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Journal of Prime Research in Mathematics, 18(2) (2022), 42–54

Journal of Prime Research in Mathematics

On the solutions of nonlinear Caputo–Fabrizio fractional partial differential equations arising in applied mathematics

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Abstract

This paper proposes a new semi-analytical method known as the variational iteration transform method (VITM) to obtain the solutions of the nonlinear Caputo–Fabrizio fractional partial differential equations arising in applied mathematics. For nonlinear equations in general, there is no method that gives an exact solution and, therefore, only approximate analytical solutions can be derived by using procedures such as linearization or perturbation. This method is combined form of the Aboodh transform and the variational iteration method. The advantage of VITM is the simplicity of the computations and the non-requirement of linearization or smallness assumptions. Moreover, this method enables us to overcome the difficulties arising in identifying the general Lagrange multiplier. For further illustrations of the efficiency and reliability of VITM, some numerical applications are pesented. The numerical results showed that the proposed method is efficient and precise to obtain the solutions of nonlinear fractional partial differential equations.

Keywords: fractional partial differential equations, Caputo–Fabrizio fractional derivative, Aboodh transform, variational iteration transform method, general lagrange multiplier.
2010 MSC: 34A08, 26A33, 34K28, 35C10.

1. Introduction

Fractional differential equations appear in many branches of physics, engineering and applied mathematics. It has turned out that many phenomena in engineering, physics, chemistry, biology, medical sciences and other sciences can be described very successfully by models using mathematical tools from fractional calculus [3, 8, 9, 10, 13, 14].

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Received : 25 May 2022*; Accepted:* 14 July 2022*; Published Online:* 26 September 2022.

Fractional calculus plays an important role in many fields of pure and applied mathematics where fractional integrals and derivatives in association with different integral transforms, are used to solve different types of differential and integral equations. In the last few years, many research workers have paid attention to study the solutions of fractional differential equations by using various methods. Therefore, the search for a solution to these equations is an important aspect of scientific research.

It is known that integral transforms such as Laplace, Sumudu, natural, Elzaki, Shehu and Aboodh are not able to solve the nonlinear differential equations. Therefore, we find that some researchers are working to combine these transforms with many methods, among them we find the variational iterative method. This method was proposed and developed by the Chinese mathematician Je-Haun-He in the early 1990s [6, 7], it was proposed the first time to solve mechanical problems. This method has been used to solve a wide variety of linear and nonlinear problems with successive approximations that rapidly converge to the exact solution if it exists.

The motivation of this paper is to study the nonlinear fractional partial differential equations involving Caputo–Fabrizio fractional operator with a reliable computationally effective numerical scheme, which is compilation of variational iterative method with Aboodh transform in the sense of Caputo–Fabrizio fractional derivative. This method known as variatonal iteraive transform method (VITM)

The remainder of this paper is organized as follows: Section 2, introduces preliminary definitions about fractional calculus and results related to the Aboodh transform of the Caputo-Fabrizio fractional derivative. Section 3, gives the methodology of the VITM to solve the nonlinear Caputo-Fabrizio fractional partial differential equations. Section 4, presents three numerical applications to illustrate the efficiency and accuracy of the VITM. Section 5, discusses numerical results obtained. Section 6, summarizes the findings of this paper.

2. Definitions and preliminary results

This section presents the basic definitions and several properties of the theory of fractional calculus which has been recently developed by [4, 11].

Definition 2.1. The Caputo-Fabrizio fractional derivative of order $0 < \alpha < 1$ for a function $u \in H^1(0,T)$ is defined by

$$^{CF}D^{\alpha}u(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t u'(\tau)\exp\left(-\frac{\alpha\,(t-\tau)}{1-\alpha}\right)d\tau, t \geq 0, \tag{2.1}$$

where $M(\alpha)$ is a normalization constant that depends on α.

Definition 2.2. The Caputo-Fabrizio fractional integral of order $0 < \alpha < 1$ for a function $u \in H^1(0,T)$ is defined by

$$^{CF}I^{\alpha}u(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t u(\tau)d\tau, t \geq 0. \tag{2.2}$$

Note that, according to the previous definition, the Caputo-Fabrizio fractional integral of a function u of order $0 < \alpha < 1$ is an average between function u and its integral of order one.

Imposing

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1,$$

we get an explicit formula for $M(\alpha)$, as follows

$$M(\alpha) = \frac{2}{2-\alpha}, 0 \leq \alpha \leq 1.$$

For this reason, we have the following definition of the fractional derivative of order $0 < \alpha \leq 1$.

Definition 2.3. The Caputo-Fabrizio fractional derivative of order $0 < \alpha \leq 1$ for a function $u \in H^1(0,T)$ is given by

$$^{CF}D^{\alpha}u(t) = \frac{1}{1-\alpha}\int_0^t u'(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau, t \geq 0. \tag{2.3}$$

Definition 2.4. The Caputo-Fabrizio fractional derivative of order $\alpha + n$ when $0 < \alpha \leq 1$ and $n \geq 1$ is defined by

$$^{CF}D^{\alpha+n}u(t) =^{CF} D^{\alpha}(D^n u(t)). \tag{2.4}$$

Now, we present a new result related to the Aboodh transform of the Caputo-Fabrizio fractional derivative. The Aboodh transform is a new integral transform that is applied to solve ordinary and partial differential equations, defined and developed by [1].

Definition 2.5. The Aboodh transform is defined over the set of functions

$$U = \{u(t) : |u(t)| < M\exp(-vt), \text{if } t \in [0,\infty[, M, k_1, k_2 > 0, k_1 < v < k_2\},$$

by the following integral

$$\mathcal{A}[u(t)] = K(v) = \frac{1}{v}\int_0^{\infty} u(t)\exp(-vt)\,dt, t \geq 0, k_1 < v < k_2,$$

where v is the factor of the variable t.

Definition 2.6. The inverse Aboodh transform of the function $u(t)$ is defined by

$$\mathcal{A}^{-1}[K(v)] = u(t), \text{ for } t \geq 0. \tag{2.5}$$

This is equivalent to

$$u(t) = \mathcal{A}^{-1}[K(v)] = \frac{1}{2\pi i}\int_{\rho - i\infty}^{\rho + i\infty} v\exp(vt)\,K(v)dv, \tag{2.6}$$

where ρ is a real constant and the integral in equation (2.6) is taken along $v = \rho$ in the complex plane $v = x + iy$.

Some basic properties of the Aboodh transform are given as follows.

Property 1: The Aboodh transform is a linear operator. That is, if λ and μ are non-zero constants, then

$$\mathcal{A}[\lambda u(t) \pm \mu v(t)] = \lambda\mathcal{A}[u(t)] \pm \mu\mathcal{A}[v(t)].$$

Property 2: If $u^{(n)}(t)$ is the $n-$th derivative of the function $u(t) \in A$ with respect to t then its Aboodh transform is given by

$$\mathcal{A}\left[u^{(n)}(t)\right] = v^n K(v) - \sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{v^{2-n+k}}.$$

Property 3: (Convolution property) Suppose $K(v)$ and $G(v)$ are the Aboodh transforms of $u(t)$ and $v(t)$, respectively, both defined in the set A. Then the Aboodh transform of their convolution is given by

$$\mathcal{A}[(u * v)(t)] = vK(v)G(v),$$

where the convolution of two functions is defined by

$$(u * v)(t) = \int_0^t u(t)v(t-\tau)d\tau = \int_0^t u(t-\tau)v(\tau)d\tau.$$

Property 4: In the following, we give the Aboodh transform for some special functions.

$$\begin{aligned}
\mathcal{A}(1) &= \frac{1}{v^2}, \\
\mathcal{A}(t) &= \frac{1}{v^3}, \\
\mathcal{A}\left(\frac{t^n}{n!}\right) &= \frac{1}{v^{n+2}}, n = 0, 1, 2, ... \\
\mathcal{A}\left(\exp(at)\right) &= \frac{1}{v^2 - av}.
\end{aligned}$$

Theorem 2.7. *The Aboodh transform of the Caputo-Fabrizio fractional derivative of the function $u(t)$ of order $\alpha + n$, where $0 < \alpha \leq 1$ and $n \in \mathbb{N} \cup \{0\}$, is given by*

$$\mathcal{A}\left[{}^{CF}D^{\alpha+n}u(t)\right] = \frac{v}{v - \alpha(v-1)}\left[v^n \mathcal{A}\left(u(t)\right) - \sum_{k=0}^{n} \frac{u^{(k)}(0)}{v^{2-n+k}}\right]. \tag{2.7}$$

Proof. Using the equations (2.3) and (2.4), we have

$$\begin{aligned}
\mathcal{A}\left[{}^{CF}D^{\alpha+n}u(t)\right] &= \mathcal{A}\left[{}^{CF}D^{\alpha}(D^n u(t))\right] \\
&= \frac{1}{1-\alpha}\frac{1}{v}\int_0^\infty \exp(-vt)\left(\int_0^t u^{(n+1)}(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau\right)dt \\
&= \frac{1}{1-\alpha}\frac{1}{v}\int_0^\infty \exp(-vt)\left(u^{(n+1)}(t) * \exp\left(-\frac{\alpha t}{1-\alpha}\right)\right)dt \\
&= \frac{1}{1-\alpha}\mathcal{A}\left(u^{(n+1)}(t) * \exp\left(-\frac{\alpha t}{1-\alpha}\right)\right).
\end{aligned}$$

Hence, from the properties (2), (3) and (4), we have

$$\begin{aligned}
\mathcal{A}\left[{}^{CF}D^{\alpha+n}u(t)\right] &= \frac{1}{1-\alpha}v\mathcal{A}\left(u^{(n+1)}(t)\right)\mathcal{A}\left(\exp\left(-\frac{\alpha t}{1-\alpha}\right)\right) \\
&= \frac{1}{(1-\alpha)v + \alpha}\left[v^{n+1}\mathcal{A}\left(u(t)\right) - \sum_{k=0}^{n}\frac{u^{(k)}(0)}{v^{1-n+k}}\right] \\
&= \frac{v}{v - \alpha(v-1)}\left[v^n\mathcal{A}\left(u(t)\right) - \sum_{k=0}^{n}\frac{u^{(k)}(0)}{v^{2-n+k}}\right].
\end{aligned}$$

Hence, the proof is completed. □

3. Analysis of the VITM

This section gives our main result related to the VITM to solve the nonlinear Caputo-Fabrizio fractional partial differential equations

Theorem 3.1. *Let us consider the nonlinear fractional partial differential equation in a general form*

$${}^{CF}D_t^{\alpha+n}u(x,t) + \mathcal{L}u(x,t) + \mathcal{N}u(x,t) = f(x,t), \tag{3.1}$$

with the initial conditions

$$u^{(k)}(x,0) = g_k(x), k = 0, 1, ..., n-1, \tag{3.2}$$

where ${}^{CF}D^{\alpha+n}$ *is the Caputo-Fabrizio fractional derivative operator of order* $\alpha+n$ *with* $0<\alpha\leq 1$ *and* $n\in\mathbb{N}\cup\{0\}$, $\mathcal{L}$ *and* $\mathcal{N}$ *are a linear and nonlinear operators, respectively, and* f *is the source term.*

Then, the exact solution of equations (3.1)-(3.2) using the VITM, is given as a limit of the successive approximations $u_N(x,t), N=0,1,2,...$, *in other words*

$$u(x,t)=\lim_{N\to\infty}u_N(x,t).$$

Proof. According to the variational iteration transform [15], the correction functional of equation (3.1), is given as

$$u_{N+1}(x,t)=u_N(x,t)+\int_0^t\lambda(t-\tau)\left[\begin{array}{c}{}^{CF}D^{\alpha+n}u_N(x,\tau)+\mathcal{L}u_N(x,\tau)\\+\mathcal{N}u_N(x,\tau)-f(x,\tau)\end{array}\right]d\tau, \tag{3.3}$$

where $\lambda(t-\tau)$ is a general lagrange multiplier, the subscript $N\geq 0$ denotes the N^{th} approximation.

Taking the Aboodh transform and using convolution property in equation (3.3), we have

$$\begin{aligned}\mathcal{A}\left[u_{N+1}(x,t)\right] &= \mathcal{A}\left[u_N(x,t)\right]+\mathcal{A}\left[\int_0^t\lambda(t-\tau)\left[\begin{array}{c}{}^{CF}D^{\alpha+n}u_N(x,\tau)+\mathcal{L}u_N(x,\tau)\\+\mathcal{N}u_N(x,\tau)-f(x,\tau)\end{array}\right]d\tau\right]\\ &= \mathcal{A}\left[u_N(x,t)\right]+v\mathcal{A}\left[\lambda(t)\right]\mathcal{A}\left[\begin{array}{c}{}^{CF}D^{\alpha+n}u_N(x,t)+\mathcal{L}u_N(x,t)\\+\mathcal{N}u_N(x,t)-f(x,t)\end{array}\right].\end{aligned} \tag{3.4}$$

Applying the Theorem 2.1, the equation (3.4) becomes

$$\begin{aligned}\mathcal{A}\left[u_{N+1}(x,t)\right] &= \mathcal{A}\left[u_N(x,t)\right]+v\mathcal{A}\left[\lambda(t)\right]\left(\frac{v}{v-\alpha(v-1)}\left[v^n\mathcal{A}\left(u_N(x,t)\right)\right.\right.\\ &\left.\left.-\sum_{k=0}^{n}\frac{u^{(k)}(x,0)}{v^{2-n+k}}\right]+\mathcal{A}\left[\mathcal{L}u_N(x,t)+\mathcal{N}u_N(x,t)-f(x,t)\right]\right).\end{aligned} \tag{3.5}$$

The optimal value of λ can be identified by making the equation (3.5) stationary with respect to $u_N(x,t)$

$$\begin{aligned}\delta\left(\mathcal{A}\left[u_{N+1}(x;t)\right]\right) &= \delta\left(\mathcal{A}\left[u_N(x,t)\right]\right)+v\delta\left(\mathcal{A}\left[\lambda(t)\right]\left(\frac{v}{v-\alpha(v-1)}\left[v^n\mathcal{A}\left(u_N(x,t)\right)\right.\right.\right.\\ &\left.\left.\left.-\sum_{k=0}^{n}\frac{g_k(x)}{v^{2-n+k}}\right]+\mathcal{A}\left[\mathcal{L}u_N(x,t)+\mathcal{N}u_N(x,t)-f(x,t)\right]\right)\right).\end{aligned} \tag{3.6}$$

Considering $\mathcal{A}\left[\mathcal{L}u_N(x,t)+\mathcal{N}u_N(x,t)\right]$ as a restricted variation, i.e.,

$$\delta\left(\mathcal{A}\left[\mathcal{L}u_N(x,t)+\mathcal{N}u_N(x,t)\right]\right)=0,$$

we have

$$1+\frac{v^{n+2}}{v-\alpha(v-1)}\mathcal{A}\left[\lambda(t)\right]=0,$$

which implies that

$$\mathcal{A}\left[\lambda(t)\right]=-\frac{v-\alpha(v-1)}{v^{n+2}}. \tag{3.7}$$

Using (3.7) in equation (3.5) and taking the inverse Aboodh transform, we attain a new correction functional

$$u_{N+1}(x,t)=\sum_{k=0}^{n}g_k(x)\frac{t^k}{k!}-\mathcal{A}^{-1}\left(\frac{v-\alpha(v-1)}{v^{n+1}}\mathcal{A}\left[\mathcal{L}u_N(x,t)+\mathcal{N}u_N(x,t)-f(x,t)\right]\right), \tag{3.8}$$

where the primary iteration $u_0(x,t)$ can be determined by

$$u_0(x,t) = \sum_{k=0}^{n} g_k(x)\frac{t^k}{k!}.$$

The successive approximations rapidly converge to the exact solution of equation (3.1) as $N \to \infty$, that is

$$u(x,t) = \lim_{N\to\infty} u_N(x,t).$$

Hence, the proof is completed. □

4. Numerical Applications

This section presents the analytical technique based on VITM for solving some nonlinear Caputo-Fabrizio fractional parial differential equations with a known exact solution.

Example 4.1. Let us consider the nonlinear fractional gas dynamic equation

$$^{CF}D_t^{\alpha}u(x,t) + \frac{1}{2}\left(u^2(x,t)\right)_x = u(x,t) - u^2(x,t), \tag{4.1}$$

with the initial condition

$$u(x,0) = e^{-x}, \tag{4.2}$$

where $^{CF}D_t^{\alpha}u(x,t)$ is the Caputo-Fabrizio fractional derivative of order α with $0 < \alpha \leq 1$ and $(x,t) \in \mathbb{R}\times\mathbb{R}^+$.

The exact solution of equations (4.1)-(4.2) for $\alpha = 1$ is [2]

$$u(x,t) = e^{t-x}.$$

Applying the same methodology of the VITM described in Section 3 to the equations (4.1)-(4.2), we have the iteration formula

$$u_{N+1}(x,t) = e^{-x} - \mathcal{A}^{-1}\left(\frac{v-\alpha(v-1)}{v}\mathcal{A}\left[\frac{1}{2}\left(u_N^2(x,t)\right)_x - u_N(x,t) + u_N^2(x,t)\right]\right), \tag{4.3}$$

where the primary iteration $u_0(x,t)$ is given by

$$u_0(x,t) = e^{-x}. \tag{4.4}$$

We then have the following successive approximations by (4.3)

$$\begin{aligned}
u_1(x,t) &= (2-\alpha+\alpha t)\, e^{-x},\\
u_2(x,t) &= \left(1 + (1-\alpha)(2-\alpha) + \alpha(3-2\alpha)\, t + \alpha^2\frac{t^2}{2!}\right) e^{-x},\\
u_3(x,t) &= \Big(1 + (1-\alpha) + (1-\alpha)^2(2-\alpha) + (\alpha + \alpha(1-\alpha)(5-3\alpha))\, t\\
&\quad + \alpha^2(4-3\alpha)\frac{t^2}{2!} + \alpha^3\frac{t^3}{3!}\Big) e^{-x},\\
&\ \ \vdots
\end{aligned} \tag{4.5}$$

Taking $\alpha = 1$ in (4.5), we get the following successive approximations

$$\begin{aligned}
u_1(x,t) &= (1+t)\, e^{-x},\\
u_2(x,t) &= \left(1 + t + \frac{t^2}{2!}\right) e^{-x},\\
u_3(x,t) &= \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right) e^{-x},\\
&\ \ \vdots
\end{aligned}$$

which converges to the exact solution

$$u(x,t) = e^{t-x}.$$

Example 4.2. Let us consider the nonlinear fractional reaction-diffusion-convection equation

$$ {}^{CF}D_t^{\alpha} u(x,t) = u_{xx}(x,t) - u_x(x,t) + u(x,t)u_x(x,t) - u^2(x,t) + u(x,t), \tag{4.6}$$

with the initial condition

$$u(x,0) = e^x, \tag{4.7}$$

where ${}^{CF}D_t^{\alpha} u(x,t)$ is the Caputo-Fabrizio fractional derivative of order α with $0 < \alpha \leq 1$ and $(x,t) \in \mathbb{R}\times\mathbb{R}^+$.
The exact solution of equations (4.6)-(4.7) for $\alpha = 1$ is [12]

$$u(x,t) = e^{t+x}$$

Applying the same methodology of the VITM described in Section 3 to the equations (4.6)-(4.7), we have the iteration formula

$$\begin{aligned} u_{N+1}(x,t) &= e^x - \mathcal{A}^{-1}\left(\frac{v-\alpha(v-1)}{v}\mathcal{A}\left[-u_{Nxx}(x,t) + u_{Nx}(x,t) - u_N(x,t)u_{Nx}(x,t)\right.\right. \\ &\quad \left.\left. + u_N^2(x,t) - u_N(x,t)\right]\right), \end{aligned} \tag{4.8}$$

where the primary iteration $u_0(x,t)$ is given by

$$u_0(x,t) = e^x. \tag{4.9}$$

We then have the following successive approximations by (4.8)

$$\begin{aligned} u_1(x,t) &= (2-\alpha+\alpha t)\, e^x, \\ u_2(x,t) &= \left(1 + (1-\alpha)(2-\alpha) + \alpha(3-2\alpha)\, t + \alpha^2\frac{t^2}{2!}\right) e^x, \\ u_3(x,t) &= \left(1 + (1-\alpha) + (1-\alpha)^2(2-\alpha) + (\alpha + \alpha(1-\alpha)(5-3\alpha))\, t\right. \\ &\quad \left. + \alpha^2(4-3\alpha)\frac{t^2}{2!} + \alpha^3\frac{t^3}{3!}\right) e^x, \\ &\vdots \end{aligned} \tag{4.10}$$

Taking $\alpha = 1$ in (4.10), we get the following successive approximations

$$\begin{aligned} u_1(x,t) &= (1+t)\, e^x, \\ u_2(x,t) &= \left(1 + t + \frac{t^2}{2!}\right) e^x, \\ u_3(x,t) &= \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right) e^x, \\ &\vdots \end{aligned}$$

which converges to the exact solution

$$u(x,t) = e^{t+x}.$$

Example 4.3. Let us consider the nonlinear fractional wave-like equation with variable coefficients

$$ {}^{CF}D_t^{\alpha+1} u(x,t) = x^2\frac{\partial}{\partial x}(u_x(x,t)u_{xx}(x,t)) - x^2 u_{xx}^2(x,t) - u(x,t), \tag{4.11}$$

with the initial conditions

$$u(x,0) = 0, u_t(x,0) = x^2, \tag{4.12}$$

where ${}^{CF}D_t^{\alpha+1}u(x,t)$ is the Caputo-Fabrizio fractional derivative of order $\alpha + 1$ with $0 < \alpha \leq 1$ and $(x,t) \in]0,1[\times \mathbb{R}^+$.

The exact solution of equations (4.11)-(4.12) for $\alpha = 1$ is [5]

$$u(x,t) = x^2 \sin t.$$

Applying the same methodology of the VITM described in Section 3 to the equations (4.11)-(4.12), we have the iteration formula

$$\begin{aligned} u_{N+1}(x,t) &= x^2 t - \mathcal{A}^{-1}\left(\frac{v-\alpha(v-1)}{v^2}\mathcal{A}\left[-x^2\frac{\partial}{\partial x}(u_{Nx}(x,t)u_{Nxx}(x,t)) + x^2 u_{Nxx}^2(x,t)\right.\right. \\ &\quad \left.\left. + u_N(x,t)\right]\right), \end{aligned} \tag{4.13}$$

where the primary iteration $u_0(x,t)$ is given by

$$u_0(x,t) = x^2 t. \tag{4.14}$$

We then have the following successive approximations by (4.13)

$$\begin{aligned} u_1(x,t) &= x^2\left(t - (1-\alpha)\frac{t^2}{2!} - \alpha\frac{t^3}{3!}\right), \\ u_2(x,t) &= x^2\left(t - (1-\alpha)\frac{t^2}{2!} - \left(\alpha - (1-\alpha)^2\right)\frac{t^3}{3!} + 2\alpha(1-\alpha)\frac{t^4}{4!} + \alpha^2\frac{t^5}{5!}\right), \\ u_3(x,t) &= x^2\left(t - (1-\alpha)\frac{t^2}{2!} - \left(\alpha - (1-\alpha)^2\right)\frac{t^3}{3!} + (1-\alpha)\left(2\alpha - (1-\alpha)^2\right)\frac{t^4}{4!}\right. \\ &\quad \left. - \left(2\alpha(1-\alpha)^2 - \left(\alpha^2 - \alpha(1-\alpha)^2\right)\right)\frac{t^5}{5!} - 3\alpha^2(1-\alpha)\frac{t^6}{6!} - \alpha^3\frac{t^7}{7!}\right), \\ &\vdots \end{aligned} \tag{4.15}$$

Taking $\alpha = 1$ in (4.15), we get the following successive approximations

$$\begin{aligned} u_1(x,t) &= x^2\left(t - \frac{t^3}{3!}\right), \\ u_2(x,t) &= x^2\left(t - \frac{t^3}{3!} + \frac{t^5}{5!}\right), \\ u_3(x,t) &= x^2\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right), \\ &\vdots \end{aligned}$$

which converges to the exact solution

$$u(x,t) = x^2 \sin t.$$

5. Results and Discussion

This section discusses the numerical results obtained in the previous section. All results are calculated using the MATLAB software package. Figures 1, 3 and 5 demonstrate the surface graphs of the exact solution and approximate solutions for different values of $\alpha = 1, 0.9, 0.8$ obtained by VITM, for the equations ((4.1), (4.6) and (4.11), respectively. While Figures 2, 4 and 6 demonstrate the behavior of the exact solution and approximate solutions for different values of $\alpha = 1, 0.9, 0.8, 0.7$ obtained by VITM, for the equations (4.1), (4.6) and (4.11), respectively. It has been observed from these figures that the exact solutions and the approximate solutions at $\alpha = 1$ for Applications 4.1–4.3 are in excellent agreement with each other, respectively. Also, it is seen that, as the value of α approaches to 1, the approximate solutions approaches to the exact solutions. The efficiency of the present method was clearly noted as only up to four-order solutions were used to evaluate the approximate solutions. Tables 1–3 demonstrate the numerical values of the exact solution and approximate solutions for different values of $\alpha = 1, 0.9, 0.8, 07$ obtained by VITM, also show the absolute error between VITM and the exact solution by $|u_{exact} - u_{VITM}|$ for the equations (4.1), (4.6) and (4.11), respectively. From Tables 1–3 it is evident that with the increase in the fractional order α, the accuracy increases. Thus the series solutions for equations (4.1), (4.6) and (4.11) converge. It can also be seen that the solution obtained by using the present method is in complete agreement with the exact solution.

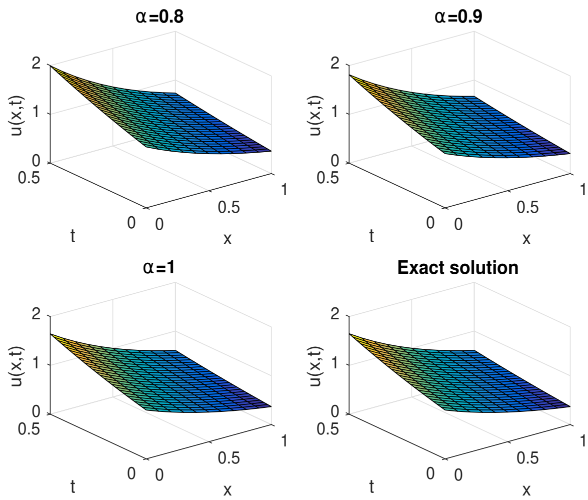

Figure 1: 3D graphs of the approximate solutions and exact solution of Application 4.1

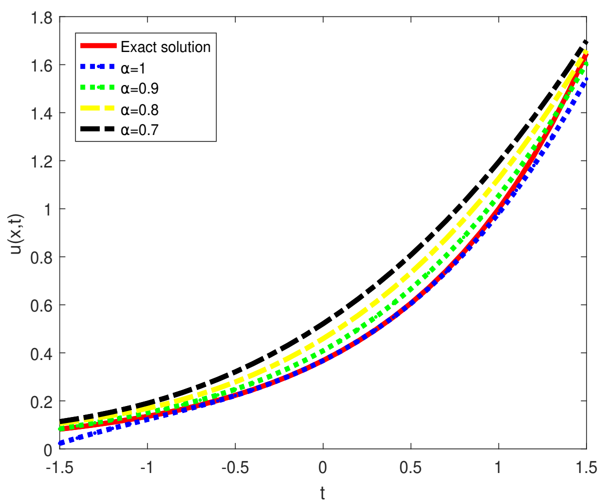

Figure 2: 2D graphs of the approximate solutions and exact solution of Application 4.1

t	$u_{\alpha=0.7}$	$u_{\alpha=0.8}$	$u_{\alpha=0.9}$	$u_{\alpha=1}$	u_{exact}	$\lvert u_{exact} - u_{VITM}\rvert$
0.01	0.52612	0.46361	0.41281	0.37158	0.37158	1.5359×10^{-10}
0.03	0.53589	0.47270	0.42111	0.37908	0.37908	1.2491×10^{-8}
0.05	0.54579	0.48196	0.42957	0.38674	0.38674	9.6768×10^{-8}
0.07	0.55584	0.49136	0.43819	0.39455	0.39455	3.7325×10^{-7}
0.09	0.56603	0.50092	0.44697	0.40252	0.40252	1.0241×10^{-6}

Table 1: Numerical results of Application 4.1 at $x = 1$

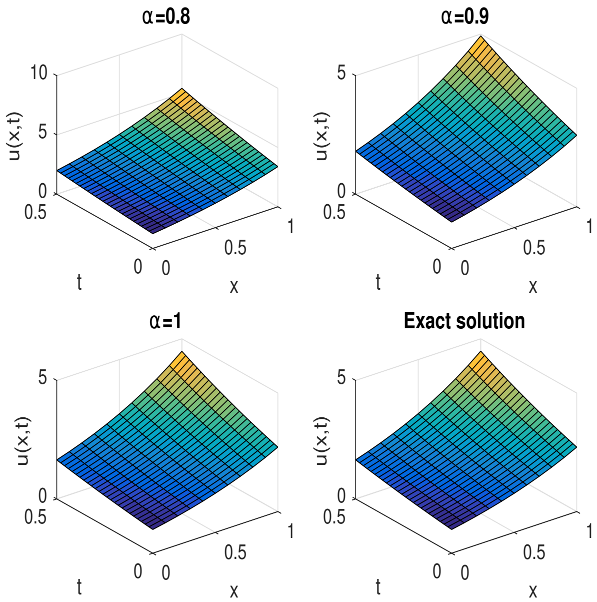

Figure 3: 3D graphs of the approximate solutions and exact solution of Application 4.2

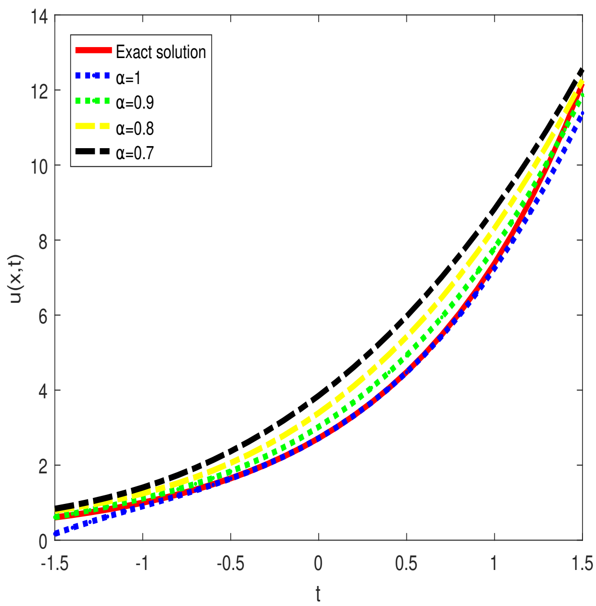

Figure 4: 2D graphs of the approximate solutions and exact solution of Application 4.2

t	$u_{\alpha=0.7}$	$u_{\alpha=0.8}$	$u_{\alpha=0.9}$	$u_{\alpha=1}$	u_{exact}	$\lvert u_{exact} - u_{VITM} \rvert$
0.01	3.8875	3.4256	3.0502	2.7456	2.7456	1.1349×10^{-9}
0.03	3.9597	3.4928	3.1116	2.8011	2.8011	9.2295×10^{-8}
0.05	4.0329	3.5612	3.1741	2.8577	2.8577	7.1502×10^{-7}
0.07	4.1071	3.6307	3.2378	2.9154	2.9154	2.7579×10^{-6}
0.09	4.1824	3.7013	3.3027	2.9743	2.9743	7.5669×10^{-6}

Table 2: Numerical results of Application 4.2 at $x = 1$

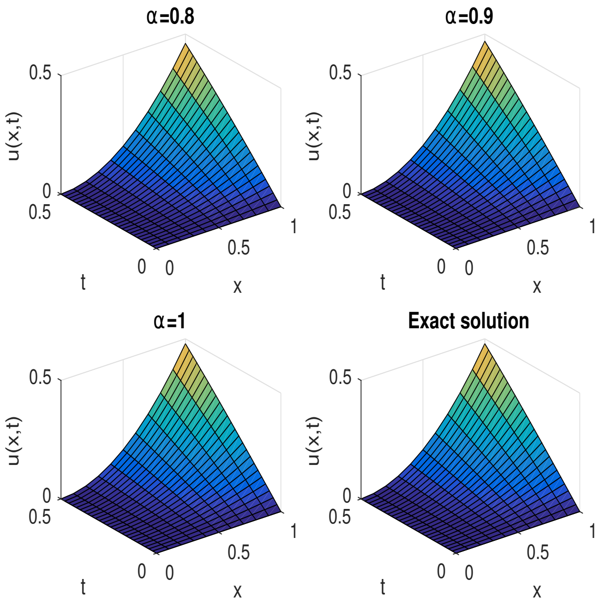

Figure 5: 3D graphs of the approximate solutions and exact solution of Application 4.3

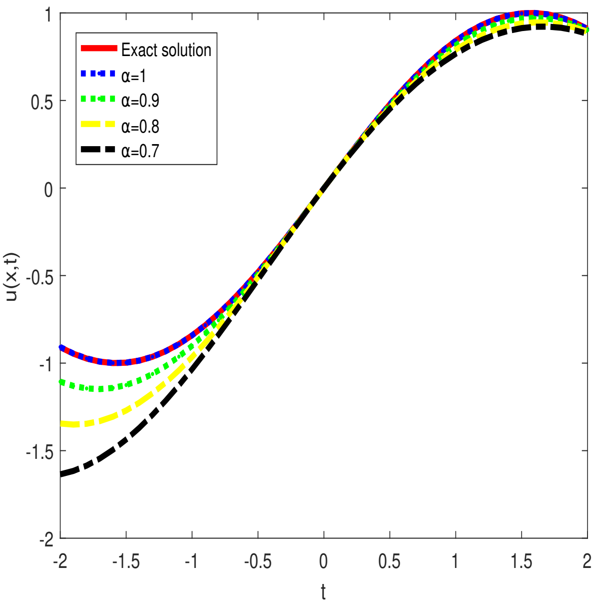

Figure 6: 2D graphs of the approximate solutions and exact solution of Application 4.3

t	$u_{\alpha=0.7}$	$u_{\alpha=0.8}$	$u_{\alpha=0.9}$	$u_{\alpha=1}$	u_{exact}	$\lvert u_{exact} - u_{VITM} \rvert$
0.01	0.0099849	0.0099899	0.0099949	0.0099998	0.0099998	2.7557×10^{-24}
0.03	0.029862	0.029907	0.029951	0.029996	0.029996	5.4241×10^{-20}
0.05	0.049612	0.049734	0.049857	0.049979	0.049979	5.3822×10^{-18}
0.07	0.069231	0.069467	0.069704	0.069943	0.069943	1.1120×10^{-16}
0.09	0.088712	0.089099	0.089487	0.089879	0.089879	1.0675×10^{-15}

Table 3: Numerical results of Application 4.3 at $x = 1$

6. Conclusion

This paper deals with the solutions of nonlinear Caputo-Fabrizio fractional partial differential equations arising in applied mathematics. Further, we have developed VITM to obtain approximate solutions of the Caputo–Fabrizio fractional differential equations successfully. The approximate solutions are compared with exact solutions and also with other existing solutions in the literature. It is observed that the obtained successive approximations solutions using the VITM converge very rapidly to the exact solution and so its high degree of accuracy. Thus, we can conclude that the proposed method is reliable, simple and effective to find approximate solutions of many nonlinear fractional partial differential equations that appear in various scientific fields, dynamical systems and mathematical sciences.

Acknowledgements:

The author is grateful to the anonymous referees for a careful checking of the details and for helpful comments that improved this paper.

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