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Generalized Identities and Inequalities of Čebyšev and Ky Fan Type for ∇−convex function

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Abstract

In the present article we establish three generalizations, first generalization is related to discrete Čebyšev identity for function of higher order ∇ divided difference with two independent variables and give its special case as a sequence of higher order ∇ divided difference. Moreover, we deduce results of discrete inequality of Čebyšev involving higher order ∇−convex function. The second and third generalizations are for integral Čebyšev and integral Ky Fan identities for function of higher order derivatives with two independent variables and discuss its inequalities using ∇−convex function. Generalized results give similar results of Pečarić's article [23] and recapture some established results.

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1. Introduction and Preliminaries

Over past few decades, there were some reviewers by [18] and [19] respectively, which traced completely the chronological and historical development of Čebyšev identities and its related inequalities. These research works are remarkable due to in many instances incorrect quotations of results, sometimes by change of several mathematical scholars–have been uncritically transferred paper to paper and book to book. It is well known that the famous Čebyšev functional is applied in many fields such as numerical quadrature, probability, transform theory, special functions and statistical problems (see [7]).

J an interval in $\mathbb{R}$, throughout the article. Also throughout the article we would use the following notations for $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = (0, \infty)$. We start this article from a significant result of Čebyšev [5, 6] may be stated as (see also [24], p. 197).

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Proposition 1.1. *Let $f, h : [a, b] \to \mathbb{R}$ be integrable functions and $r : [a, b] \to \mathbb{R}_+$, where r is also integrable. In the same direction, if f and h are monotone, then the inequality*

$$\int_a^b r(y)dy \int_a^b f(y)h(y)r(y)dy - \int_a^b h(y)r(y)dy \int_a^b f(y)r(y)dy \geq 0 \tag{1.1}$$

holds, if there is existence of integrals. In the opposite directions, if $f(y)$ and $h(y)$ are monotone, then (1.1) *is also valid for reverse inequality. Equality holds in* (1.1) *in the both cases, iff either h or f is constant function almost every places.*

A discrete form of above proposition can be present as following (see [24]).

Proposition 1.2. *Let r be a nonnegative $m-$tuple and $\mathbf{a}, \mathbf{b}$ be real $m-$tuples monotone in the same direction. Then*

$$\sum_{j=1}^{M} r_j a_j b_j \sum_{j=1}^{M} r_j - \sum_{j=1}^{M} r_j b_j \sum_{j=1}^{M} r_j a_j \geq 0 \tag{1.2}$$

holds. In the opposite directions, if $\mathbf{a}$ and $\mathbf{b}$ are monotone , then (1.2) *is also valid for reverse inequality. Equality holds in* (1.2) *in the both cases, iff either $a_1 = a_2 = \cdots = a_m$ or $b_1 = b_2 = \cdots = b_m$.*

For further details on the materials of Čebyšev's inequality we suggest books [20] and [24] and we also suggest related materials [1, 2, 3].

In [21] Ostrowski obtained the result which is connected to inequality of Čebyšev as follows:

Proposition 1.3. *Let $r : J \to \mathbb{R}_+$ an integrable function and $f, h \in C^1(J)$ be both monotone functions. Then $\exists\ \nu, \zeta \in J,\ \ni$*

$$T(f, h, r) = f'(\nu)h'(\zeta)T(y - a, y - a, r), \tag{1.3}$$

where

$$T(f, h, r) = \int_a^b f(y)h(y)r(y)dy \int_a^b r(y)dy - \int_a^b h(y)r(y)dy \int_a^b f(y)r(y)dy. \tag{1.4}$$

For other generalizations of Proposition 1.3, (see [22]). By using the functional, J. Pečarić has given the main generalization of Proposition 1.3 in [23] which is as follows:

$$\mathcal{C}(f, r) = \int_a^b \int_a^b f(y, y)r(y, z)dz\, dy - \int_a^b \int_a^b f(y, z)r(y, z)dz\, dy, \tag{1.5}$$

remember that above functions f and r are integrable.

Proposition 1.4. *Let r be the integrable function and defined by $r : J^2 \to \mathbb{R}$, $\ni$*

$$Y(y, y) = \overline{Y}(y, y) \quad \forall\, y \in J$$

and let either

$$Y(y, z) \geq 0, \quad a \leq z \leq y \leq b; \qquad \overline{Y}(y, z) \geq 0, \quad a \leq y \leq z \leq b$$

or their reverse inequalities be true, there

$$Y(y, z) = \int_y^b \int_a^z r(s, t)\, dt\, ds$$

and

$$\overline{Y}(y, z) = \int_a^y \int_z^b r(s, t)\, dt\, ds.$$

If $f : J^2 \to \mathbb{R}$ has $f_{(1,0)} = \frac{\partial}{\partial y} f(y, z)$, $f_{(0,1)} = \frac{\partial}{\partial z} f(y, z)$, and $f_{(1,1)} = \frac{\partial^2}{\partial y \partial z} f(y, z)$ continuous partial derivatives, then there is existence of $\nu,\ \zeta \in J^2$, $\ni$

$$\mathcal{C}(f, r) = \mathcal{C}((y - a)(z - a), r) f_{(1,1)}(\nu, \zeta). \tag{1.6}$$

Now we need some definitions and notations that can be extracted from [10, 13, 16, 17, 24], but we should remember the basic definitions from [14, 15].

The *mth order divided difference* of function $f : J \to \mathbb{R}$, at different elements $y_j, y_{j+1}, \ldots, y_{j+m} \in J = [a, b] \subset \mathbb{R}$, where $j \in \mathbb{N}$ is stated as:

$$\begin{aligned} [y_l; f] &= f(y_l), \quad l \in \{j, j+1, \ldots, j+m\} \\ [y_j, \ldots, y_{j+m}; f] &= \frac{[y_{j+1}, \ldots, y_{j+m}; f] - [y_j, \ldots, y_{j+m-1}; f]}{y_{j+m} - y_j}. \end{aligned}$$

We denote $[y_j, \ldots, y_{j+m}; f]$ by $\Delta_{(m)} f(y_j)$.

A function $f : J \to \mathbb{R}$, is known as $m-$*convex* or *mth order convex* , if $\Delta_{(m)} f(y_j) \geq 0$ holds $\forall (m+1)$ different points $y_j, \ldots, y_{j+m} \in J$. Further that if mth order derivative of function exists, then function is convex of order m iff $f^{(m)} \geq 0$.

A function $f : J \to \mathbb{R}$, is known as $m - \nabla-$convex, if $\nabla_{(m)} f(y_j) = (-1)^m \Delta_{(m)} f(y_j) \geq 0$ holds, where $\forall (m+1)$ different points $y_j, y_{j+1}, \ldots, y_{j+m}$. Further that if mth order derivative of function exists, then function is $\nabla-$convex of order m iff $(-1)^m f^{(m)} \geq 0$.

Let $f : J \times L \to \mathbb{R}$, be function, then (m, n)*th order divided difference* of function at different elements $y_j, y_{j+1}, \ldots, y_{j+m} \in J$, $z_l, z_{l+1}, \ldots, z_{l+n} \in L$ for some $j, l \in \mathbb{N}$, is stated as

$$\Delta_{(m,n)} f(y_j, z_l) = [y_j, \ldots, y_{j+m}; [z_l, \ldots, z_{l+n}; f]].$$

A function $f : J \times L \to \mathbb{R}$, is known as (m, n)*th order convex*, if $\forall$ different elements $y_j, \ldots, y_{j+m} \in J$ and $z_l, \ldots, z_{l+n} \in L$ we have $\Delta_{(m,n)} f(y_j, z_l) \geq 0$. Further that the f is $(m, n)-$th order convex iff $f_{(m,n)} \geq 0$, if the partial derivative $\frac{\partial^{m+n} f}{\partial y^m \partial z^n}$ denoted by $f_{(m,n)}$ and exists.

The finite difference of function $f : J \times L \to \mathbb{R}$ of order (m,n), where $h, k \in \mathbb{R}$ and $y \in J$, $z \in L$, is stated as

$$\begin{aligned} \Delta^{m,n}_{h,k} f(y, z) &= \Delta^m_h (\Delta^n_k f(y, z)) = \Delta^n_k (\Delta^m_h f(y, z)) \\ &= \sum_{j=0}^{m} \sum_{l=0}^{n} (-1)^{m-j+n-l} \binom{m}{j} \binom{n}{l} f(y + jh, z + lk), \end{aligned}$$

where $y + jh, z + lk \in J, L$ respectively and $j \in \{0, 1, \ldots, m\}$ and $l \in \{0, 1, \ldots, n\}$. Moreover, a function $f : J \times L \to \mathbb{R}$ is called the $(m, n)-$convex, if the following condition holds $\Delta^{m,n}_{h,k} f(y, z) \geq 0$, $\forall\ y \in J$, $z \in L$.

Finite difference and Divided difference of (m, n) order, of a sequence (a_{jl}) are stated as $\Delta^{m,n} a_{jl} = \Delta^{m,n}_{1,1} f(y_j, z_l)$ and $\Delta_{(m,n)} a_{jl} = \Delta_{(m,n)} f(y_j, z_l)$ respectively, where $j \in \{1, 2, \ldots, m\}$, $l \in \{1, 2, \ldots, n\}$. If $y_j = j$, $z_l = l$, then $f : \{1, \ldots, m\} \times \{1, \ldots, n\} \to \mathbb{R}$ is the function which is $f(j, l) = a_{jl}$. Moreover, a sequence (a_{jl}) is called a $(m, n)-$convex, if following condition holds $\Delta^{m,n} a_{jl} \geq 0$ for $m, n \geq 0$ and $j, l \in \{1, 2, \ldots\}$.

Further, in the current paper we would use following notations for some real sequence (a_m), $m \in \mathbb{N}$ and $n \in \{2, 3, \ldots\}$:

$$\nabla^{(1)} a_m = \nabla a_m = a_m - a_{m+1}, \quad \nabla^{(n)} a_m = \nabla(\nabla^{(n-1)} a_m).$$

We would recall the function $f : J \times L \to \mathbb{R}$ is known as $(m, n) - \nabla-$convex, if $\nabla_{(m,n)} f(y_j, z_l) = (-1)^{m+n} \Delta_{(m,n)} f(y_j, z_l) \geq 0$, holds $\forall$ different points $y_j, \ldots, y_{j+m} \in J$, $z_l, \ldots, z_{l+n} \in L$.

Now its time to describe discrete Čebyšev identity and inequality which is stated as following [23]. Let

$$\mathcal{C}^{\Delta}(a, r) = \sum_{j=1}^{M} \sum_{l=1}^{M} r_{jl} a_{jj} - \sum_{j=1}^{M} \sum_{l=1}^{M} r_{jl} a_{jl}, \tag{1.7}$$

where $r_{jl}, a_{jl} \in \mathbb{R}$; $j, l \in \{1, 2, \ldots, M\}$.

Proposition 1.5. *The following given inequality*

$$\mathcal{C}^{\Delta}(a, r) \geq 0 \tag{1.8}$$

holds $\forall$ *real numbers* a_{jl}, *for* $j, l \in \{1, 2, \dots, M\}$ *such that* $\Delta^{(1,1)} a_{jl} \geq 0$ *for* $j, l \in \{1, \dots, M-1\}$ *iff*

$$Y_{l+1,l} = \overline{Y}_{l,l+1} \quad l \in \{1, 2, \dots, M-1\}$$

and

$$Y_{jl} \geq 0, \quad j \in \{l+1, l+2, \dots, m\} \quad for \quad l \in \{1, 2, \dots, M-1\}$$
$$\overline{Y}_{jl} \geq 0, \quad j \in \{1, 2, \dots, l-1\} \quad for \quad l \in \{2, 3, \dots, M\}$$

hold. The reverse inequality of above inequality (1.8) *be also valid for* $j, l \in \{1, 2, \dots, M-1\}$, *if* $\Delta^{(1,1)} a_{jl} \leq 0$, *where*

$$Y_{jl} = \sum_{r=j}^{M} \sum_{s=1}^{l} r_{rs} \quad and \quad \overline{Y}_{jl} = \sum_{r=1}^{j} \sum_{s=l}^{M} r_{rs}.$$

Ky Fan [8] proposed the following result in 1952, as a problem (see also [18]):

Proposition 1.6. *Let* $(y, z) \mapsto v(y, z)$ *be a function of non-negative Lebesgue integrable over* $\{(y, z) : a \leq y \leq b \;\; ; \;\; a \leq z \leq b\}$ *square, and let* D *be a posotive constant* $\ni \int_a^b v(y, z) dy \leq D$ *for almost all* $z \in [a, b]$ *and* $\int_a^b v(y, z) dz \leq D$ *for almost all* $y \in [a, b]$. *If* f *and* h *finite valued functions and both are non-increasing and non-negative in the interval* $[a, b]$, *then*

$$\int_a^b \int_a^b v(y, z) f(y) h(z) dy\, dz \leq D \int_a^b f(y) h(y) dy \tag{1.9}$$

holds.

Remark 1.7. If $v(y, z) = constant$, then (1.9) becomes special case of inequality (1.1).

In [23] J. Pečarić considered the following expression for f, r and q integrable functions for generalization of result of Ky Fan

$$\mathcal{R}(f, r, q) = \int_a^b f(y, y) q(y) dy - \int_a^b \int_a^b r(y, z) f(y, z) dy\, dz \tag{1.10}$$

and gave the result as follows.

Proposition 1.8. *Let* $q : J \to \mathbb{R}$ *and* $r : J^2 \to \mathbb{R}$ *be both integrable functions,* $\ni$

$$Q_1(y, z) \leq S_1(max\{y, z\}); \quad Q_1(a, z) = S(z), \quad Q_1(y, a) = S_1(y), \quad \forall\, y,\, z \in [a, b]$$

where $Q_1(y, z) = \int_y^b \int_z^b r(s, t) dt\, ds, \quad S_1(y) = \int_y^b q(t) dt.$
If $f : J^2 \to \mathbb{R}$ *has* $f_{(0,1)}$, $f_{(1,0)}$, *and* $f_{(1,1)}$ *continuous partial derivatives on* J^2. *Then there is existence of* $(\nu, \zeta) \in J^2$, $\ni$

$$\mathcal{R}(f, r, q) = f_{(1,1)}(\nu, \zeta) \mathcal{R}((y-a)(z-a), r, q) \quad for \quad \nu,\, \zeta \in [a, b].$$

We give following Proposition for our article from above Proposition.

Proposition 1.9. *Let* $q : J \to \mathbb{R}$ *and* $r : J^2 \to \mathbb{R}$ *both are integrable functions,* $\ni$

$$Q(y, z) \leq S(max\{y, z\}); \quad Q(b, z) = S(z), \quad Q(y, b) = S(y), \quad \forall\, y,\, z \in [a, b]$$

where $S(y) = \int_a^y q(t) dt, \quad Q(y, z) = \int_a^y \int_a^z r(s, t) dt\, ds.$
If $f \in C^2(J^2)$, *then there exists* $(\nu, \zeta) \in J^2$ *such that*

$$\overline{\mathcal{R}}(f, r, q) = f_{(1,1)}(\nu, \zeta) \mathcal{R}((b-y)(b-z), r, q). \tag{1.11}$$

Under the suppositions of Proposition 1.9, we would like to use some notations for easy to present the statements of following upcoming theorems

$$Q^{(j,l)}(y,z) = \int_a^y \int_a^z r(s,t)\frac{(y-s)^j}{j!}\frac{(z-t)^l}{l!}dtds, \tag{1.12}$$

$$\overline{Q}^{(j,l)}(y,z) = \int_a^y \int_a^z r(s,t)\frac{(y-s)^j}{j!}\frac{(z-s)^l}{l!}dtds, \tag{1.13}$$

$$S^{(j,l)}(y) = \int_a^y q(s)\frac{(y-s)^j}{j!}\frac{(b-s)^l}{l!}ds, \tag{1.14}$$

$$\begin{aligned} \curlyvee(y,z) &= \int_a^{max\{y,z\}} \int_a^b r(s,t)\frac{(y-s)^M}{M!}\frac{(z-s)^N}{N!}dt\,ds \\ &- \int_a^y \int_a^z r(s,t)\frac{(y-s)^M}{M!}\frac{(z-t)^N}{N!}dt\,ds, \end{aligned} \tag{1.15}$$

$$\begin{aligned} \overline{\curlyvee}(y,z) &= \int_a^{max\{y,z\}} q(s)\frac{(y-s)^M}{M!}\frac{(z-s)^N}{N!}\,ds \\ &- \int_a^y \int_a^z r(s,t)\frac{(y-s)^M}{M!}\frac{(z-t)^N}{N!}dt\,ds. \end{aligned} \tag{1.16}$$

Let $q : J \to \mathbb{R}$ and $f, r : J^2 \to \mathbb{R}$ be functions and in which r, q are integrable and there should be existence of $f_{(M,N)}$ and absolutely continuous (in the Carathéodory's sense [25]), then for this article $\overline{\mathcal{C}}(f,r)$ and $\overline{\mathcal{R}}(f,r,q)$ defined as:

$$\begin{aligned} \overline{\mathcal{C}}(f,r) &= \mathcal{C}(f,r) - \sum_{j=0}^{M}\sum_{l=0}^{N} f_{(j,l)}(b,b)\left[\overline{Q}^{(j,l)}(b,b) - Q^{(j,l)}(b,b)\right] \\ &- \sum_{l=0}^{N}\int_a^b f_{(M+1,l)}(y,b)\left[\overline{Q}^{(M,l)}(y,b) - Q^{(M,l)}(y,b)\right]dy \\ &- \sum_{j=0}^{M}\int_a^b f_{(j,N+1)}(b,z)\left[\overline{Q}^{(j,N)}(b,z) - Q^{(j,N)}(b,z)\right]dz, \end{aligned} \tag{1.17}$$

where $\mathcal{C}(f,r)$ is stated in (1.5).

$$\begin{aligned} \overline{\mathcal{R}}(f,r,q) &= \mathcal{R}(f,r,q) - \sum_{l=0}^{N}\sum_{j=0}^{M} f_{(j,l)}(b,b)\left[S^{(j,l)}(b) - Q^{(j,l)}(b,b)\right] \\ &- \sum_{l=0}^{N}\int_a^b f_{(M+1,l)}(y,b)\left[S^{(M,l)}(y) - Q^{(M,l)}(y,b)\right]\,dy \\ &- \sum_{j=0}^{M}\int_a^b f_{(j,N+1)}(b,z)\left[S^{(j,N)}(z) - Q^{(j,N)}(b,z)\right]\,dz, \end{aligned} \tag{1.18}$$

where $\mathcal{R}(f,r,q)$ is stated in (1.10).

This article divide into four main sections. The second section is devoted to generalized discrete identity and inequality of Čebyšev and in 3rd and 4th sections, we would like to discuss about the generalized integral identities and inequalities of Čebyšev and Ky Fan respectively.

2. Generalized Discrete Identity and Inequality of Čebyšev

In present section we will get discrete identity and inequality of Čebyšev in sequential manner.

We would like to introduce following notation in this article

$$(y_k - y_j)^{\{n+1\}} = (y_k - y_j)(y_{k-1} - y_j)\dots(y_{k-n} - y_j), \quad (y_k - y_j)^{\{0\}} = 1.$$

We are required the following proposition and lemma from [11], to obtain the main identity of the present section .

Proposition 2.1. *Let m, M be integers; $m \leq M$ and let r_j be real numbers for $j \in \{1, 2, \cdots, M\}$. Let $f(y_j)$ be a function and y_j non-mutual elements from interval J, $j \in \{1, 2, \dots, M\}$. Then following identity holds*

$$\begin{aligned}\sum_{j=1}^{M} r_j f(y_j) &= \sum_{k=0}^{m-1}\left(\sum_{l=1}^{M-k} r_l (y_M - y_l)^{\{k\}}\right)\nabla_{(k)} f(y_{M-k}) \\ &+ \sum_{k=1}^{M-m}\left(\sum_{l=1}^{k} r_l (y_{k+m-1} - y_l)^{\{m-1\}}\right)\nabla_{(m)} f(y_k)(y_{k+m} - y_k).\end{aligned} \tag{2.1}$$

Lemma 2.2. *Let $r_{jl} \in \mathbb{R}$ and $f : J^2 \to \mathbb{R}$ be discrete function, where $j, l \in \{1, 2, \dots, M\}$, then following identity holds*

$$\begin{aligned}&\sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl} f(y_j, z_l) \\ &= \sum_{k=0}^{n-1}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{M-k} r_{sp}(z_M - z_p)^{\{k\}}(y_M - y_s)^{\{t\}}\nabla_{(t,k)} f(y_{M-t}, z_{M-k}) \\ &+ \sum_{k=0}^{n-1}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{M-k} r_{sp}(z_M - z_p)^{\{k\}}(y_{t+m-1} - y_s)^{\{m-1\}}\nabla_{(m,k)} f(y_t, z_{M-k})\times \\ &\times (y_{t+m} - y_t) \\ &+ \sum_{k=1}^{M-n}\sum_{t=0}^{m-1}\sum_{s=1}^{M-t}\sum_{p=1}^{k} r_{sp}(z_{k+n-1} - z_p)^{\{n-1\}}(y_M - y_s)^{\{t\}}\nabla_{(t,n)} f(y_{M-t}, z_k)\times \\ &\times (z_{k+n} - z_k) \\ &+ \sum_{k=1}^{M-n}\sum_{t=1}^{M-m}\sum_{s=1}^{t}\sum_{p=1}^{k} r_{sp}(z_{k+n-1} - z_p)^{\{n-1\}}(y_{t+m-1} - y_s)^{\{m-1\}}\nabla_{(m,n)} f(y_t, z_k)\times \\ &\times (y_{t+m} - y_t)(z_{k+n} - z_k).\end{aligned} \tag{2.2}$$

Theorem 2.3. *Let $f : J^2 \to \mathbb{R}$ be function and let $(y_j, z_l) \in J^2 = [a, b] \times [a, b]$ be mutually different points,* *where $(j, l = 1, 2, \ldots, M)$. Let r_{jl} $(j, l = 1, 2, \ldots, M)$, be real numbers. Then*

$$
\begin{aligned}
\mathcal{C}^{\nabla}(f,r) &= \sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl} f(y_j, z_j) - \sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl} f(y_j, z_l) \qquad (2.3)\\
&= \sum_{k=0}^{n-1}\sum_{t=0}^{m-1} \nabla_{(t,k)} f(y_{M-t}, z_{M-k}) \times \\
&\times \Bigg[\sum_{s=1}^{M-max\{t,k\}} \sum_{p=1}^{M-k} r_{sp}(z_M - z_s)^{\{k\}}(y_M - y_s)^{\{t\}} \\
&- \sum_{s=1}^{M-t}\sum_{p=1}^{M-k} r_{sp}(z_M - z_p)^{\{k\}}(y_M - y_s)^{\{t\}} \Bigg] \\
&+ \sum_{k=0}^{n-1}\sum_{t=1}^{M-m} \nabla_{(m,k)} f(y_t, z_{M-k})(y_{t+m} - y_t) \times \\
&\times \Bigg[\sum_{s=1}^{max\{t,k\}} \sum_{p=1}^{M-k} r_{sp}(z_M - z_s)^{\{k\}}(y_{t+m-1} - y_s)^{\{m-1\}} \\
&- \sum_{s=1}^{t}\sum_{p=1}^{M-k} r_{sp}(z_M - z_p)^{\{k\}}(y_{t+m-1} - y_s)^{\{m-1\}} \Bigg] \\
&= \sum_{k=1}^{M-n}\sum_{t=0}^{m-1} \nabla_{(t,n)} f(y_{M-t}, z_k)(z_{k+n} - z_k) \times \\
&\times \Bigg[\sum_{s=1}^{M-max\{t,k\}} \sum_{p=1}^{k} r_{sp}(z_{k+n-1} - z_s)^{\{n-1\}}(y_M - y_s)^{\{t\}} \\
&- \sum_{s=1}^{M-t}\sum_{p=1}^{k} r_{sp}(z_{k+n-1} - z_p)^{\{n-1\}}(y_M - y_s)^{\{t\}} \Bigg] \\
&+ \sum_{k=1}^{M-n}\sum_{t=1}^{M-m} \nabla_{(m,n)} f(y_t, z_k)(y_{t+m} - y_t)(z_{k+n} - z_k) \times \\
&\times \Bigg[\sum_{s=1}^{max\{t,k\}} \sum_{p=1}^{k} r_{sp}(z_{k+n-1} - z_s)^{\{n-1\}}(y_{t+m-1} - y_s)^{\{m-1\}} \\
&- \sum_{s=1}^{t}\sum_{p=1}^{k} r_{sp}(z_{k+n-1} - z_p)^{\{n-1\}}(y_{t+m-1} - y_s)^{\{m-1\}} \Bigg]
\end{aligned}
$$

holds, where $\nabla_{(m,n)} f(y, z) = (-1)^{m+n}\Delta_{(m,n)} f(y, z)$.

Proof. By considering the following expression we begin the proof of this theorem

$$\sum_{j=1}^{M}\sum_{l=1}^{M} \tilde{r}_{jl} f(y_j, z_j)$$

where $\tilde{r}_{jl}$ is defined as

$$\tilde{r}_{jl} = \begin{cases} \sum_{p=1}^{M} r_{jp}, & j = l, \\ 0, & j \neq l. \end{cases}$$

$$\sum_{j=1}^{M}\sum_{l=1}^{M} \tilde{r}_{jl} f(y_j, z_j) = \sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl} f(y_j, z_j)$$

We have

$$\sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl} f(y_j, z_j) = \sum_{j=1}^{M}\left(\sum_{l=1}^{M} q_l G_j(z_j)\right),$$

where $r_{jl} = q_l$ and $G_j : z \mapsto f(y_j, z)$. Using (2.1) in the inner sum we get

$$\begin{aligned}
\sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl} f(y_j, z_j) &= \sum_{j=1}^{M}\sum_{k=0}^{n-1}\left(\sum_{l=1}^{M-k} q_l (z_M - z_j)^{\{k\}}\right)\nabla_{(k)} G_j(z_{M-k}) \\
&+ \sum_{j=1}^{M}\sum_{k=1}^{M-n}\left(\sum_{l=1}^{k} q_l (z_{k+n-1} - z_j)^{\{n-1\}}\right)\nabla_{(n)} G_j(z_k)(z_{k+n} - z_k) \\
&= \sum_{k=0}^{n-1}\left(\sum_{j=1}^{M}\left(\sum_{l=1}^{M-k} q_l (z_M - z_j)^{\{k\}}\right)\nabla_{(k)} G_j(z_{M-k})\right) \\
&+ \sum_{k=1}^{M-n}\left(\sum_{j=1}^{M}\left(\sum_{l=1}^{k} q_l (z_{k+n-1} - z_j)^{\{n-1\}}\right)\nabla_{(n)} G_j(z_k)(z_{k+n} - z_k)\right) \\
&= \sum_{k=0}^{n-1}\left(\sum_{j=1}^{M} w_j F(y_j)\right) + \sum_{k=1}^{M-n}\left(\sum_{j=1}^{M} v_j H(y_j)\right),
\end{aligned}$$

where $w_j = \sum_{l=1}^{M-k} q_l (z_M - z_j)^{\{k\}} = \sum_{l=1}^{M-k} r_{jl} (z_M - z_j)^{\{k\}}$, $v_j = \sum_{l=1}^{k} q_l (z_{k+n-1} - z_j)^{\{n-1\}}$, $F(y_j) = \nabla_{(k)} G_j(z_{M-k})$, and $H(y_j) = \nabla_{(n)} G_j(z_k)(z_{k+n} - z_k)$.

Using again (2.1) on inner sums, then we have

$$\begin{aligned}
&\sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl} f(y_j, z_j) \\
&= \sum_{k=0}^{n-1}\sum_{p=0}^{m-1}\left(\sum_{j=1}^{M-p} w_j (y_M - y_j)^{\{p\}}\right)\nabla_{(p)} F(y_{M-p}) \\
&+ \sum_{k=0}^{n-1}\sum_{p=1}^{M-m}\left(\sum_{j=1}^{p} w_j (y_{p+m-1} - y_j)^{\{m-1\}}\right)\nabla_{(m)} F(y_p)(y_{p+m} - y_p)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{M-n}\sum_{t=0}^{m-1}\left(\sum_{j=1}^{M-t} v_j(y_M-y_j)^{\{t\}}\right)\nabla_{(t)}H(y_{M-t})\\
&+\sum_{k=1}^{M-n}\sum_{t=1}^{M-m}\left(\sum_{j=1}^{t} v_j(y_{t+m-1}-y_j)^{\{m-1\}}\right)\nabla_{(m)}H(y_t)(y_{t+m}-y_t)\\
&=\sum_{k=0}^{n-1}\sum_{p=0}^{m-1}\sum_{j=1}^{M-max\{p,k\}}\sum_{l=1}^{M-k} r_{jl}(z_M-z_j)^{\{k\}}(y_M-y_j)^{\{p\}}\nabla_{(p,k)}f(y_{M-p},z_{M-k})\\
&+\sum_{k=0}^{n-1}\sum_{p=1}^{M-m}\sum_{j=1}^{max\{p,k\}}\sum_{l=1}^{M-k} r_{jl}(z_M-z_j)^{\{k\}}(y_{p+m-1}-y_j)^{\{m-1\}}\nabla_{(m,k)}f(y_p,z_{M-k})\times\\
&\times(y_{p+m}-y_p)\\
&+\sum_{k=1}^{M-n}\sum_{t=0}^{m-1}\sum_{j=1}^{M-max\{t,k\}}\sum_{l=1}^{k} r_{jl}(z_{k+n-1}-z_j)^{\{n-1\}}(y_M-y_j)^{\{t\}}\nabla_{(t,n)}f(y_{M-t},z_k)\times\\
&\times(z_{k+n}-z_k)\\
&+\sum_{k=1}^{M-n}\sum_{t=1}^{M-m}\sum_{j=1}^{max\{t,k\}}\sum_{l=1}^{k} r_{jl}(z_{k+n-1}-z_j)^{\{n-1\}}(y_{t+m-1}-y_j)^{\{m-1\}}\times\\
&\times\nabla_{(m,n)}f(y_t,z_k)(y_{t+m}-y_t)(z_{k+n}-z_k).
\end{aligned}
$$

If change $j\to s$, $l\to p$ in all sums and put $p\to t$ in first and second sums, then we obtain the required result by putting the values of $\sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl}f(y_j,z_j)$ and Lemma 2.2 in $\mathcal{C}^{\nabla}(f,r)=\sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl}f(y_j,z_j)-\sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl}f(y_j,z_l)$. □

Remark 2.4. If put $y_j=j$, $z_l=l$ and $f(y_j,z_l)=f(j,l)=a_{jl}$ in Theorem 2.3, then get upcoming next corollary.

Corollary 2.5. *Let r_{jl} and a_{jl} $(l,j=1,2,3,\ldots,M)$ be real numbers, then*

$$
\begin{aligned}
\mathcal{C}^{\nabla}(a,r)&=\sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl}a_{jj}-\sum_{j=1}^{M}\sum_{l=1}^{M} r_{jl}a_{jl}\\
&=\sum_{k=0}^{n-1}\sum_{t=0}^{m-1}\nabla_{(t,k)}a_{(M-t,M-k)}\left[\sum_{s=1}^{M-max\{t,k\}}\sum_{p=1}^{M-k} r_{sp}\frac{(M-s)^{\{t\}}}{t!}\frac{(M-s)^{\{k\}}}{k!}\right.\\
&\left.-\sum_{s=1}^{M-t}\sum_{p=1}^{M-k} r_{sp}\frac{(M-s)^{\{t\}}}{t!}\frac{(M-p)^{\{k\}}}{k!}\right]
\end{aligned}
$$

$$+\sum_{k=0}^{n-1}\sum_{t=1}^{M-m}\nabla_{(m,k)}a_{(t,M-k)}\left[\sum_{s=1}^{max\{t,k\}}\sum_{p=1}^{M-k}r_{sp}\frac{(M-s)^{\{k\}}}{k!}\frac{(t-s+m-1)^{\{m-1\}}}{(m-1)!}\right.$$
$$\left.-\sum_{s=1}^{t}\sum_{p=1}^{M-k}r_{sp}\frac{(M-p)^{\{k\}}}{k!}\frac{(t-s+m-1)^{\{m-1\}}}{(m-1)!}\right]$$
$$+\sum_{k=1}^{M-n}\sum_{t=0}^{m-1}\nabla_{(t,n)}a_{(M-t,k)}\left[\sum_{s=1}^{M-max\{t,k\}}\sum_{p=1}^{k}r_{sp}\frac{(M-s)^{\{t\}}}{t!}\frac{(k-s+n-1)^{\{n-1\}}}{(n-1)!}\right.$$
$$\left.-\sum_{s=1}^{M-t}\sum_{p=1}^{k}r_{sp}\frac{(M-s)^{\{t\}}}{t!}\frac{(k-p+n-1)^{\{n-1\}}}{(n-1)!}\right]$$
$$+\sum_{k=1}^{M-n}\sum_{t=1}^{M-m}\nabla_{(m,n)}a_{(t,k)}\left[\sum_{s=1}^{max\{t,k\}}\sum_{p=1}^{k}r_{sp}\frac{(k-s+n-1)^{\{n-1\}}}{(n-1)!}\frac{(t-s+m-1)^{\{m-1\}}}{(m-1)!}\right.$$
$$\left.-\sum_{s=1}^{t}\sum_{p=1}^{k}r_{sp}\frac{(k-p+n-1)^{\{n-1\}}}{(n-1)!}\frac{(t-s+m-1)^{\{m-1\}}}{(m-1)!}\right]$$

holds, where $\nabla_{(m,n)}a_{jl}$ *represents nabla divided difference of order* (m,n) *of the sequence* (a_{jl}).

Before starting the next theorem, we would like to state few notations, under suppositions of Theorem 2.3

$$\overline{\mathcal{C}}^{\nabla}(f,r)=\mathcal{C}^{\nabla}(f,r)-\sum_{k=0}^{n-1}\sum_{t=0}^{m-1}\nabla_{(t,k)}f(y_{M-t},z_{M-k})\times \tag{2.4}$$
$$\times\left[\sum_{s=1}^{M-max\{t,k\}}\sum_{p=1}^{M-k}r_{sp}(z_M-z_s)^{\{k\}}(y_M-y_s)^{\{t\}}\right.$$
$$\left.-\sum_{s=1}^{M-t}\sum_{p=1}^{M-k}r_{sp}(z_M-z_p)^{\{k\}}(y_M-y_s)^{\{t\}}\right]$$
$$-\sum_{k=0}^{n-1}\sum_{t=1}^{M-m}\nabla_{(m,k)}f(y_t,z_{M-k})(y_{t+m}-y_t)\times$$
$$\times\left[\sum_{s=1}^{max\{t,k\}}\sum_{p=1}^{M-k}r_{sp}(z_M-z_s)^{\{k\}}(y_{t+m-1}-y_s)^{\{m-1\}}\right.$$
$$\left.-\sum_{s=1}^{t}\sum_{p=1}^{M-k}r_{sp}(z_M-z_p)^{\{k\}}(y_{t+m-1}-y_s)^{\{m-1\}}\right]$$

$$- \sum_{k=1}^{M-n} \sum_{t=0}^{m-1} \nabla_{(t,n)} f(y_{M-t}, z_k)(z_{k+n} - z_k) \times$$
$$\times \left[\sum_{s=1}^{M-max\{t,k\}} \sum_{p=1}^{k} r_{sp}(z_{k+n-1} - z_s)^{\{n-1\}}(y_M - y_s)^{\{t\}} \right.$$
$$\left. - \sum_{s=1}^{M-t} \sum_{p=1}^{k} r_{sp}(z_{k+n-1} - z_p)^{\{n-1\}}(y_M - y_s)^{\{t\}} \right],$$

$$\Upsilon^{\nabla}(t,k) = \left[\sum_{s=1}^{max\{t,k\}} \sum_{p=1}^{k} r_{sp}(z_{k+n-1} - z_s)^{\{n-1\}}(y_{t+m-1} - y_s)^{\{m-1\}} \right.$$
$$\left. - \sum_{s=1}^{t} \sum_{p=1}^{k} r_{sp}(z_{k+n-1} - z_p)^{\{n-1\}}(y_{t+m-1} - y_s)^{\{m-1\}} \right]. \quad (2.5)$$

Theorem 2.6. *Let (y_j) and (z_l) $(j, l = 1, 2, \dots, M)$ be real sequences and monotonic in the same sense and f is $\nabla-$convex function of order (m, n) and $r_{jl} \in \mathbb{R}$ $(j, l = 1, 2, \dots, M)$. Then*

$$\overline{\mathcal{C}}^{\nabla}(f, r) \geq 0 \quad if \quad \Upsilon^{\nabla}(t,k) \geq 0; \quad t = m+1, m+2, \dots, M \quad , \quad k = n+1, n+2, \dots, M.$$

where $\overline{\mathcal{C}}^{\nabla}(f, r)$ and $\Upsilon^{\nabla}(t,k)$ are stated respectively in (2.4) and (2.5).

Proof. This result can easily obtain using (2.3). □

Remark 2.7. If put $y_j = j$, $z_l = l$ and $f(y_j, z_l) = f(j, l) = a_{jl}$ in previous theorem for $m = n = 1$ then we get similar result for $\nabla-$convex function of Theorem 3 of paper [23] and hence in this result for $a_{jl} = f(a_j, b_l)$ we can also get similar result for $\nabla-$convex function of Corollary 2 of paper [23].

Theorem 2.8. *Let $(y_j, z_l) \in J^2 = [a, b] \times [a, b]$ $(l, j = 1, 2, \dots, M)$, be mutually distinct elements and $r_{jl} \in \mathbb{R}$ $(l, j = 1, 2, \dots, M)$ and suppose that $f, h : J^2 \to \mathbb{R}$ be $(m, n) - \nabla-$convex functions, $\ni$ inequalities*

$$\Upsilon^{\nabla}(t,k) \geq 0; \quad t = m+1, m+2, \dots, M \quad , \quad k = n+1, n+2, \dots, M. \quad (2.6)$$

and

$$L\nabla_{(m,n)} h(y_j, z_l) \leq \nabla_{(m,n)} f(y_j, z_l) \leq U\nabla_{(m,n)} h(y_j, z_l) \quad (2.7)$$

hold, then below are valid

$$L\overline{\mathcal{C}}^{\nabla}(h, r) \leq \overline{\mathcal{C}}^{\nabla}(f, r) \leq U\overline{\mathcal{C}}^{\nabla}(h, r), \quad (2.8)$$

where $\Upsilon^{\nabla}(t,k)$ is stated in (2.5) and U, L belong to some real constants.

Proof. Let $F_1(y_j, z_l) = f(y_j, z_l) - Lh(y_j, z_l)$ and $F_2(y_j, z_l) = Uh(y_j, z_l) - f(y_j, z_l)$, then $\nabla_{(m,n)} F_1(y_j, z_l) \geq 0$ and $\nabla_{(m,n)} F_2(y_j, z_l) \geq 0$, now using Theorem 2.6 we can get Theorem 2.8. □

Remark 2.9. If reverse inequalities hold in (2.6) and (2.7), then inequalities in (2.8) remain hold. Further that if the reverse inequalities in (2.8) are also valid, if reverse of inequality holds in (2.6).

Remark 2.10. If put $y_j = j$, $z_l = l$ and $f(y_j, z_l) = f(j, l) = a_{jl}$ and $h(j, l) = b_{jl}$ in previous theorem then we get similar result for $\nabla-$convex function of Theorem 4 of paper [23].

3. Generalized Integral Identity and Inequality of Čebyšev

In the current section, we start from the following lemma [16] which will be helpful for proving the upcoming important theorem.

Lemma 3.1. *Let $r, f : J^2 \to \mathbb{R}$ be both functions, r be an integrable function and $f \in C^{(M+1,N+1)}(J^2)$, where $j \in \{0, 1, \cdots, M+1\}$, $l \in \{0, 1, \ldots, N+1\}$. Then the following identity holds*

$$\begin{aligned}
&\int_a^b \int_a^b r(y,z) f(y,z) dz\, dy \\
&= \sum_{j=0}^{M} \sum_{l=0}^{N} \int_a^b \int_a^b (-1)^{j+l} r(s,t) \frac{(b-s)^j}{j!} \frac{(b-t)^l}{l!} f_{(j,l)}(b,b) dt\, ds \\
&+ \sum_{l=0}^{N} \int_a^b \int_a^y \int_a^b (-1)^{M+1+l} r(s,t) \frac{(y-s)^M}{M!} \frac{(b-t)^l}{l!} f_{(M+1,l)}(y,b) dt\, ds\, dy \\
&+ \sum_{j=0}^{M} \int_a^b \int_a^b \int_a^y (-1)^{j+N+1} r(s,t) \frac{(b-s)^j}{j!} \frac{(z-t)^N}{N!} f_{(j,N+1)}(b,z) dt\, ds\, dz \\
&+ \int_a^b \int_a^b \int_a^y \int_a^z (-1)^{M+N} r(s,t) \frac{(y-s)^M}{M!} \frac{(z-t)^N}{N!} f_{(M+1,N+1)}(y,z) dt\, ds\, dz\, dy.
\end{aligned}$$

Theorem 3.2. *Let $r, f : J^2 \to \mathbb{R}$ be both functions, where r is an integrable and there should be existence of partial derivatives $f_{(M+1,N)}$ and $f_{(M,N+1)}$ that are absolutely continuous. Then*

$$\begin{aligned}
\mathcal{C}(f,r) &= \int_a^b \int_a^b r(y,z) f(y,y) dz\, dy - \int_a^b \int_a^b r(y,z) f(y,z) dz\, dy \\
&= \sum_{j=0}^{M} \sum_{l=0}^{N} (-1)^{j+l} f_{(j,l)}(b,b) \left[\overline{Q}^{(j,l)}(b,b) - Q^{(j,l)}(b,b) \right] \\
&+ \sum_{l=0}^{N} \int_a^b (-1)^{M+1+l} f_{(M+1,l)}(y,b) \left[\overline{Q}^{(M,l)}(y,b) - Q^{(M,l)}(y,b) \right] dy \\
&+ \sum_{j=0}^{M} \int_a^b (-1)^{j+N+1} f_{(j,N+1)}(b,z) \left[\overline{Q}^{(j,N)}(b,z) - Q^{(j,N)}(b,z) \right] dz \\
&+ \int_a^b \int_a^b (-1)^{M+N} f_{(M+1,N+1)}(y,z) \curlyvee (y,z) dz\, dy, \qquad (3.1)
\end{aligned}$$

where $Q^{(j,l)}$, $\overline{Q}^{(j,l)}$, and $\curlyvee(y,z)$ are stated in (1.12), (1.13), and (1.15) respectively.

Proof. For fixed y we define a function $f(y,z) = F_y(z)$. Now we write Taylor expansion of $F_y(z)$ as follows:

$$\begin{aligned}
f(y,z) &= F_y(z) = \sum_{l=0}^{N} F^{(l)}(b) \frac{(z-b)^l}{l!} + \int_b^z F^{(N+1)}(t) \frac{(z-t)^N}{N!} dt \\
&= \sum_{l=0}^{N} (-1)^l \frac{(b-z)^l}{l!} f_{(0,l)}(z,b) + \int_z^b (-1)^{N+1} \frac{(t-z)^N}{N!} f_{(0,N+1)}(z,t)\, dt,
\end{aligned}$$

where $F^{(l)}(b) = f_{(0,l)}(y,b)$ and $F^{(N+1)}(t) = f_{(0,N+1)}(y,t)$.

Now, for $z = y$ we have

$$f(y,y) = \sum_{l=0}^{N} (-1)^l \frac{(b-y)^l}{l!} f_{(0,l)}(y,b) + \int_y^b (-1)^{N+1} \frac{(t-y)^N}{N!} f_{(0,N+1)}(y,t)\, dt.$$

Multiplying above equation by $r(y,z)$ and integrate it by z over the limit a to b, then

$$\begin{aligned}\int_a^b r(y,z)f(y,y)dz &= \sum_{l=0}^{N}(-1)^l f_{(0,l)}(y,b)\int_a^b r(y,z)\frac{(b-y)^l}{l!}dz \\ &+ \int_a^b \left(\int_y^b r(y,z)(-1)^{N+1} f_{(0,N+1)}(y,t)\frac{(t-y)^N}{N!}dt\right)dz.\end{aligned} \tag{3.2}$$

Now we further use representation of functions $y \mapsto f_{(0,l)}(y,b)$ and $y \mapsto f_{(0,N+1)}(y,t)$ by Taylor expansions:

$$\begin{aligned}f_{(0,l)}(y,b) &= \sum_{j=0}^{M}(-1)^j f_{(j,l)}(b,b)\frac{(b-y)^j}{j!} + \int_y^b (-1)^{M+1} f_{(M+1,l)}(s,b)\frac{(s-y)^M}{M!}ds, \\ f_{(0,N+1)}(y,t) &= \sum_{j=0}^{M}(-1)^j f_{(j,N+1)}(b,t)\frac{(b-y)^j}{j!} + \int_y^b (-1)^{M+1} f_{(M+1,N+1)}(s,t)\frac{(s-y)^M}{M!}ds.\end{aligned}$$

Putting these above formulae in equation (3.2), then

$$\begin{aligned}&\int_a^b r(y,z)f(y,y)dz \\ &= \sum_{l=0}^{N}(-1)^l \left(\sum_{j=0}^{M}(-1)^j f_{(j,l)}(b,b)\frac{(b-y)^j}{j!}\right. \\ &+ \left.\int_y^b (-1)^{M+1} f_{(M+1,l)}(s,b)\frac{(s-y)^M}{M!}ds\right)\int_a^b r(y,z)\frac{(b-y)^l}{l!}dz \\ &+ \int_a^b \left(\int_y^b r(y,z)(-1)^{N+1}\left(\sum_{j=0}^{M}(-1)^j f_{(j,N+1)}(b,t)\frac{(b-y)^j}{j!}\right.\right. \\ &+ \left.\left.\int_y^b (-1)^{M+1} f_{(M+1,N+1)}(s,t)\frac{(s-y)^M}{M!}ds\right)\frac{(t-y)^N}{N!}dt\right)dz\end{aligned}$$

$$\begin{aligned}&= \sum_{l=0}^{N}\left(\sum_{j=0}^{M}(-1)^{j+l} f_{(j,l)}(b,b)\frac{(b-y)^j}{j!}\right)\int_a^b r(y,z)\frac{(b-y)^l}{l!}dz \\ &+ \sum_{l=0}^{N}\left(\int_y^b (-1)^{M+1+l} f_{(M+1,l)}(s,b)\frac{(s-y)^M}{M!}ds\right)\int_a^b r(y,z)\frac{(b-y)^l}{l!}dz \\ &+ \int_a^b\int_y^b r(y,z)\left(\sum_{j=0}^{M}(-1)^{j+N+1} f_{(j,N+1)}(b,t)\frac{(b-y)^j}{j!}\right)\frac{(t-y)^N}{N!}dt\,dz \\ &+ \int_a^b\int_y^b\left(\int_y^b r(y,z)(-1)^{M+N} f_{(M+1,N+1)}(s,t)\frac{(s-y)^M}{M!}ds\right)\frac{(t-y)^N}{N!}dt\,dz.\end{aligned}$$

Now integrate $r(y,z)f(y,y)$ by y over the limit a to b and obtain:

$$\begin{aligned}
&\int_a^b\int_a^b r(y,z)f(y,y)dz\,dy\\
&=\int_a^b\left[\sum_{l=0}^{N}\left(\sum_{j=0}^{M}(-1)^{j+l}f_{(j,l)}(b,b)\frac{(b-y)^j}{j!}\right)\int_a^b r(y,z)\frac{(b-y)^l}{l!}dz\right]dy\\
&+\int_a^b\left[\sum_{l=0}^{N}\left(\int_y^b(-1)^{M+1+l}f_{(M+1,l)}(s,b)\frac{(s-y)^M}{M!}ds\right)\int_a^b r(y,z)\frac{(b-y)^l}{l!}dz\right]dy\\
&+\int_a^b\left[\int_a^b\int_y^b r(y,z)\left(\sum_{j=0}^{M}(-1)^{j+N+1}f_{(j,N+1)}(b,t)\frac{(b-y)^j}{j!}\right)\frac{(t-y)^N}{N!}dt\,dz\right]dy\\
&+\int_a^b\left[\int_a^b\int_y^b\left(\int_y^b r(y,z)(-1)^{M+N}f_{(M+1,N+1)}(s,t)\frac{(s-y)^M}{M!}ds\right)\frac{(t-y)^N}{N!}dt\,dz\right]dy.
\end{aligned}$$

Now changing the order of summation in first summand, and use integral linearity property and obtain.

$$\sum_{j=0}^{M}\sum_{l=0}^{N}\int_a^b\int_a^b r(y,z)(-1)^{j+l}f_{(j,l)}(b,b)\frac{(b-y)^j}{j!}\frac{(b-y)^l}{l!}dz\,dy.$$

The 2nd summand is rewritten as,

$$\begin{aligned}
&\int_a^b\left[\sum_{l=0}^{N}\left(\int_y^b(-1)^{M+1+l}f_{(M+1,l)}(s,b)\frac{(s-y)^M}{M!}ds\right)\int_a^b r(y,z)\frac{(b-y)^l}{l!}dz\right]dy\\
&=\int_a^b\left[\sum_{l=0}^{N}\left(\int_y^b\int_a^b r(y,z)\frac{(b-y)^l}{l!}(-1)^{M+1+l}f_{(M+1,l)}(s,b)\frac{(s-y)^M}{M!}dz\,ds\right)\right]dy\\
&=\sum_{l=0}^{N}\int_a^b\int_y^b\int_a^b r(y,z)(-1)^{M+1+l}f_{(M+1,l)}(s,b)\frac{(s-y)^M}{M!}\frac{(b-y)^l}{l!}dz\,ds\,dy\\
&=\sum_{l=0}^{N}\int_a^b\int_a^s\int_a^b r(y,z)(-1)^{M+1+l}f_{(M+1,l)}(s,b)\frac{(s-y)^M}{M!}\frac{(b-y)^l}{l!}dz\,dy\,ds.
\end{aligned}$$

Applying Fubini theorem for variables s and y in the last step. Let us first, the change of variable y from a to b while the changing of variable s from y to b. After the change of order of integration, s is changed a to b while y is changed a to s. In the similar manner the 3rd summand may be rewritten as:

$$\begin{aligned}
&\int_a^b\left[\int_a^b\int_y^b r(y,z)\left(\sum_{j=0}^{M}(-1)^{j+N+1}f_{(j,N+1)}(b,t)\frac{(b-y)^j}{j!}\right)\frac{(t-y)^N}{N!}dt\,dz\right]dy\\
&=\sum_{j=0}^{M}\int_a^b\int_a^b\int_y^b r(y,z)(-1)^{j+N+1}f_{(j,N+1)}(b,t)\frac{(b-y)^j}{j!}\frac{(t-y)^N}{N!}dt\,dz\,dy\\
&=\sum_{j=0}^{M}\int_a^b\int_a^b\int_a^t r(y,z)(-1)^{j+N+1}f_{(j,N+1)}(b,t)\frac{(b-y)^j}{j!}\frac{(t-y)^N}{N!}dz\,dt\,dy\\
&=\sum_{j=0}^{M}\int_a^b\int_a^b\int_a^t r(y,z)(-1)^{j+N+1}f_{(j,N+1)}(b,t)\frac{(b-y)^j}{j!}\frac{(t-y)^N}{N!}dz\,dy\,dt.
\end{aligned}$$

In above using Fubini theorem twice. First, changing t and z, then changing t and y. Therefore, the last summand may be rewritten as:

$$\int_a^b \left[\int_a^b \int_y^b \left(\int_y^b r(y,z)(-1)^{M+N} f_{(M+1,N+1)}(s,t)\frac{(s-y)^M}{M!}ds\right)\frac{(t-y)^N}{N!}dt\,dz\right]dy$$
$$= \int_a^b \int_a^b \int_y^b \int_y^b r(y,z)(-1)^{M+N} f_{(M+1,N+1)}(s,t)\frac{(s-y)^M}{M!}\frac{(t-y)^N}{N!}ds\,dt\,dz\,dy$$
$$= \int_a^b \int_a^b \int_a^{max\{s,t\}} \int_a^b r(y,z)(-1)^{M+N} f_{(M+1,N+1)}(s,t)\frac{(s-y)^M}{M!}\frac{(t-y)^N}{N!}dz\,dy\,dt\,ds.$$

Now, adding up all these summand results to obtain

$$\int_a^b \int_a^b r(y,z) f(y,y) dz\,dy$$
$$= \sum_{j=0}^M \sum_{l=0}^N \int_a^b \int_a^b r(y,z)(-1)^{j+l} f_{(j,l)}(b,b)\frac{(b-y)^j}{j!}\frac{(b-y)^l}{l!}dz\,dy$$
$$+ \sum_{l=0}^N \int_a^b \int_a^s \int_a^b r(y,z)(-1)^{M+1+l} f_{(M+1,l)}(s,b)\frac{(s-y)^M}{M!}\frac{(b-y)^l}{l!}dz\,dy\,ds$$
$$+ \sum_{j=0}^M \int_a^b \int_a^b \int_a^t r(y,z)(-1)^{j+N+1} f_{(j,N+1)}(b,t)\frac{(b-y)^j}{j!}\frac{(t-y)^N}{N!}dz\,dy\,dt$$
$$+ \int_a^b \int_a^b \int_a^{max\{s,t\}} \int_a^b r(y,z)(-1)^{M+N} f_{(M+1,N+1)}(s,t)\frac{(s-y)^M}{M!}\frac{(t-y)^N}{N!}dz\,dy\,dt\,ds.$$

After changing $y \leftrightarrow s$, $z \leftrightarrow t$ on right side, then get:

$$\int_a^b \int_a^b r(y,z) f(y,y) dz\,dy$$
$$= \sum_{j=0}^M \sum_{l=0}^N \int_a^b \int_a^b r(s,t)(-1)^{j+l} f_{(j,l)}(b,b)\frac{(b-s)^{j+l}}{j!l!}dt\,ds$$
$$+ \sum_{l=0}^N \int_a^b \int_a^y \int_a^b r(s,t)(-1)^{M+1+l} f_{(M+1,l)}(y,b)\frac{(y-s)^M}{M!}\frac{(b-s)^l}{l!}dt\,ds\,dy$$
$$+ \sum_{j=0}^M \int_a^b \int_a^b \int_a^z r(s,t)(-1)^{j+N+1} f_{(j,N+1)}(b,z)\frac{(b-s)^j}{j!}\frac{(z-s)^N}{N!}dt\,ds\,dz$$
$$+ \int_a^b \int_a^b \int_a^{max\{y,z\}} \int_a^b r(s,t)(-1)^{M+N} f_{(M+1,N+1)}(y,z)\frac{(y-s)^M}{M!}\frac{(z-s)^N}{N!}dt\,ds\,dz\,dy.$$

Now by defined notations, finally we obtain

$$\begin{aligned}
&\int_a^b \int_a^b r(y,z) f(y,y) dz\, dy \\
&= \sum_{j=0}^{M} \sum_{l=0}^{N} (-1)^{j+l} f_{(j,l)}(b,b) \overline{Q}^{(j,l)}(b,b) \\
&+ \sum_{l=0}^{N} \int_a^b (-1)^{M+1+l} f_{(M+1,l)}(y,b) \overline{Q}^{(M,l)}(y,b)\, dy \\
&+ \sum_{j=0}^{M} \int_a^b (-1)^{j+N+1} f_{(j,N+1)}(b,z) \overline{Q}^{(j,N)}(b,z)\, dz \\
&+ \int_a^b \int_a^b (-1)^{M+N} f_{(M+1,N+1)}(y,z) \int_a^{max\{y,z\}} \int_a^b r(s,t) \frac{(y-s)^M}{M!} \frac{(z-s)^N}{N!} dt\, ds\, dz\, dy,
\end{aligned}$$

where $\overline{Q}^{(j,l)}$ is stated in (1.13). Use above expression for $\int_a^b \int_a^b r(y,z) f(y,y) dz\, dy$ and Lemma 3.1 in

$$\mathcal{C}(f,r) = \int_a^b \int_a^b r(y,z) f(y,y) dz\, dy - \int_a^b \int_a^b r(y,z) f(y,z) dz\, dy,$$

we get our required identity. □

Remark 3.3. If put $f(y,z) = f(y)h(z)$ and $r(y,z) = r(y)r(z)$ in Theorem 3.2, then we can give following corollary.

Corollary 3.4. *Let $r, f, h : J \to \mathbb{R}$ be three functions $\ni$ r is an integrable and there should be existence of derivatives $f^{(M)}$ and $h^{(N)}$ with absolutely continuous, then*

$$\begin{aligned}
T(f,h,r) &= T(Q_M(f), Q_N(h), r) + T(\Upsilon_M(f), Q_N(h), r) + T(Q_M(f), \Upsilon_N(h), r) \\
&+ \int_a^b r(y)\, dy \int_a^b \int_a^b \int_a^{max\{y,z\}} \frac{f^{(M+1)}(y)(y-s)^M}{M!} \frac{h^{(N+1)}(z)(z-s)^N}{N!} (-1)^{M+N} \\
&\times r(s)\, ds\, dz\, dy - \int_a^b (-1)^{M+N}\, \Upsilon_M\, (f)(y) r(y)\, dy \int_a^b \Upsilon_N(h)(y) r(y)\, dy, \qquad (3.3)
\end{aligned}$$

where $Q_k(g)(y) = \sum_{j=0}^{k} \frac{(b-y)^j g^{(j)}(b)}{j!}$, $\Upsilon_k(g)(y) = \int_a^y \frac{(y-s)^M g^{(M+1)}(s)}{M!} ds$, $k \in \mathbb{N}$, and g is a function and $T(f,h,r)$ is stated in (1.4).

Proof. Applying Taylor formula for h,f, we can easily obtain (3.3). □

Corollary 3.5. *Let $r, f : J^2 \to \mathbb{R}$ be both functions, r is an integrable and there should be existence of partial derivatives $f_{(M+1,N)}$ and $f_{(M,N+1)}$ with absolutely continuous, then for $\frac{1}{\rho} + \frac{1}{\tau} = 1$; $\rho, \tau > 1$; we get*

$$|\, \overline{\mathcal{C}}(f,r) \,| \leq \left(\int_a^b \int_a^b |\, \Upsilon\, (y,z)|^{\tau}\, dz\, dy \right)^{\frac{1}{\tau}} \left(\int_a^b \int_a^b |(-1)^{M+N} f_{(M+1,N+1)}(y,z)|^{\rho}\, dz\, dy \right)^{\frac{1}{\rho}}, \qquad (3.4)$$

where $\overline{\mathcal{C}}(f,r)$ and $\Upsilon(y,z)$ are stated in (1.17) *and* (1.15) *respectively.*

Proof. Applying inequality of Hölder for integrals on Theorem 3.2, we may obtain (3.4). □

Theorem 3.6. *Let $r, f : J^2 \to \mathbb{R}$ be two functions, r is integrable and f is $(M+1, N+1) - \nabla -$convex. Then*

$$\overline{\mathcal{C}}(f,r) \geq 0 \quad if \quad \Upsilon(y,z) \geq 0 \quad \forall\, y\,, z \in [a,b],$$

where $\overline{\mathcal{C}}(f,r)$ and $\Upsilon(y,z)$ are defined in (1.17) *and* (1.15) *respectively.*

Proof. If function f is $\nabla-$convex function of order $(M+1, N+1)$ and on domain, it can be approximated uniformly by polynomials containing nonnegative $(M+1,N+1)$th order partial derivatives. From polynomials of Bernstein

$$B^{m,n}(y,z) = \sum_{j=0}^{m}\sum_{l=0}^{n}\binom{n}{l}\binom{m}{j} f(a_j,b_l)(y-a)^j(z-a)^l(b-z)^{n-l}(b-y)^{m-j},$$

(where $k=(b-a)/n$ and $h=(b-a)/m$) converge to function f uniformly in the domain J^2 limits as $n\to\infty$, $m\to\infty$ provided that function is continuous. Furthermore, if function f $(M+1,N+1)$th order $\nabla-$convex function, where polynomial containing non-negative $(M+1,N+1)$th order partial derivatives, that is, $(-1)^{M+N}B^{m,n}_{(M+1,N+1)}\geq 0$, applying following formula it may be prove using method of induction:

$$\begin{aligned}
&(-1)^{M+N}B^{m,n}_{(M+1,N+1)}(y,z) = \\
&(N+1)!(M+1)!\binom{n}{N+1}\binom{m}{M+1}\sum_{j=0}^{m-M-1}\sum_{l=0}^{n-N-1}\binom{n-N-1}{l}\binom{m-M-1}{j}\times \\
&\times\left(\Delta^{(M+1,N+1)}_{h,k}f(a+jh,a+lk)\right)(z-a)^l(b-z)^{n-N-1-l}(y-a)^j(b-y)^{m-M-1-j} \\
&= ((N+1)!)^2((M+1)!)^2h^{M+1}k^{N+1}\binom{n}{N+1}\binom{m}{M+1}\sum_{j=0}^{m-M-1}\sum_{l=0}^{n-N-1}\binom{m-M-1}{j}\times \\
&\times\binom{n-N-1}{l}\left(\nabla_{(M+1,N+1)}f(a_j,b_l)\right)(z-a)^l(b-z)^{n-N-1-l}(y-a)^j(b-y)^{m-M-1-j},
\end{aligned}$$

where $a_j=a+jh$, $b_l=a+lk$ and as (a_j) and (b_l) increasing sequences.
Since f is $(M+1,N+1)th$ order $\nabla-$convex, so $\nabla_{(M+1,N+1)}f\geq 0$. Since $\Upsilon(y,z)$ is continuous and $(-1)^{M+N}B^{m,n}_{(M+1,N+1)}\geq 0$ in the domain J^2 so by (1.17), get

$$\begin{aligned}
\overline{\mathcal{C}}(B^{m,n},r) &= \int_a^b\int_a^b(-1)^{M+N}B^{m,n}_{(M+1,N+1)}(y,z)\left[\int_a^{max\{y,z\}}\int_a^b r(s,t)\frac{(z-s)^N}{N!}\frac{(y-s)^M}{M!}dt\,ds\right. \\
&- \left.\int_a^y\int_a^z r(s,t)\frac{(y-s)^M}{M!}\frac{(z-t)^N}{N!}dt\,ds\right]dz\,dy \geq 0,
\end{aligned}$$

or we can write

$$\overline{\mathcal{C}}(B^{m,n},r) = \int_a^b\int_a^b \Upsilon(y,z)(-1)^{M+N}B^{m,n}_{(M+1,N+1)}(y,z)dz\,dy \geq 0. \tag{3.5}$$

Now convergence of $B^{m,n}_{(M+1,N+1)}$ uniformly to $f_{(M+1,N+1)}$ by letting $n,m\to\infty$ through an appropriate sequence, provides the required result. □

Theorem 3.7. *Let $r,f:J^2\to\mathbb{R}$ be functions, where $f\in C^{(M+1,N+1)}$ be a $(M+1,N+1)th$ order $\nabla-$convex function on the interval J^2 and r is integrable. If*

$$\Upsilon(y,z)\geq 0$$

holds $\forall\, y,\, z\in[a,b]$, $\exists\ \nu,\ \zeta\in[a,b]$, $\ni$

$$\overline{\mathcal{C}}(f,r) = \mathcal{C}\,(G_0,r)\ (-1)^{M+N}f_{(M+1,N+1)}(\nu,\zeta), \tag{3.6}$$

where

$$G_0(y,z) = (-1)^{M+N}\frac{(b-y)^{M+1}(b-z)^{N+1}}{(M+1)!(N+1)!} \tag{3.7}$$

and $\overline{\mathcal{C}}(f,r)$, $\Upsilon(y,z)$ are stated in (1.17) and (1.15) respectively.

Proof. Since

$$\overline{\mathcal{C}}(f,r)=\int_a^b\int_a^b(-1)^{M+N}\curlyvee(y,z)f_{(M+1,N+1)}(y,z)dz\,dy,$$

by applying Mean Value Theorem for the purpose of double integrals, then obtain

$$\overline{\mathcal{C}}(f,r)=(-1)^{N+M}f_{(M+1,N+1)}(\nu,\zeta)\int_a^b\int_a^b\curlyvee(y,z)dz\,dy.$$

In above equation, if put $f(y,z)=G_0(y,z)$ then we can write as:

$$\overline{\mathcal{C}}\,(G_0,r)=\mathcal{C}\,(G_0,r)=\int_a^b\int_a^b\curlyvee(y,z)\,dz\,dy$$

and hence we get what we wanted. □

Remark 3.8. For $M=N=0$, Theorem 3.7 recaptures the Proposition 1.4.

Remark 3.9. By putting $f(y,z)=h(z)f(y)$ and $r(y,z)=r(y)r(z)$ in Theorem 3.7 with $M=N=0$, then can obtain similar result for $\nabla-$convex function of (1.3).

Theorem 3.10. *Let $f,h:J^2\to\mathbb{R}$ be both functions and $r:J^2\to\mathbb{R}$ is integrable, $\ni$ f is $(M+1,N+1)-$ $\nabla-$convex function and $h\in C^{(M+1,N+1)}(J^2)$ with $h_{(M+1,N+1)}\neq 0$ on J^2. If*

$$\curlyvee(y,z)\geq 0,\quad\forall\, y,\ z\in[a,b]$$

holds, $\exists\ \nu,\ \zeta\in[a,b]$, $\ni$

$$\overline{\mathcal{C}}(f,r)=\frac{f_{(M+1,N+1)}(\nu,\zeta)}{h_{(M+1,N+1)}(\nu,\zeta)}\overline{\mathcal{C}}(h,r),$$

where $\curlyvee(y,z)$ and $\overline{\mathcal{C}}(f,r)$ are stated in (1.15) and (1.17) respectively.

Proof. Method 1:
Applying Mean Value Theorem of Integral and (3.6), then

$$\begin{aligned}\overline{\mathcal{C}}(f,r)&=\int_a^b\int_a^b(-1)^{M+N}\frac{f_{(M+1,N+1)}(y,z)}{h_{(M+1,N+1)}(y,z)}h_{(M+1,N+1)}(y,z)\curlyvee(y,z)dz\,dy\\&=\frac{f_{(M+1,N+1)}(\nu,\zeta)}{h_{(M+1,N+1)}(\nu,\zeta)}\int_a^b\int_a^b\curlyvee(y,z)(-1)^{M+N}h_{(M+1,N+1)}(y,z)dz\,dy\\&=\frac{f_{(M+1,N+1)}(\nu,\zeta)}{h_{(M+1,N+1)}(\nu,\zeta)}\overline{\mathcal{C}}(h,r).\end{aligned}$$

□

Proof. Method 2:
Let $u\in C^{(M+1,N+1)}$ be $(M+1,N+1)$th order $\nabla-$convex function on the interval $J\times L$, iz stated as:

$$u=\overline{\mathcal{C}}(h,r)f-\overline{\mathcal{C}}(f,r)h$$

by applying Theorem 3.7, $\exists\ \nu,\ \zeta\ \in J$, $\ni$

$$0=\overline{\mathcal{C}}(u,r)=(-1)^{M+N}u_{(M+1,N+1)}(\nu,\zeta)\overline{\mathcal{C}}(G_0,r),$$

or

$$[\overline{\mathcal{C}}(h,r)f_{(M+1,N+1)}(\nu,\zeta)-\overline{\mathcal{C}}(f,r)h_{(M+1,N+1)}(\nu,\zeta)]\overline{\mathcal{C}}(G_0,r)=0.$$

This gives required result. □

Remark 3.11. If we put $M = N = 0$ in Theorem 3.10, then we get similar result for $\nabla-$convex function of Theorem 2 of [23].

Theorem 3.12. *Let $r, f : J^2 \to \mathbb{R}$ be both functions, f is $(M+1, N+1)$th order $\nabla-$convex and r is integrable. $\exists\ \nu,\ \zeta \in [a,b]$, $\ni$*

$$\overline{\mathcal{C}}(f,r) = (-1)^{M+N}\ \Upsilon\ (\nu,\zeta)\left(f_{(M,N)}(b,b) - f_{(M,N)}(b,a) - f_{(M,N)}(a,b) + f_{(M,N)}(a,a)\right),$$

where $\Upsilon(y,z)$ and $\overline{\mathcal{C}}(f,r)$ are stated in (1.15) *and* (1.17)*.*

Proof. Since $(-1)^{M+N}B^{m,n}_{(M+1,N+1)} \geq 0$ in the interval J^2 and $\Upsilon(y,z)$ is continuous, here $B^{m,n}$ is polynomial of Bernstien, using same statement that was applied in the proof of the Theorem 3.7, we start from 3.5, we obtain

$$\begin{aligned}
&\overline{\mathcal{C}}(B^{m,n},r) = \int_a^b\int_a^b (-1)^{M+N}\ \Upsilon\ (y,z)B^{m,n}_{(M+1,N+1)}(y,z)\,dz\,dy\\
&= (-1)^{M+N}\ \Upsilon\ (\nu_{m,n},\zeta_{m,n})\int_a^b\int_a^b B^{m,n}_{(M+1,N+1)}(y,z)\,dz\,dy\\
&= (-1)^{N+M}\ \Upsilon\ (\nu_{m,n},\zeta_{m,n})\left(B^{m,n}_{(M,N)}(b,b) - B^{m,n}_{(M,N)}(a,b) - B^{m,n}_{(M,N)}(b,a) + B^{m,n}_{(M,N)}(a,a)\right).
\end{aligned}$$

The points $y_{m,n} = (\nu_{m,n},\zeta_{m,n})$ have a limit point (ν,ζ) in J^2 as $m,n \to \infty$, so the uniform convergence of $B^{m,n}_{(M,N)}$ to $f_{(M,N)}$ by letting $m,n \to \infty$ through an appropriate sequence, gives our desired result. □

Remark 3.13. For the case $M = N = 0$ in Theorem 3.12, we can obtain the similar result for $\nabla-$convex function of Theorem 6 of [23].

4. Generalized Integral Identity and Inequality of Ky Fan

According to MathSciNet, Ky Fan (1914-2010) published 126 research papers and books. The contributions of Ky Fan in mathematics, have provided many of influence in development of convex analysis, nonlinear analysis, linear algebra, operator theory, mathematical economics, approximation theory and mathematical programming (see [12]). In literature, there are different kinds of inequalities due to Ky Fan worked in several fields; cf. [4].

Now in this section we have to obtain some important identities and inequalities as below:

Theorem 4.1. *Let $q : J \to \mathbb{R}$ and $f, r : J^2 \to \mathbb{R}$ be functions, $\exists$ q and r are integrable and there should be existence of partial derivatives $f_{(M+1,N)}$ and $f_{(M,N+1)}$ with absolutely continuous, then*

$$\begin{aligned}
\mathcal{R}(f,r,q) \;=\;& \sum_{l=0}^{N}\sum_{j=0}^{M}(-1)^{j+l}f_{(j,l)}(b,b)\left[S^{(j,l)}(b) - Q^{(j,l)}(b,b)\right]\\
&+ \sum_{l=0}^{N}\int_a^b (-1)^{M+1+l}f_{(M+1,l)}(y,b)\left[S^{(M,l)}(y) - Q^{(M,l)}(y,b)\right]\,dy\\
&+ \sum_{j=0}^{M}\int_a^b (-1)^{j+N+1}f_{(j,N+1)}(b,z)\left[S^{(j,N)}(z) - Q^{(j,N)}(b,z)\right]\,dz\\
&+ \int_a^b\int_a^b (-1)^{M+N}\overline{\Upsilon}(y,z)f_{(M+1,N+1)}(y,z)\,dz\,dy,
\end{aligned}$$

where $S^{(j,l)}$, $Q^{(j,l)}$, and $\overline{\Upsilon}(y,z)$ are stated in (1.14), (1.12)*, and* (1.16) *respectively.*

Proof. By applying the substitution

$$\int_a^b r(y,z)dz = q(y).$$

we may prove of this theorem in the similar manner as done in proof of Theorem 3.2. □

Remark 4.2. By putting $f(y,z) = f(y)h(z)$ and $r(y,z) = \frac{q(y)q(z)}{\int_a^b q(t)\,dt}$ in Theorem 4.1, here q is integrable and $\int_a^b q(t)\,dt \neq 0$, then we can give corollary as below.

Corollary 4.3. *Let $f, h, q : J \to \mathbb{R}$ be functions, $\exists$ q is integrable, where $\int_a^b q(t)\,dt \neq 0$ and there should be existence of derivatives $f^{(M)}$ and $g^{(N)}$ and are absolutely continuous. Then*

$$\begin{aligned} T(f,h,q) &= T(Q_M(f), Q_N(h), q) + T(\Upsilon_M(f), Q_N(h), q) + T(Q_M(f), \Upsilon_N(h), q) \\ &+ \int_a^b \int_a^b \int_a^{max\{y,z\}} (-1)^{M+N} \frac{f^{(M+1)}(y)(y-s)^M}{M!} \frac{h^{(N+1)}(z)(z-s)^N}{N!} \\ &\times q(s)\,ds\,dz\,dy - \int_a^b (-1)^{M+N}\, \Upsilon_M\,(f)(y)q(y)\,dy \int_a^b \Upsilon_N(h)(y)q(y)\,dy, \end{aligned}$$

where $Q_k(g)(y) = \sum_{j=0}^k \frac{(b-y)^j g^{(j)}(b)}{j!}$, $\Upsilon_k(g)(y) = \int_a^y \frac{(y-s)^M g^{(M+1)}(s)}{M!}\,ds$, $k \in \mathbb{N}$, and here g is a function and $T(f,h,r)$ is defined in (1.4).

Corollary 4.4. *Let $q : J \to \mathbb{R}$ and $f, r : J^2 \to \mathbb{R}$ be functions and also r and q are integrable and there should be existence of partial derivatives $f_{(M+1,N)}$ and $f_{(M,N+1)}$ with absolutely continuous. Then for $\frac{1}{\rho} + \frac{1}{\tau} = 1$; $\rho, \tau > 1$; we have*

$$|\,\overline{\mathcal{R}}(f,r,q)\,| \leq \left(\int_a^b \int_a^b |(-1)^{M+N} f_{(M+1,N+1)}(y,z)|^\rho\,dz\,dy\right)^{\frac{1}{\rho}} \left(\int_a^b \int_a^b |\overline{\Upsilon}(y,z)|^\tau\,dz\,dy\right)^{\frac{1}{\tau}},$$

where $\overline{\mathcal{R}}(f,r,q)$ and $\overline{\Upsilon}(y,z)$ are stated in (1.18) *and* (1.16) *respectively.*

Theorem 4.5. *Let $q : J \to \mathbb{R}$ and $f, r : J^2 \to \mathbb{R}$ be three functions and also r and q are integrable and function f is $(M+1, N+1)$th order $\nabla-$convex. Then*

$$\overline{\mathcal{R}}(f,r,q) \geq 0 \quad \textit{if} \quad \overline{\Upsilon}(y,z) \geq 0, \quad \forall\, y,\, z \in [a,b],$$

where $\overline{\mathcal{R}}(f,r,q)$ and $\overline{\Upsilon}(y,z)$ are stated in (1.18) *and* (1.16) *respectively.*

Remark 4.6. We can prove Theorem 4.5 in the similar manner as done in proof of Theorem 3.6.

Theorem 4.7. *Let $q : J \to \mathbb{R}$ and $f, r : J^2 \to \mathbb{R}$ be functions and also r and q are integrable and function f is $(M+1, N+1)$th order $\nabla-$convex and assuming $\forall\, y,\, z \in [a,b]$*

$$\overline{\Upsilon}(y,z) \geq 0.$$

$\exists\ \nu,\ \zeta\ \in\ [a,b]$, $\ni$

$$\overline{\mathcal{R}}(f,r,q) = (-1)^{N+M} f_{(M+1,N+1)}(\nu,\zeta)\mathcal{R}\,(G_0,r,q)\,,$$

where $\overline{\mathcal{R}}(f,r,q)$ and G_0 are stated in (1.18) *and* (3.7) *respectively.*

Remark 4.8. We can give proof of Theorem 4.7 in the similar way as done in the proof of Theorem 3.7. Further that we recapture the Proposition 1.9 from Theorem 4.7 by putting $M = N = 0$.

Theorem 4.9. *Let $q : J \to \mathbb{R}$ and $f, h, r : J^2 \to \mathbb{R}$ be four functions and also q and r are integrable and function f is $(M+1, N+1) - \nabla-$convex and $h \in C^{(M+1,N+1)}(J^2)$ with $h_{(M+1,N+1)} \neq 0$ in the interval J^2 and assuming $\forall\, y,\, z \in [a, b]$*

$$\overline{\Upsilon}(y, z) \geq 0.$$

Then $\exists\ \nu,\, \zeta \in [a, b]$, $\ni$

$$\overline{\mathcal{R}}(f, r, q) = \frac{f_{(M+1,N+1)}(\nu, \zeta)}{h_{(M+1,N+1)}(\nu, \zeta)}\, \overline{\mathcal{R}}(h, r, q),$$

where $\overline{\mathcal{R}}$ and $\overline{\Upsilon}$ are defined in (1.18) *and* (1.16) *respectively.*

Remark 4.10. We can give proof of Theorem 4.9 in the similar ways as done in the proof of Theorem 3.10 by two different methods. Further, if put $M = N = 0$ in Theorem 4.9, then we get similar result for $\nabla-$convex function of Theorem 16 of [23].

5. Conclusion

In this article, we have obtained the generalization of discrete Čebyšev identity for function in the interval J^2 involving higher order ∇ divided difference of two independent variables and also got the similar result as Theorem 2.3 for sequence of higher order ∇ divided difference for two dimension if substitute $y_j = j$, $z_l = l$ and $f(y_j, z_l) = f(j, l) = a_{jl}$ in Theorem 2.3 and also found results of discrete inequality of Čebyšev by using $(m, n) - \nabla-$convex functions of two independent variables on J^2. Moreover, we have obtained the generalizations of integral Čebyšev and Ky Fan's identities for differentiable function of higher order for two independent variables and also found results of integral inequalities of Čebyšev and Ky Fan by using $(M + 1, N + 1) - \nabla-$convex function of two independent variables. From the obtained generalizations, we have given similar results of article [23] and recaptured some established results.

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