)\cdot\left(2n-1\right)\cdot\left(2n-2\right)\ \cdots\ 3\cdot 2}{\left[\left(2n+1\right)^2+\left(a\right)^2\right]\cdot\left[\left(2n-1\right)^2+\left(a\right)^2\right]\cdots\left[\left(3\right)^2+\left(a\right)^2\right]}\ I(1), \\ I\left(2n-1\right) &= \frac{\left(2n-1\right)\cdot\left(2n-2\right)\cdot\left(2n-3\right)\cdot\left(2n-4\right)\ \cdots\ 3\cdot 2}{\left[\left(2n-1\right)^2+\left(a\right)^2\right]\cdot\left[\left(2n-3\right)^2+\left(a\right)^2\right]\cdots\left[\left(3\right)^2+\left(a\right)^2\right]}I(1). \end{aligned}$$

$$\begin{aligned} I\left(0\right) &= \int_0^{\pi} e^{ax}\ dx = \frac{e^{\pi a}-1}{a}; \\ I\left(1\right) &= \int_0^{\pi} e^{ax} sin^2 x\ dx = \frac{2(e^{\pi a}-1)}{\left[\left(a\right)^2+4\right]\ a}. \end{aligned}$$

$$\begin{aligned} I\left(2n\right) &= \frac{2n\cdot\left(2n-1\right)\cdot\left(2n-2\right)\cdot\left(2n-3\right)\ \cdots\ 2\cdot 1}{\left[\left(2n\right)^2+\left(a\right)^2\right]\cdot\left[\left(2n-2\right)^2+\left(a\right)^2\right]\cdots\left[\left(2\right)^2+\left(a\right)^2\right]}\times\frac{e^{\pi a}-1}{a}, \\ I\left(2n+1\right) &= \frac{\left(2n+1\right)\cdot\left(2n\right)\cdot\left(2n-1\right)\cdot\left(2n-2\right)\ \cdots\ 3\cdot 2}{\left[\left(2n+1\right)^2+\left(a\right)^2\right]\cdot\left[\left(2n-1\right)^2+\left(a\right)^2\right]\cdots\left[\left(3\right)^2+\left(a\right)^2\right]}\times\frac{2(e^{\pi a}-1)}{\left[\left(a\right)^2+4\right]\ a}, \\ I\left(2n-1\right) &= \frac{\left(2n-1\right)\cdot\left(2n-2\right)\cdot\left(2n-3\right)\cdot\left(2n-4\right)\ \cdots\ 3\cdot 2}{\left[\left(2n-1\right)^2+\left(a\right)^2\right]\cdot\left[\left(2n-3\right)^2+\left(a\right)^2\right]\cdots\left[\left(3\right)^2+\left(a\right)^2\right]}\times\frac{2(e^{\pi a}-1)}{\left[\left(a\right)^2+4\right]\ a}. \end{aligned}$$

For any n $>\frac{1}{2}$ and $0 \le x \le \pi$, $e^{ax}\sin^{2n+1}x \le e^{ax}\sin^{2n}x \le e^{ax}\sin^{2n-1}$ So, the following inequality also holds:

$$\int_0^{\pi} e^{ax} sin^{2n+1}x\ dx\ \le \int_0^{\pi} e^{ax} sin^{2n}x\ dx\ \le \int_0^{\pi} e^{ax} sin^{2n-1}x\ dx.$$

So,

$$I(2n+1) \le I(2n) \le I(2n-1) \implies 1 \le \frac{I(2n)}{I(2n+1)} \le \frac{I(2n-1)}{I(2n+1)},$$

$$\frac{I(2n-1)}{I(2n+1)} = \frac{\left[\left(2n+1\right)^2+\left(a\right)^2\right]}{\left(2n+1\right).\left(2n\right)} = \frac{4n^2+4n+1+a^2}{4n^2+2n},$$

$$\lim_{n\to\infty}\frac{I(2n-1)}{I(2n+1)} = \lim_{n\to\infty}\frac{n^2(4+\frac{4}{n}+\frac{a^2+1}{n^2})}{n^2(4+\frac{2}{n})} =\ 1.$$

So,

$$\lim_{n\to\infty} 1 \le \frac{I(2n)}{I(2n+1)} \le 1.$$

By the squeeze theorem [7], [8]

$$\lim_{n\to\infty}\frac{I(2n)}{I(2n+1)} =\ 1.$$

$$\therefore \lim_{n\to\infty} \frac{1}{2n+1}\frac{(a)^2+4}{2}\prod_{k=1}^{n}\frac{(2k+1)^2+(a)^2}{(2k)^2+(a)^2}=1,$$

$$\lim_{n\to\infty}\ \lim_{a\to 0} \frac{1}{2n+1}\frac{(a)^2+4}{2}\prod_{k=1}^{n}\frac{(2k+1)^2+(a)^2}{(2k)^2+(a)^2}=1,$$

$$\lim_{n\to\infty} \frac{2}{2n+1}\prod_{k=1}^{n}\frac{(2k+1)^2}{(2k)^2}=1,$$

$$\lim_{n\to\infty} \frac{2}{2n+1}\prod_{k=1}^{n}\frac{(2k+1)(2k-1)}{(2k)^2}\ \frac{(2k+1)}{(2k-1)}=1.$$

But, by use of Wallis' Product [5] we have,

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{(2k)^2}{(2k+1)(2k-1)} = \frac{\pi}{2}.$$

So,

$$\lim_{n\to\infty} \frac{2}{2n+1}\frac{2}{\pi}\prod_{k=1}^{n}\frac{(2k+1)}{(2k-1)}=1 \implies \lim_{n\to\infty} \frac{2(2n+1)}{2n+1}\frac{2}{\pi}=1 \implies \pi=4.$$

Again,

$$\lim_{n\to\infty}\frac{I(2n)}{I(2n+1)}=1,$$

$$\lim_{n\to\infty} \frac{1}{2n+1}\frac{(a)^2+4}{2}\prod_{k=1}^{n}\frac{(2k+1)^2+(a)^2}{(2k)^2+(a)^2}=1,$$

$$\lim_{n\to\infty}\ \mathit{lim}_{a\to i} \frac{1}{2n+1}\frac{(i)^2+4}{2}\prod_{k=1}^{n}\frac{(2k+1)^2+(i)^2}{(2k)^2+(i)^2}=1,$$

$$\lim_{n\to\infty} \frac{1}{2n+1}\times\frac{3}{2}\prod_{k=1}^{n}\frac{(2k+1)^2-1}{(2k)^2-1}=1,$$

$$\lim_{n\to\infty} \frac{1}{2n+1}\times\frac{3}{2}\prod_{k=1}^{n}\frac{(2k+2)(2k)}{(2k+1)(2k-1)}=1,$$

$$\lim_{n\to\infty} \frac{1}{2n+1}\times\frac{3}{2}\prod_{k=1}^{n}\frac{(2k+2)(2k)}{(2k+1)(2k-1)}=1,$$

$$\lim_{n\to\infty} \frac{1}{2n+1}\times\frac{3}{2}\prod_{k=1}^{n}\frac{(2k+2)}{(2k+1)}\prod_{k=1}^{n}\frac{2k}{(2k-1)}=1,$$

$$\lim_{n\to\infty} \frac{1}{2n+1}\times\frac{3}{2}\times\left(\frac{4}{3}\times\frac{6}{5}\times\frac{8}{7}\times\dots\right)\times\left(\frac{2}{1}\times\frac{4}{3}\times\frac{6}{5}\times\frac{8}{7}\times\dots\right)=1,$$

$$\lim_{n\to\infty} \frac{1}{2n+1}\times\frac{3}{2}\times\frac{1}{2}\prod_{k=1}^{n}\frac{2k}{(2k-1)}\prod_{k=1}^{n}\frac{2k}{(2k-1)}=1,$$

$$\lim_{n\to\infty} \frac{1}{2n+1}\times\frac{3}{2}\times\frac{1}{2}\prod_{k=1}^{n}\frac{(2k)^2}{(2k-1)(2k+1)}\prod_{k=1}^{n}\frac{(2k+1)}{(2k-1)}=1,$$

$$\lim_{n\to\infty} \frac{1}{2n+1}\times\frac{3}{2}\times\frac{1}{2}\times\frac{\pi}{2}\prod_{k=1}^{n}\frac{(2k+1)}{(2k-1)}=1,$$

$$\lim_{n\to\infty} \frac{2n+1}{2n+1}\times\frac{3}{2}\times\frac{1}{2}\times\frac{\pi}{2}=1 \implies \pi=\frac{8}{3}.$$

3. Contradiction Demystified

The above mentioned postulate [6] for infinite product or infinite sum states that "associativity and commutativity holds for infinite sum or product if and only if the sum or product is absolutely convergent". But in this argument, we applied associative and commutative laws for the infinite product namely,

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{(2k+1)^2+(a)^2}{(2k)^2+(a)^2}$$

for $a = 0$ and $a = i$. We will now test convergence of these products to check whether the operations were valid or not.

For $a = 0$:

$$\prod_{k=1}^{\infty}\frac{(2k+1)^2+(a)^2}{(2k)^2+(a)^2}=\prod_{k=1}^{\infty}\frac{(2k+1)^2}{(2k)^2}=\left(\prod_{k=1}^{\infty}\left(1+\frac{1}{2k}\right)\right)^2 \tag{3.1}$$

But a theorem states that, A product $\prod(1+a_n)$ with positive terms a_n is convergent if, and only if, the series $\sum a_n$ converges.[9].

As the series $\sum_{k=1}^{\infty}\frac{1}{2k}$ is divergent, the product $\prod_{k=1}^{\infty}\frac{(2k+1)^2}{(2k)^2}$ is also divergent.

For $a = i$: (Comparison Test)

$$\prod_{k=1}^{\infty}\frac{(2k+1)^2-1}{(2k)^2-1}\geq\prod_{k=1}^{\infty}\frac{(2k+1)^2-1}{(2k)^2}=\prod_{k=1}^{\infty}\frac{(2k)(2k+2)}{(2k)^2}$$

So,

$$\prod_{k=1}^{\infty}\frac{(2k+1)^2-1}{(2k)^2-1}\geq\prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right) \tag{3.2}$$

But the series $\sum_{k=1}^{\infty}\frac{1}{k}$ is divergent, so $\prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)$ is also divergent, i.e., the product $\prod_{k=1}^{\infty}\frac{(2k+1)^2-1}{(2k)^2-1}$ must be divergent by direct comparison test.

So, it is now clear that in the arguments we applied associative and commutative laws for the infinite products which are not convergent, so the operations are invalid, and hence contradictory results arise.

4. Conclusion

This study examines the invalidity of associative and commutative rules for infinite series, concentrating on two contradicting values of pi. The argument demonstrates that using infinite products can lead to serious inconsistencies, such as when the value of pi turns out to be 4. The Ramanujan Summation is a good example of this, where applying these associativity and commutativity rules to the sum of all positive integers yields a negative number, i.e., using an argument that appears to be legitimate but isn't. The contradiction shows that operations on infinite series or infinite products should be handled carefully. As mentioned above, for infinite sums or products, mathematical errors (e.g., contradictions) upon application of associativity and commutativity frequently arise, i.e., "associativity and commutativity hold for infinite sum or product if and only if the sum or product is convergent".

An easy example to understand where associativity doesn't hold is the infinite series $S = 1 - 1 + 1 - 1 + \cdots$ since, $S = 1 - 1 + 1 - 1 + \cdots = (1 - 1) + (1 - 1) + \cdots = 0$, while $S = 1 - 1 + 1 - 1 + ... = [1 - (1 - 1) + (1 - 1) + \cdots] = 1$.

Or the infinite product $P = 2 \div 2 \times 2 \div 2 \times 2 \div 2 \times \cdots$ since, $\log P = log\ 2 - \log\ 2 + \log\ 2 - \log\ 2 + \cdots = (\log\ 2 - \log\ 2) + (\log\ 2 - \log\ 2) + \cdots = 0$; $P = 1$ while $\log\ P = \log\ 2 - \log\ 2 + \log\ 2 - \log\ 2 + \cdots = \log\ 2 - [(\log\ 2 - \log\ 2) + (\log\ 2 - \log\ 2) + \cdots] = \log\ 2$; $P = 2$.

Or the Ramanujan Summation, where the sum of all positive integers turns out to be $-1/12$ as a result of associative law applied to infinite series which doesn't even make sense. Further, it is interesting that, $\sum_{n=1}^{\infty} n = -\frac{1}{12}$ has been used to derive the equations in 'string theory, quantum field theory and in some aspects of complex analysis.

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