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Journal of Prime Research in Mathematics, 19(1) (2023), 73–81

Journal of Prime Research in Mathematics

Group Distance Magic Labeling of Product of Graphs

Wasim Ashraf[a,*], Hani Shaker[a]

[a]*Department of Mathematics, COMSATS University Islamabad, Lahore Campus, 54000, Pakistan.*

Abstract

A graph is a tool used to build the interconnection network that a system requires. Such networks interoperability is ensured by specific labeling. There are several labelings in the literature, however the Group Distance Magic Labeling is better for graphs. A graph G is described as $\mathcal{H}$-distance magic graph if for an abelian group $\mathcal{H}$, there exist one-one mapping ℓ between group elements and vertex set of graph G such that $\sum_{x\in N(u)} \ell(x) = \mu$ for all $u \in V$, where μ is the magic constant belongs to abelian group $\mathcal{H}$ and $N(u)$ is u's free neighborhood. In this article, we prove direct product of anti-prism graphs with nth order cycles are $\mathbb{Z}_{2st}$, $\mathbb{Z}_2 \times \mathbb{Z}_{st}$, $\mathbb{Z}_3 \times \mathbb{Z}_{2t}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{\frac{2}{3}st}$ distance magic graphs.

Keywords: Group Distance Magic Labeling, Anti-prism, Cycles, Direct Product. **2010 Mathematics Subject Classification:** 05C10, 05C35, 05C57

1. Introduction and Preliminaries

Graph labeling is the process of assigning labels to vertices or edges, or both, based on entries from a certain set of elements. The distance magic labeling (also known as Sigma labeling) is defined as a bijection $\lambda : V(G) \to \{1, 2, 3, ..., n\}$ such that for every $x \in V$

$$w(x) = \sum_{y\in N_G(x)} \lambda(y) = k,$$

where $N_G(x)$, neighbouring area of Vertex is the collection of vertices close to x, $w(x)$ is each vertex's weight in G and the magic constant is k [1, 8]. Dalibor Froncek created the group distance magic labeling (GDML) in 2013 [7], based on the principle of distance magic labeling.

*Corresponding author
Email addresses: `wasimashraf1947@gmail.com` (Wasim Ashraf), `hani.uet@gmail.com` (Hani Shaker)

Received : 4 December 2022*; Accepted:* 8 May 2023*; Published Online:* 12 September 2023

For an abelian group $\mathcal{H}$, there exist one-to-one mapping ℓ between group elements and vertex set of group G such that

$$w(x) = \sum_{y \in N_G(x)} \lambda(y) = \mu,$$

for all $x \in V$, the magic constant μ belongs to abelian group $\mathcal{H}$ and $N(x)$ is x's free neigborhood.
In general, the labels for graph G's vertices are assigned using an abelian group components. It is the proved fact that every graph having DM-labeling is also having GDM-labeling with respect to modulo group $\mathbb{Z}_n$, where n is the order of the graph, but the problem of finding group distance magic labeling still retains its interest for other abelian groups other than $\mathbb{Z}_n$. An other interesting aspect of the problem is the opposite of this truth is untrue in general.
There are different types of graph labeling that includes vertex magic total labeling, Graceful and Harmonious labelings, edge magic total labeling, DML, GDML and OGDML. DML, GDML and OGDML are very famous labelings in the field of graph theory. A lot of work has been done on them since nineties. Mostly we will focus on DML, GDML and OGDML for different family of graphs.
The story of DML was started when Vilfred introduced the concept of sigma labeling in his doctoral thesis in 1994. He used the term sigma labeling since the sum of each vertex's weights is constant. For the sigma labeling, Mirka Miller et al. used the name 1-vertex MVL in 2003. She introduced 1-vertex MVL due to distance involved among the vertices of graph. For the same labeling, Sugeng et al. used the name DML instead of sigma labeling and 1-vertex MVL. DML was used due to relation between magic square and magic labelings. After that, DML is the universal term used for such labelings.
There are many basic results of distance magic graphs which are the path P_t iff $t = 1$ or $t = 3$, the cycle C_m iff $m = 4$, the complete graph K_n iff $n = 1$, the tree T iff either T is P_1 or T is P_3 and the wheel W_n iff $n = 4$. M. Kashif Shafiq et al [9]. proved DML for the union of different graphs. They worked on m-copies of complete graphs for different conditions on order and copies of graphs. There exist DML in $uK_{(v[w])}$ if w is even or uvw is odd, $u \geq 1, w > 1$ and $v > 1$. There does not exist DML in $uK_{(v[w])}$ if wv is odd, $v \equiv 3(mod4)$ and u is even.
Dalibor Froncek [7] proved GDML for Cartesian product, direct product of cycles for different conditions on order of graph, that is $C_n \times C_m$ $(n \leq m)$ admits GDML iff nm is even or n, m both even. In 2015, Marcin Anholcer et al. made some outstanding work on the graph direct product in GDML [2]. They proved GDML for $C_n \times C_m$ for $\mathbb{Z}_m \times \mathbb{Z}_n$ if $m, n \equiv 0(mod4)$. They also proved, the direct product of a r-regular graph G of order n with C_4 is GDML. They proved GDML for $C_q \times C_p$ for group $\mathbb{Z}_t \times \mathcal{A}$ if $p, q \equiv 0(mod4)$. The direct product of C_p with C_q is not GDML for any abelian group Γ and $p, q \not\equiv 0(mod4)$. The direct product of a r_1-regular graph G_1 with graph G_2 which is a r_2- regular is $\Gamma_1 \times \Gamma_2$-DM. They also proved GDML for $G \times H$.
Sylwia Cichacz [5] extended this labeling for lexicographic product of regular graphs with cycles, composition of regular graphs with complete bipartite graphs. She gave the formula $\mu = \frac{n+1}{2}$ for regular graph G. According to her, the lexicographic product of G having nth order with C_4 is GDML for Γ having $4n$ order like that $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathcal{A}$ [3, 4]. The lexicographic product of $K_{m,n}$ for even m and odd n with C_4 is GDML for Γ having $4(m + n)$ order. She proved GDML in $G \times C_4$ where G is Eulerian graph of the order $4n$ abelian group Γ and the odd order n. Just lately in [6], the authors established the required and adequate circumstances exist for cartesian product of C_{2^m} and C_{2^n} being $Z_2{}^{m+n}$-DM labeling. Additionally, they demonstrated cartesian product of C_n with C_m is not Γ-DM labeling, abelian group Γ of even order mn with exponent r such that If $2rminf(m), f(n) < lcm(m, n)$. Wasim Ashraf et al [10]. worked on GDML for the direct product of anti-prism family of graphs. They used different non-isomorphic modulo groups for this family of graphs. According to them, there exist GDML in graph $A_m \times A_n$ $m, n \geq 3$ for using $\mathbb{Z}_{4mn}$, $\mathbb{Z}_2 \times \mathbb{Z}_{2mn}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{4n}$ modulo groups.
The anti-prism family of graphs is used in this study to obtain the group distance magic labeling with regard modulo group and its product. For the anti-prism, We give you the $\mathbb{Z}_{2n}$ and $\mathbb{Z}_2 \times \mathbb{Z}_n$-DM labeling. For the anti-prism graphs' direct product, we also provide the $\mathbb{Z}_{2n}$ and $\mathbb{Z}_2 \times \mathbb{Z}_n$-DM labeling. A graph $G \times H$ is the direct product of graphs G and H with $V(G) \times V(H)$ as follow

$$E(G \times H) = \{(x,y)(x',y') \quad | \quad x,y \in V(G), x',y' \in V(H), xy \in E(G), x'y' \in E(H)\},$$

(x, y) and (x', y') are adjoining in $G \times H$ iff x is adjoining to y in G and x' is adjoining to y' in H [4].

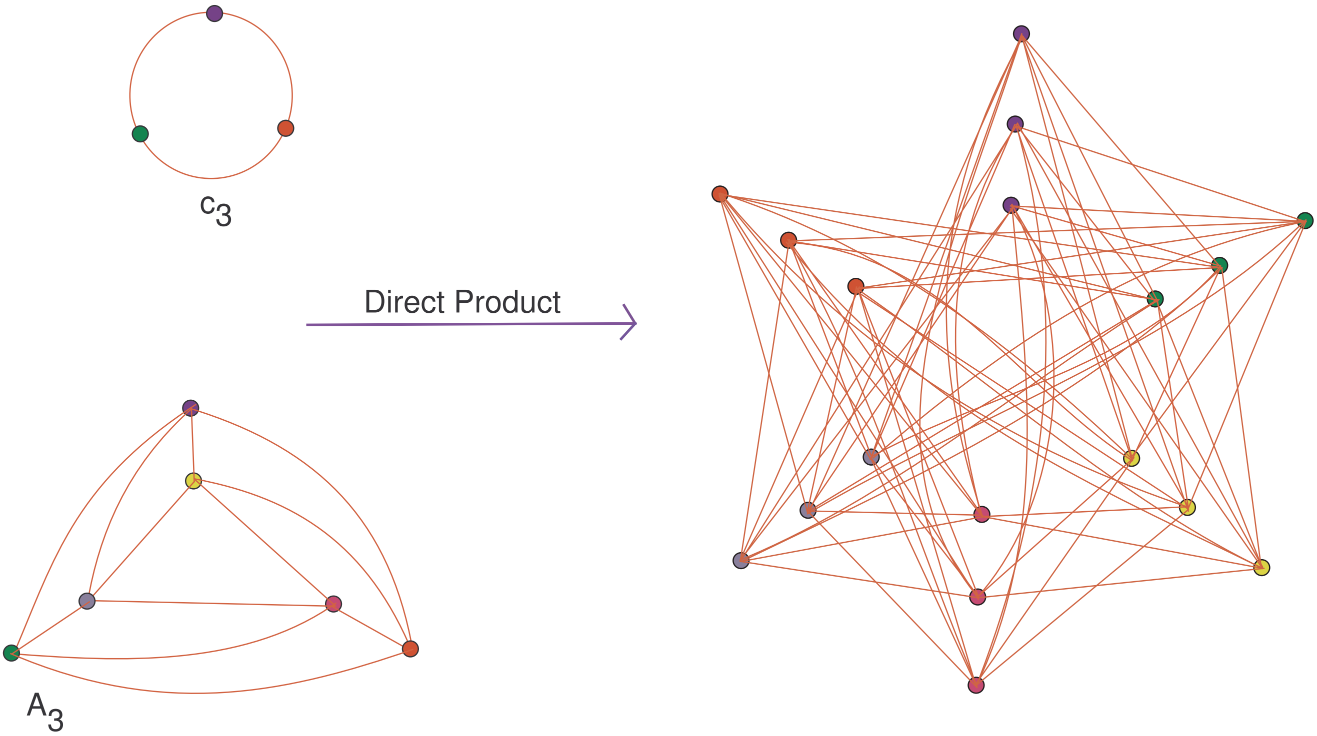

Figure 1: Direct Product of A_3 with C_3

2. Main results

In theorems given below, we offer GDML for the direct product of Anti-prism graphs with Cycles.

Theorem 2.1. *For A_s and C_t, where A_s be anti-prism graph and C_t be cycle such that $s \leq t$. The graph $A_s \times C_t$ admits a $\mathbb{Z}_{2st}$-DM labeling $\forall\ s,t \geq 3$ with magic constant $2((s-2)t-2)$ for $s \geq 3$, where $\mathbb{Z}_{2st}$ be the module group of order $2st$.*

Proof. The vertex and edge sets that represent A_s and C_t are as follows:

$$\begin{aligned} V(A_s) &= \{x_i, y_i | 0 \leq i \leq s-1\}, \\ E(A_s) &= \{x_i x_{i+1}, y_i y_{i+1}, x_i y_i, x_i y_{i+1} | 0 \leq i \leq s-2\} \cup \{x_0 x_{s-1}, y_0 y_{s-1}, y_0 x_{s-1}, x_{s-1} y_{s-1}\}, \\ V(C_t) &= \{x'_i, 0 \leq i \leq t-1\}, \\ E(C_t) &= \{x'_i x'_{i+1}, 0 \leq i \leq t-2\} \cup \{x'_0 x'_{t-1}\}. \end{aligned}$$

We get the vertex set below using the direct product concept, which represents $A_s \times C_t$,

$$V(A_s \times C_t) = \{(x_i, x'_j), (y_i, x'_j) | 0 \leq i \leq s-1\ ,\ 0 \leq j \leq t-1\}$$

Define $\ell : V(A_s \times C_t) \to \mathbb{Z}_{2st}$ in the following way,

$$\begin{aligned} \ell(x_i, x'_j) &= 2ti + 2j, && \text{for} \quad 0 \leq i \leq s-1, 0 \leq j \leq t-1, \\ \ell(y_i, x'_j) &= (2st-1) - 2(ti+j), && \text{for} \quad 0 \leq i \leq s-1, 0 \leq j \leq t-1. \end{aligned}$$

In our graph $A_s \times C_t$, we have two types of vertices as per the vertex representation given above i.e. (x_i, x'_j) and (y_i, x'_j). Let us apply the labeling ℓ to compute the weight of both types of vertices. Now each vertex of the type (x_i, x'_j) is adjacent with following eight vertices:

$$(x_{i+1}, x'_{j+1}), (x_{i+(s-1)}, x'_{j+1}), (y_i, x'_{j+1}), (y_{i+1}, x'_{j+1}), (x_{i+1}, x'_{j+2}), (x_{i+(s-1)}, x'_{j+2}), (y_i, x'_{j+2}), (y_{i+1}, x'_{j+2}),$$

and each vertex of the type (y_i, x'_j) is adjacent with following eight vertices:

$$(x_i, x'_{j+1}), (x_{i+(s-1)}, x'_{j+1}), (y_{i+1}, x'_{j+1}), (y_{i+(s-1)}, x'_{j+1}), (x_i, x'_{j+2}), (x_{i+(s-1)}, x'_{j+2}), (y_{i+1}, x'_{j+2}), (y_{i+(s-1)}, x'_{j+2}).$$

We can calculate the weights of vertices (x_i, x'_j) and (y_i, x'_j) in such a way,

$$\begin{aligned} w(x_i, x'_j) &= \ell(x_{i+1}, x'_{j+1}) + \ell(x_{i+(s-1)}, x'_{j+1}) + \ell(y_i, x'_{j+1}) + \ell(y_{i+1}, x'_{j+1}) + \ell(x_{i+1}, x'_{j+2}) \\ &+ \ell(x_{i+(s-1)}, x'_{j+2}) + \ell(y_i, x'_{j+2}) + \ell(y_{i+1}, x'_{j+2}) \\ &= 2t(i+1) + 2(j+1) + 2t(i+(s-1)) + 2(j+1) + (2st-1) - 2(ti+j+1) + (2st-1) \\ &- 2(t(i+1)+j+1) + 2t(i+1) + 2(j+2) + 2t(i+(s-1)) + 2(j+2) + (2st-1) \\ &- 2(ti+j+2) + (2st-1) - 2(t(i+1)+j+2) = 12st - 4t - 4. \end{aligned}$$

$$\begin{aligned} w(y_i, x'_j) &= \ell(x_i, x'_{j+1}) + \ell(x_{i+(s-1)}, x'_{j+1}) + \ell(y_{i+1}, x'_{j+1}) + \ell(y_{i+(s-1)}, x'_{j+1}) + \ell(x_i, x'_{j+2}) \\ &+ \ell(x_{i+(s-1)}, x'_{j+2}) + \ell(y_{i+1}, x'_{j+2}) + \ell(y_{i+(s-1)}, x'_{j+2}) \\ &= 2t(i) + 2(j+1) + 2t(i+(s-1)) + 2(j+1) + (2st-1) - 2(ti+t+j+1) + (2st-1) \\ &- 2(t(i+(s-1))+j+1) + 2t(i+1) + 2(j+2) + 2t(i+(s-1)) + 2(j+2) + (2st-1) \\ &- 2(ti+t+j+2) + (2st-1) - 2(t(i+(s-1))+j+2) = 8st - 4t - 4. \end{aligned}$$

But $w(x_i, x'_j)$, $w(y_i, x'_j) \in \mathbb{Z}_{2st}$, so these weights are under $\mod 2st$, i.e.

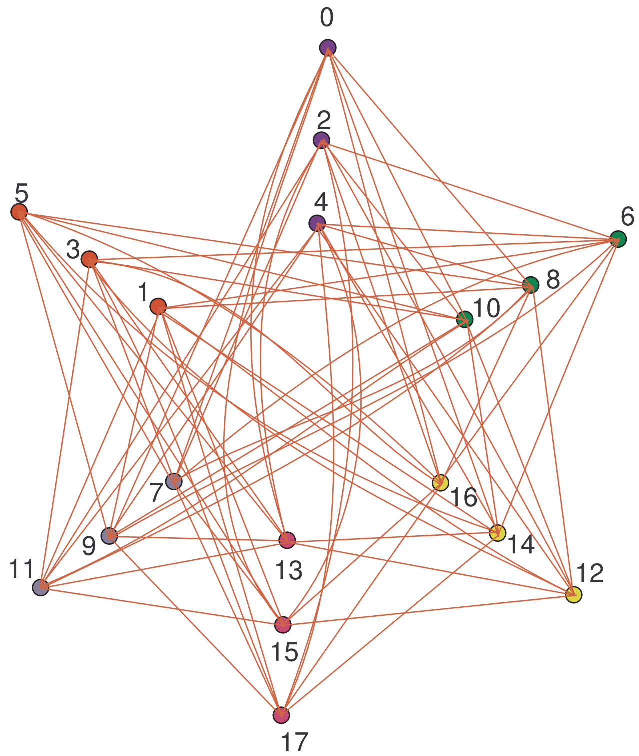

Figure 2: Direct Product of A_3 with C_3

$$w(x_i, x'_j) \equiv 12st - 4t - 4 \mod 2st \equiv 2st - 4t - 4 \mod 2st$$
$$w(y_i, x'_j) \equiv 8st - 4t - 4 \mod 2st \equiv 2st - 4t - 4 \mod 2st$$

The magic constant of $\mathbb{Z}_{2st}$-distance magic graph $A_s \times C_t$ under ℓ is,

$$\mu = 2((s-2)t - 2) \quad \text{for} \quad s \geq 3.$$

□

Since $Z_{2st} \cong Z_2 \times Z_{st}$ if $gcd(2, st) = 1$ which are used for GDML of graph $A_s \times C_t$ in 2.1. Now we discuss module group $Z_2 \times Z_{st}$ if $gcd(2, st) \neq 1$ for GDML of graph $A_s \times C_t$ in the following theorem.

Theorem 2.2. *For A_s and C_t, the graph $A_s \times C_t$ admits a $\mathbb{Z}_2 \times \mathbb{Z}_{st}$-distance magic labeling with magic constant $(0, (s-2)t - 4)$ for $s \geq 3,\ t \geq 3$, where $\mathbb{Z}_2 \times \mathbb{Z}_{st}$ be the module group of order $2st$.*

Proof. The vertex and edge sets that represent A_s, C_t and the vertex set $A_s \times C_t$ using the direct product concept given in theorem 2.1.
Define $\ell : V(A_s \times C_t) \to \mathbb{Z}_2 \times \mathbb{Z}_{st}$ as follows,

$$\begin{aligned} \ell(x_i, x'_j) &= (0, ti + j), && \text{for} \quad 0 \leq i \leq s-1, 0 \leq j \leq t-1, \\ \ell(y_i, x'_j) &= (1, t(s-i) - 1 - j), && \text{for} \quad 0 \leq i \leq s-1, 0 \leq j \leq t-1. \end{aligned}$$

We can calculate the weight of vertices (x_i, x'_j) and (y_i, x'_j) in such a way,

$$\begin{aligned} w(x_i, x'_j) &= \ell(x_{i+1}, x'_{j+1}) + \ell(x_{i+(s-1)}, x'_{j+1}) + \ell(y_i, x'_{j+1}) + \ell(y_{i+1}, x'_{j+1}) + \ell(x_{i+1}, x'_{j+2}) \\ &+ \ell(x_{i+(s-1)}, x'_{j+2}) + \ell(y_i, x'_{j+2}) + \ell(y_{i+1}, x'_{j+2}) \\ &= (0, t(i+1) + j + 1) + (0, t(i + (s-1)) + j + 1) + (1, t(s-i) - 1 - (j+1)) \\ &+ (1, t(s-(i+1)) - 1 - (j+1)) + (0, t(i+1) + j + 2) + (0, t(i + (s-1)) + j + 2) \\ &+ (1, t(s-i) - 1 - (j+2)) + (1, t(s-(i+1)) - 1 - (j+2)) = (4, 6st - 2t - 4). \end{aligned}$$

$$\begin{aligned} w(y_i, x'_j) &= \ell(x_i, x'_{j+1}) + \ell(x_{i+(s-1)}, x'_{j+1}) + \ell(y_{i+1}, x'_{j+1}) + \ell(y_{i+(s-1)}, x'_{j+1}) + \ell(x_i, x'_{j+2}) \\ &+ \ell(x_{i+(s-1)}, x'_{j+2}) + \ell(y_{i+1}, x'_{j+2}) + \ell(y_{i+(s-1)}, x'_{j+2}) \\ &= (0, t(i) + j + 1) + (0, t(i + (s-1)) + j + 1) + (1, t(s-i-1) - 1 - (j+1)) \\ &+ (1, t(s-(i+(s-1))) - 1 - (j+1)) + (0, t(i) + j + 2) + (0, t(i + (s-1)) + j + 2) \\ &+ (1, t(s-i-1) - 1 - (j+2)) + (1, t(s-(i+(s-1))) - 1 - (j+2)) = (4, 4st - 2t - 4). \end{aligned}$$

But $w(x_i, x'_j)$, $w(y_i, x'_j) \in \mathbb{Z}_2 \times \mathbb{Z}_{st}$ so it will be considered under $\mod st$.

$$w(x_i, x'_j) \equiv (4, 6st - 2t - 4) \mod st \equiv (0, st - 2t - 4) \mod st$$

$$w(y_i, x'_j) \equiv (4, 4st - 2t - 4) \mod st \equiv (0, st - 2t - 4) \mod st$$

The magic constant of $\mathbb{Z}_2 \times \mathbb{Z}_{st}$ -distance magic graph $A_s \times C_t$ under ℓ is,

$$\mu = (0, (s-2)t - 4), \quad \text{for} \quad s \geq 3, \quad t \geq 4.$$

Theorem 2.3. *For A_3 and C_t, such that $t = 3k$, $k \geq 1$. The graph $A_3 \times C_t$ admits a $\mathbb{Z}_3 \times \mathbb{Z}_{2t}$-distance magic labeling with magic constant $(0, 2(t-2))$, where $\mathbb{Z}_3 \times \mathbb{Z}_{2t}$ is module group of order $6t$.*

Proof. The vertex and edge sets that represent A_3, C_t and the vertex set $A_3 \times C_t$ using the direct product concept given in theorem 2.1.
Define $\ell : V(A_3 \times C_t) \to \mathbb{Z}_3 \times \mathbb{Z}_{2t}$ as follows,

$$\begin{aligned} \ell(x_i, x'_j) &= (i, 2j), && \text{for} \quad 0 \le i \le 2, 0 \le j \le t-1, \\ \ell(y_i, x'_j) &= (2-i, 2t-1-2j), && \text{for} \quad 0 \le i \le 2, 0 \le j \le t-1. \end{aligned}$$

We can calculate the weight of vertex (x_i, x'_j) in such a way,

$$\begin{aligned} w(x_i, x'_j) &= \ell(x_{i+1}, x'_{j+1}) + \ell(x_{i+2}, x'_{j+1}) + \ell(y_i, x'_{j+1}) + \ell(y_{i+1}, x'_{j+1}) \\ &+ \ell(x_{i+1}, x'_{j+2}) + \ell(x_{i+2}, x'_{j+2}) + \ell(y_i, x'_{j+2}) + \ell(y_{i+1}, x'_{j+2}) \\ &= (i+1, 2(j+1)) + (i+2, 2(j+1)) + (2-i, 2t-1-2(j+1)) \\ &+ (2-(i+1), 2t-1-2(j+1)) + (i+1, 2(j+2)) + (i+2, 2(j+2)) \\ &+ (2-i, 2t-1-2(j+2)) + (2-(i+1), 2t-1-2(j+2)) = (12, 8t-4). \end{aligned}$$

$$\begin{aligned} w(y_i, x'_j) &= \ell(x_i, x'_{j+1}) + \ell(x_{i+2}, x'_{j+1}) + \ell(y_{i+1}, x'_{j+1}) + \ell(y_{i+2}, x'_{j+1}) + \ell(x_i, x'_{j+2}) \\ &+ \ell(x_{i+2}, x'_{j+2}) + \ell(y_{i+1}, x'_{j+2}) + \ell(y_{i+2}, x'_{j+2}) \\ &= (i, 2(j+1)) + (i+2, 2(j+1)) + (2-i-1, 2t-1-2(j+1)) \\ &+ (2-(i+2), 2t-1-2(j+1)) + (i, 2(j+2)) + (i+2, 2(j+2)) \\ &+ (2-i-1, 2t-1-2(j+2)) + (2-(i+2), 2t-1-2(j+2)) = (6, 8t-4). \end{aligned}$$

But $w(x_i, x'_j)$, $w(y_i, x'_j) \in \mathbb{Z}_3 \times \mathbb{Z}_{2t}$ so it will be considered under $\mod 2t$.

$$w(x_i, x'_j) \equiv (12, 8t-4) \mod 2t \equiv (0, 2t-4) \mod 2t$$

$$w(y_i, x'_j) \equiv (6, 8t-4) \mod 2t \equiv (0, 2t-4) \mod 2t$$

The magic constant of $\mathbb{Z}_3 \times \mathbb{Z}_{2t}$-distance magic graph $A_3 \times C_t$ under ℓ is,

$$\mu = (0, 2(t-2)).$$

□

Theorem 2.4. *For A_s and C_t, such that $s = 3(k+1)$, $t = 3k'$, $k, k' \ge 1$. The graph $A_s \times C_t$ admits a $\mathbb{Z}_3 \times \mathbb{Z}_{\frac{2}{3}st}$-distance magic labeling with magic constant $(0, 2((s-2)\frac{t}{3} - 2))$, where $\mathbb{Z}_3 \times \mathbb{Z}_{\frac{2}{3}st}$ is the modulo group of order $2st$.*

Proof. The vertex and edge sets that represent A_s, C_t and the vertex set $A_s \times C_t$ using the direct product concept given in theorem 2.1.
Define $\ell : V(A_s \times C_t) \to \mathbb{Z}_3 \times \mathbb{Z}_{\frac{2}{3}st}$ in the following way,

$$\ell(x_i, x'_j) = \begin{cases} (0, 2(\frac{t}{3})i + 2j), & \text{for} \quad 0 \le i \le s-1, \quad j = 0, 1, ..., k'-1, \\ (1, 2(\frac{t}{3})i + 2j - 2k'), & \text{for} \quad 0 \le i \le s-1, \quad j = k', k'+1, ..., 2k'-1, \\ (2, 2(\frac{t}{3})i + 2j - 4k'), & \text{for} \quad 0 \le i \le s-1, \quad j = 2k', 2k'+1, ..., 3k'-1. \end{cases}$$

$$\ell(y_i, x'_j) = \begin{cases} (0, 2(2t - \frac{t}{3}i) - 1 - 2j), & \text{for} \quad 0 \le i \le s-1, \quad j = 0, 1, ..., k'-1, \\ (2, 2(2t - \frac{t}{3}i) + (2k'-1) - 2j), & \text{for} \quad 0 \le i \le s-1, \quad j = k', k'+1, ..., 2k'-1, \\ (1, 2(2t - \frac{t}{3}i) + (4k'-1) - 2j), & \text{for} \quad 0 \le i \le s-1, \quad j = 2k', 2k'+1, ..., 3k'-1. \end{cases}$$

We can calculate the weight of vertex (x_i, x'_j) in such a way,

$$\begin{aligned} w(x_i, x'_j) &= \ell(x_{i+1}, x'_{j+1}) + \ell(x_{i+(s-1)}, x'_{j+1}) + \ell(y_i, x'_{j+1}) + \ell(y_{i+1}, x'_{j+1}) + \ell(x_{i+1}, x'_{j+2}) \\ &+ \ell(x_{i+(s-1)}, x'_{j+2}) + \ell(y_i, x'_{j+2}) + \ell(y_{i+1}, x'_{j+2}) \\ &= (1, 2(\frac{t}{3})(i+1) + 2(j+1) - 2(1)) + (1, 2(\frac{t}{3})(i+(s-1)) \\ &+ 2(j+1) - 2(1)) + (2, 2(2t - \frac{t}{3}i) + (2(1)-1) - 2(j+1)) + (2, 2(2t - \frac{t}{3}(i+1)) \\ &+ (2(1)-1) - 2(j+1)) + (1, 2(\frac{t}{3})(i+1) + 2(j+2) - 2(1)) + (1, 2(\frac{t}{3})(i+(s-1)) \\ &+ 2(j+2) - 2(1)) + (2, 2(2t - \frac{t}{3}i) + (2(1)-1) - 2(j+2)) + (2, 2(2t - \frac{t}{3}(i+1)) \\ &+ (2(1)-1) - 2(j+2)) = (12, \frac{4}{3}st - \frac{4}{3}t + 16t - 4). \end{aligned}$$

$$\begin{aligned} w(y_i, x'_j) &= \ell(x_i, x'_{j+1}) + \ell(x_{i+(s-1)}, x'_{j+1}) + \ell(y_{i+1}, x'_{j+1}) + \ell(y_{i+(s-1)}, x'_{j+1}) + \ell(x_i, x'_{j+2}) \\ &+ \ell(x_{i+(s-1)}, x'_{j+2}) + \ell(y_{i+1}, x'_{j+2}) + \ell(y_{i+(s-1)}, x'_{j+2}) \\ &= (1, 2(\frac{t}{3})(i) + 2(j+1) - 2(1)) + (1, 2(\frac{t}{3})(i+(s-1)) + 2(j+1) \\ &- 2(1)) + (2, 2(2t - \frac{t}{3}(i+1)) + (2(1)-1) - 2(j+1)) + (2, 2(2t - \frac{t}{3}(i+(s-1))) \\ &+ (2(1)-1) - 2(j+1)) + (1, 2(\frac{t}{3})(i) + 2(j+2) - 2(1)) + (1, 2(\frac{t}{3})(i+(s-1)) \\ &+ 2(j+2) - 2(1)) + (2, 2(2t - \frac{t}{3}(i+1)) + (2(1)-1) - 2(j+2)) + (2, 2(2t - \frac{t}{3}(i \\ &+ (s-1))) + (2(1)-1) - 2(j+2)) = (12, \frac{4}{3}st - \frac{4}{3}t + 16t - 4). \end{aligned}$$

But $w(x_i, x'_j)$, $w(y_i, x'_j) \in \mathbb{Z}_3 \times \mathbb{Z}_{\frac{2}{3}st}$ so it will be considered under $\mod \frac{2}{3}st$.

$$w(x_i, x'_j) \equiv (12, \frac{4}{3}st - \frac{4}{3}t + 16t - 4) \mod \frac{2}{3}st \equiv (0, \frac{2}{3}st - \frac{4}{3}t - 4) \mod \frac{2}{3}st$$

$$w(y_i, x'_j) \equiv (12, \frac{4}{3}st - \frac{4}{3}t + 16t - 4) \mod \frac{2}{3}st \equiv (0, \frac{2}{3}st - \frac{4}{3}t - 4) \mod \frac{2}{3}st$$

The magic constant of $\mathbb{Z}_3 \times \mathbb{Z}_{\frac{2}{3}st}$-distance magic graph $A_s \times C_t$ under ℓ is,

$$\mu = (0, 2((s-2)\frac{t}{3} - 2)).$$

Theorem 2.5. *For A_s and C_t, such that either $s = 3k+1$ or $s = 3k+2$, $t = 9k'$, $k, k' \geq 1$. The graph $A_s \times C_t$ admits a $\mathbb{Z}_3 \times \mathbb{Z}_{\frac{2}{3}st}$-distance magic labeling with magic constants $(0, 2((s-2)\frac{t}{3} - 2))$ for $s = 3k+1$ and $(0, 2((s-2(k+1))t - 2))$ for $s = 3k+2$, where $\mathbb{Z}_3 \times \mathbb{Z}_{\frac{2}{3}st}$ is the module group of order $2st$.*

Proof. The vertex and edge sets that represent A_s, C_t and the vertex set $A_s \times C_t$ using the direct product concept given in theorem 2.1.

Now we have two cases as follows,

Case(i)

If $s = 3k+1$ and $t = 9k'$, $k, k' \geq 1$.

Define $\ell : V(A_s \times C_t) \to \mathbb{Z}_3 \times \mathbb{Z}_{\frac{2}{3}st}$ as follows,

$$\ell(x_i, x'_j) = \begin{cases} (0, 2(\frac{t}{3})i + 2j), & \text{for} \quad 0 \leq i \leq s-1, \quad j = 0, 1, ..., 3k'-1, \\ (1, 2(\frac{t}{3})i + 2j - 6k'), & \text{for} \quad 0 \leq i \leq s-1, \quad j = 3k', 3k'+1, ..., 6k'-1, \\ (2, 2(\frac{t}{3})i + 2j - 12k'), & \text{for} \quad 0 \leq i \leq s-1, \quad j = 6k', 6k'+1, ..., 9k'-1. \end{cases}$$

$$\ell(y_i,x_j')=\begin{cases}(0,2(\frac{s}{3}t-\frac{t}{3}i)-1-2j), & \text{for} \quad 0\le i\le s-1, \quad j=0,1,...,3k^{'}-1,\\ (2,2(\frac{s}{3}t-\frac{t}{3}i)+(6k^{'}-1)-2j), & \text{for} \quad 0\le i\le s-1, \quad j=3k^{'},3k^{'}+1,...,6k^{'}-1,\\ (1,2(\frac{s}{3}t-\frac{t}{3}i)+(12k^{'}-1)-2j), & \text{for} \quad 0\le i\le s-1, \quad j=6k^{'},6k^{'}+1,...,9k^{'}-1.\end{cases}$$

We can calculate the weight of vertex (x_i,x_j') in such a way,

$$\begin{aligned} w(x_i,x_j') &= \ell(x_{i+1},x_{j+1}')+\ell(x_{i+(s-1)},x_{j+1}')+\ell(y_i,x_{j+1}')+\ell(y_{i+1},x_{j+1}')\\ &+ \ell(x_{i+1},x_{j+8}')+\ell(x_{i+(s-1)},x_{j+8}')+\ell(y_i,x_{j+8}')+\ell(y_{i+1},x_{j+8}')\\ &= (0,2(\frac{t}{3})(i+1)+2(j+1))+(0,2(\frac{t}{3})(i+(s-1))+2(j+1))\\ &+ (0,2(\frac{s}{3}t-\frac{t}{3}i)-1-2(j+1))+(0,2(\frac{s}{3}t-\frac{t}{3}(i+1))\\ &- 1-2(j+1)))+(2,2(\frac{t}{3})(i+1)+2(j+8)-12)+(2,2(\frac{t}{3})(i+(s-1))\\ &+ 2(j+8)-12)+(1,2(\frac{s}{3}t-\frac{t}{3}i)+(12-1)-2(j+8))+(1,2(\frac{s}{3}t\\ &- \frac{t}{3}(i+1))+(12-1)-2(j+8)))=(6,4st-\frac{4}{3}t-4).\end{aligned}$$

$$\begin{aligned} w(y_i,x_j') &= \ell(x_i,x_{j+1}')+\ell(x_{i+(s-1)},x_{j+1}')+\ell(y_{i+1},x_{j+1}')+\ell(y_{i+(s-1)},x_{j+1}')+\ell(x_i,x_{j+8}')\\ &+ \ell(x_{i+(s-1)},x_{j+8}')+\ell(y_{i+1},x_{j+8}')+\ell(y_{i+(s-1)},x_{j+8}')\\ &= (0,2(\frac{t}{3})(i)+2(j+1))+(0,2(\frac{t}{3})(i+(s-1))+2(j+1))\\ &+ (0,2(\frac{s}{3}t-\frac{t}{3}(i+1))-1-2(j+1))+(0,2(\frac{s}{3}t-\frac{t}{3}(i+(s-1)))\\ &- 1-2(j+1)))+(2,2(\frac{t}{3})(i)+2(j+8)-12)+(2,2(\frac{t}{3})(i+(s-1))\\ &+ 2(j+8)-12)+(1,2(\frac{s}{3}t-\frac{t}{3}(i+1))+(12-1)-2(j+8))+(1,2(\frac{s}{3}t\\ &- \frac{t}{3}(i+(s-1)))+(12-1)-2(j+8)))=(6,\frac{4}{3}st-\frac{4}{3}t-4).\end{aligned}$$

But $w(x_i,x_j')$, $w(y_i,x_j')\in\mathbb{Z}_3\times\mathbb{Z}_{\frac{2}{3}st}$ so it will be considered under $\mod \frac{2}{3}st$.

$$w(x_i,x_j')\equiv(6,4st-\frac{4}{3}t-4) \mod \frac{2}{3}st\equiv(0,\frac{2}{3}st-\frac{4}{3}t-4) \mod \frac{2}{3}st$$

$$w(y_i,x_j')\equiv(6,\frac{4}{3}st-\frac{4}{3}t-4) \mod \frac{2}{3}st\equiv(0,\frac{2}{3}st-\frac{4}{3}t-4) \mod \frac{2}{3}st$$

case(ii)

If $s=3k+2$ and $t=9k^{'}$, $k,k^{'}\ge 1$.

Then the same labeling $\ell: V(A_s\times C_t)\to\mathbb{Z}_3\times\mathbb{Z}_{\frac{2}{3}st}$ used as in above case.

The magic constants of $\mathbb{Z}_3\times\mathbb{Z}_{\frac{2}{3}st}$-distance magic graph $A_s\times C_t$ under ℓ is,

$$\mu=\begin{cases}(0,2((s-2)\frac{t}{3}-2)), & \text{for} \quad s=3k+1,\\ (0,2((s-2(k+1))t-2)), & \text{for} \quad s=3k+2.\end{cases}$$

□

3. Conclusion

Graph theory and Groups are linked via Group Distance Magic Labeling (GDML). As a result, we must define the relationship between group $\mathbb{Z}_{2st}$ and direct product of anti-prism graphs with cycles through GDML. Our results provide the GDML of anti-prism graph of direct product with cycles and show that it is $\mathbb{Z}_{2st}$ to $\mathbb{Z}_2 \times \mathbb{Z}_{st}$, $\mathbb{Z}_3 \times \mathbb{Z}_{2t}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{\frac{2}{3}st}-$ distance magic.

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