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On the representations of the braid group and the welded braid group

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Abstract

We study the composition of F. R. Cohen's map $B_n \to B_{nk}$ with the standard representation of B_{nk}, where B_n is the braid group on n strings. We prove that the obtained representation of B_n is isomorphic to the direct sum of k copies of the standard representation of B_n. A similar work is done for the welded braid group wB_n.

Keywords: Braid group, common fixed point, Welded Braid group, Cohen.
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1. Introduction

There are many kinds of representations of B_n, the braid group on n strings. Some of the main representations of the braid group are the Burau representation, Wada representations, and the standard representation. The standard representation is known for its simplicity. In fact, it was first discovered only in 1996 by Dian-Min Tong et al. [12], and it was studied by I. Sysoeva [11].

In this paper, we consider wB_n, the welded braid group, which is a 3-dimensional analogue of B_n. Welded braid groups, also known as loop braid groups, have different definitions in terms of configuration spaces of circles in the 3-ball B^3, as automorphisms of free groups, or as tubes in $\mathbb{R}^4$. In [9], C. Damiani presented various definitions of welded braid groups and proved the equivalence of these definitions. The representation theory for these groups is recently studied. Burau representation extends, in terms of Fox derivatives and Magnus expansion, to welded braid groups [3]. In [4], Bellingeri and Soulié extended Long-Moody procedure to wB_n and compared different Burau representations for wB_n. They extended some types of Wada representations to welded braid groups. Also, they introduced the Tong-Yang-Ma representations of wB_n and its dual. The standard representation of wB_n is obtained in a similar way.

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We consider the map $B_n \to B_{nk}$ introduced by F. R. Cohen, which is defined on geometric braids by replacing each string with k strings [8]. In [1], M. N. Abdulrahim studied the composition of the Burau representation and the natural map $B_n \to B_{nk}$. Constructing a free group, G_{nm}, of rank nm, M. Abdulrahim and N. Kassem composed Cohen's map on m strings and the embedding $B_{nm} \to Aut(G_{nm})$ via Wada's map [2].

In Section two of this paper, we give basic definitions concerning B_n and wB_n, and introduce some of their main representations. Also, we present some previous known results, needed in our work, concerning monomial matrices.

In Section three, we study the composition of Cohen's map, $B_n \to B_{nk}$, and the standard representation, $B_{nk} \to GL_{nk}(\mathbb{Z}[t, t^{-1}])$. Our main result is Theorem 3.4 which shows that the linear representation obtained by composing Cohen's map and the standard representation is isomorphic to a direct sum of k copies of the standard representation of B_n.

In Section four, we extend this work to the welded braid group. We aim to construct linear representations of wB_n of high degree. Using Cohen's map on k strings, we define a map $\psi : wB_n \to wB_{nk}$, and we show that it is a well defined homomorphism. We compose ψ with the standard representation of wB_{nk}, to obtain a representation of wB_n of degree nk. The composition of ψ with representations of B_n may result in new representations of wB_n.

2. Definitions and Known Results

We recall some basic definitions of braid groups and welded braid groups. In particular, we define the standard representation of the braid group and that of the welded braid group. Also, we present Cohen's map. Then, we give some previous result of monomial matrices that are needed in our work.

Definition 2.1. [5] The braid group on n strings, B_n, is the abstract group with generators $\sigma_1, ..., \sigma_{n-1}$ and a presentation as follows:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad i = 1, 2, ..., n-2$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| \geq 2.$$

There are many representations of the braid group B_n. One of these is the standard representation of degree n.

Definition 2.2. [12] The standard representation is the representation

$$\rho : B_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$$

given by :

$$\rho(\sigma_i) = I_{i-1} \oplus \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1} \text{ , for } i = 1, 2, .., n-1.$$

Here, I_k is the $k \times k$ identity matrix.

According to [6], the fundamental group of the space of configurations of n unlinked Euclidean circles being all parallel to a fixed plane is generated by two types of moves. The move τ_i is the path permuting the i^{th} and the $(i+1)^{th}$ circles by passing over or around, and the move σ_i permutes them by passing the i^{th} circle through the $(i+1)^{th}$.

Definition 2.3. [6] The welded braid group on n generators wB_n admits a presentation with generators $\{\sigma_i, \tau_i; i \in \{1, \ldots, n-1\}\}$ together with relations:

$$\begin{cases} \sigma_i\sigma_l = \sigma_l\sigma_i & \text{if } \mid i-l \mid \geq 2 & (1) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{if } i \in \{1, \ldots, n-2\} & (2) \\ \tau_i\tau_l = \tau_l\tau_i & \text{if } \mid i-l \mid \geq 2 & (3) \\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} & \text{if } i \in \{1, \ldots, n-2\} & (4) \\ \tau_i^2 = 1 & \text{if } i \in \{1, \ldots, n-1\} & (5) \\ \sigma_i\tau_l = \tau_l\sigma_i & \text{if } \mid i-l \mid \geq 2 & (6) \\ \tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1} & \text{if } i \in \{1, \ldots, n-2\} & (7) \\ \sigma_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\sigma_{i+1} & \text{if } i \in \{1, \ldots, n-2\} & (8) \end{cases}$$

In [12], Tong, Yang and Ma investigated the representations of braid group where the i^{th} generator is sent to a non-singular matrix of the form

$$\rho(\sigma_i) = Id_{i-1} \oplus \begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus Id_{n-i-1}.$$

They proved that only two non trivial representations of this type exist (up to equivalence and transpose): the unreduced Burau representations and the Tong-Yang-Ma representation. The standard representation is the transpose of the Tong-Yang-Ma representation. This representation can be extended to welded braid groups in the following way:

Proposition 2.4. *The standard representation of the welded braid group is the representation*

$$wB_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$$

given by

$$\rho(\sigma_i) = Id_{i-1} \oplus \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} \oplus Id_{n-i-1} \quad , \quad \rho(\tau_i) = Id_{i-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus Id_{n-i-1}$$

for all natural numbers $i \in \{1, \ldots, n-1\}$.

Proof. The matrices for σ_i define the standard representation of the braid group and those of τ_i define the permutation representation of the symmetric group. It follows that we only need to check the mixed relations between the braid and symmetric generators of wB_n (see Definition 2.3). □

To construct representations of higher degree, F. R. Cohen defined a map on geometric braids that replaces each string with k strings.

Definition 2.5. ([7], [8]) The Cohen representation is the map $B_n \to B_{nk}$ defined as follows:
$\sigma_i \to (\sigma_{ki}\sigma_{ki+1}\ldots\sigma_{ki+k-1})(\sigma_{ki-1}\sigma_{ki}\ldots\sigma_{ki+k-2})\ldots(\sigma_{ki-k+1}\sigma_{ki-k+2}\ldots\sigma_{ki})$.

The image of the braid σ_i is the braid obtained by replacing each string of the geometric braid, σ_i, with k parallel strings. This tensor product of braids is a homomorphism. For more details see [7].

The images of the generators of braid group under the standard representation are monomial matrices. Thus, we introduce some results concerning this type of matrices.

Definition 2.6. [10] A Monomial matrix of order n is a regular $n \times n$-matrix which has in each row and in each column exactly one non-zero component. Monomial matrices form a group. Permutation matrices are monomial matrices in which all nonzero components are equal to 1. Its rows are a permutation of the rows of the identity matrix.

Let $P(i_1, \ldots, i_n)$ be the permutation matrix associated to the permutation $(i_1, \ldots, i_n)$.

Lemma 2.7. *[10] Every monomial matrix is a product of a diagonal matrix with a permutation matrix. In general, any monomial matrix can be written in the following way:*

$$M(a_1, \ldots, a_n; i_1, \ldots, i_n) = diag(a_1, \ldots, a_n)P(i_1, \ldots, i_n).$$

The notation for the diagonal matrix is the obvious one: $diag(a_1, \ldots, a_n)$ is the diagonal matrix with (i, i) entry a_i.

We consider the cycle decomposition of a monomial matrix $M(a_1, \ldots, a_n; i_1, \ldots, i_n)$ as the cycle decomposition of the permutation $(i_1, \ldots, i_n)$.

Note. Let $\{e_j, 1 \leq j \leq n\}$ be the standard basis and M be a monomial matrix. Then, $M(e_j) = a_{i_j} e_{i_j}$, where a_{i_j} is the non zero element in the i_j^{th} row, and i_j is the image of j under the considered permutation.

3. Composition of Cohen's map and the standard representation

In this section, we study the image of the composition map

$$\phi(n, k) : B_n \to B_{nk} \to GL_{nk}(\mathbb{Z}[t, t^{-1}]),$$

where the first map is Cohen's map (see Definition 2.5), and the second is the standard representation, ρ (see Definition 2.2).

$\phi(n, k)(\sigma_i) =$

$$= \rho((\sigma_{ki}\sigma_{ki+1} \ldots \sigma_{ki+k-1})(\sigma_{ki-1}\sigma_{ki} \ldots \sigma_{ki+k-2}) \ldots (\sigma_{ki-k+1}\sigma_{ki-k+2} \ldots \sigma_{ki})).$$

Thus, the image of σ_i under $\phi(n, k)$ is a product of k factors of the form

$$\rho(\sigma_{ki-j+1}\sigma_{ki-j+2} \ldots \sigma_{ki-j+k}),$$

starting from $j = 1$ till $j = k$. Therefore, to obtain our result we consider two steps. In the first, we compute $\rho(\sigma_{ki-j+1}\sigma_{ki-j+2} \ldots \sigma_{ki-j+k})$ for each j. In the second, we compute their product. These two steps are represented in the next two propositions. In what follows, let ρ be the standard representation of B_n.

Proposition 3.1. *For any positive integer k, and $1 \leq i \leq n-1$ we have:*

$$\rho(\sigma_{ki-j+1}\sigma_{ki-j+2} \ldots \sigma_{ki-j+k})$$

$$= I_{k(i-1)} \oplus \left(\begin{array}{c|cc|c} I_{k-j} & 0 & & 0 \\ \hline 0 & 0 & t^k & 0 \\ & I_k & 0 & \\ \hline 0 & 0 & & I_{j-1} \end{array}\right) \oplus I_{nk-k(i+1)}$$

for all $1 \leq j \leq k$.

Proof. We use induction to show that, for any natural number m such that $1 \leq m \leq k$,

$$\rho(\sigma_{ki-j+1}\sigma_{ki-j+2} \ldots \sigma_{ki-j+m}) = \rho(\sigma_{ki-j+1})\rho(\sigma_{ki-j+2}) \ldots \rho(\sigma_{ki-j+m})$$

$$= I_{k(i-1)} \oplus \left(\begin{array}{c|cc|c} I_{k-j} & 0 & & 0 \\ \hline 0 & 0 & t^m & 0 \\ & I_m & 0 & \\ \hline 0 & 0 & & I_{k+j-m-1} \end{array}\right) \oplus I_{nk-k(i+1)}.$$

For $m = 1$, it is true, by Definition 2.2. We easily see that

$$\rho(\sigma_{ki-j+1}) = I_{k(i-1)} \oplus \left(\begin{array}{c|c|c} I_{k-j} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & t \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_{k+j-2} \end{array}\right) \oplus I_{nk-k(i+1)}.$$

Assume it is true for $m = r$ with $1 < r < k$. Then, for $m = r + 1$ we have

$$\rho(\sigma_{ki-j+1})\rho(\sigma_{ki-j+2}) \dots \rho(\sigma_{ki-j+r+1}) = (\rho(\sigma_{ki-j+1}) \dots \rho(\sigma_{ki-j+r}))\rho(\sigma_{ki-j+r+1}).$$

By assumption, $\rho(\sigma_{ki-j+1}) \dots \rho(\sigma_{ki-j+r}) =$

$$= I_{k(i-1)} \oplus \left(\begin{array}{c|c|c} I_{k-j} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & t^r \\ I_r & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_{k+j-r-1} \end{array}\right) \oplus I_{nk-k(i+1)}$$

$$= I_{k(i-1)} \oplus \left(\begin{array}{c|c|c} I_{k-j} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & t^r & 0 \\ I_r & 0 & 0 \\ 0 & 0 & 1 \end{matrix} & 0 \\ \hline 0 & 0 & I_{k+j-r-2} \end{array}\right) \oplus I_{nk-k(i+1)}. \tag{1}$$

Again, from Definition 2.2, we have that $\rho(\sigma_{ki-j+r+1})$

$$= I_{k(i-1)} \oplus \left(\begin{array}{c|c|c} I_{k-j+r} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & t \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_{k+j-r-2} \end{array}\right) \oplus I_{nk-k(i+1)}$$

$$= I_{k(i-1)} \oplus \left(\begin{array}{c|c|c} I_{k-j} & 0 & 0 \\ \hline 0 & \begin{matrix} I_r & 0 & 0 \\ 0 & 0 & t \\ 0 & 1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_{k+j-r-2} \end{array}\right) \oplus I_{nk-k(i+1)} \tag{2}$$

Using (1) and (2) we get

$$\rho(\sigma_{ki-j+1})\rho(\sigma_{ki-j+2}) \dots \rho(\sigma_{ki-j+r+1}) = (\rho(\sigma_{ki-j+1}) \dots \rho(\sigma_{ki-j+r}))\rho(\sigma_{ki-j+r+1})$$

$$= I_{k(i-1)} \oplus \left(\begin{array}{c|c|c} I_{k-j} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & 0 & t^{r+1} \\ I_r & 0 & 0 \\ 0 & 1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_{k+j-r-2} \end{array}\right) \oplus I_{nk-k(i+1)}$$

$$= I_{k(i-1)} \oplus \left(\begin{array}{c|c|c} I_{k-j} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & t^{r+1} \\ I_{r+1} & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_{k+j-r-2} \end{array}\right) \oplus I_{nk-k(i+1)}.$$

So, it is true for $m = r + 1$, and thus, it is true for all m such that $1 \leq m \leq k$. □

Proposition 3.2. *Let k be a positive integer, then*

$$\rho((\sigma_{ki}\sigma_{ki+1}\dots\sigma_{ki+k-1})(\sigma_{ki-1}\sigma_{ki}\dots\sigma_{ki+k-2})\dots(\sigma_{ki-k+1}\sigma_{ki-k+2}\dots\sigma_{ki}))$$

$$= I_{k(i-1)} \oplus \begin{pmatrix} 0 & t^k I_k \\ I_k & 0 \end{pmatrix} \oplus I_{nk-k(i+1)} \quad i = 1, 2, \dots, n-1$$

Proof. Let s be the number of consecutive factors of the form $\rho(\sigma_{ki-j+1}\sigma_{ki-j+2}\dots\sigma_{ki-j+k})$. We use induction on s, $1 \leq s \leq k$, to show that

$$\rho((\sigma_{ki}\sigma_{ki+1}\dots\sigma_{ki+k-1})(\sigma_{ki-1}\sigma_{ki}\dots\sigma_{ki+k-2})\dots(\sigma_{ki-(s-1)}\dots\sigma_{ki+k-s}))$$

$$= I_{k(i-1)} \oplus \left(\begin{array}{c|c|c} I_{k-s} & 0 & 0 \\ \hline 0 & 0 & t^k I_s \\ \hline 0 & I_k & 0 \end{array}\right) \oplus I_{nk-k(i+1)}.$$

We show it is true for $s = 1$. Using the result of Proposition 3.1, we have that

$$\rho(\sigma_{ki}\sigma_{ki+1}\dots\sigma_{ki+k-1}) = I_{k(i-1)} \oplus \left(\begin{array}{c|c|c} I_{k-1} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & t^k \\ I_k & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_0 \end{array}\right) \oplus I_{nk-k(i+1)}$$

$$= I_{k(i-1)} \oplus \left(\begin{array}{c|c|c} I_{k-1} & 0 & 0 \\ \hline 0 & 0 & t^k \\ \hline 0 & I_k & 0 \end{array}\right) \oplus I_{nk-k(i+1)}.$$

Assume it is true for $s = r$ where $1 \leq r < k$. We show it is true for $s = r + 1$. By assumption, we have for $s = r$

$$\rho((\sigma_{ki}\sigma_{ki+1}\dots\sigma_{ki+k-1})(\sigma_{ki-1}\sigma_{ki}\dots\sigma_{ki+k-2})\dots(\sigma_{ki-(r-1)}\dots\sigma_{ki+k-r}))$$

$$= I_{k(i-1)} \oplus \left(\begin{array}{c|c|c} I_{k-r} & 0 & 0 \\ \hline 0 & 0 & t^k I_r \\ \hline 0 & I_k & 0 \end{array}\right) \oplus I_{nk-k(i+1)}.$$

Using Proposition 3.1 for $j = r + 1$, $\rho(\sigma_{ki-r}\dots\sigma_{ki+k-(r+1)})$

$$= I_{k(i-1)} \oplus \left(\begin{array}{c|c|c} I_{k-r-1} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & t^k \\ I_k & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_r \end{array}\right) \oplus I_{nk-k(i+1)}.$$

Using direct computations, we obtain that

$$\rho((\sigma_{ki}\sigma_{ki+1}\dots\sigma_{ki+k-1})\dots(\sigma_{ki-(r-1)}\dots\sigma_{ki+k-r})(\sigma_{ki-r}\dots\sigma_{ki+k-(r+1)}))$$

$$= I_{k(i-1)} \oplus \left(\begin{array}{c|c|c} I_{k-r-1} & 0 & 0 \\ \hline 0 & 0 & t^k I_{r+1} \\ \hline 0 & I_k & 0 \end{array}\right) \oplus I_{nk-k(i+1)}.$$

Therefore, it is true for all s such that $1 \leq s \leq k$. □

Using Proposition 3.1 and Proposition 3.2 we state the following lemma.

Lemma 3.3. *Consider the composition map $\phi(n,k) : B_n \to B_{nk} \to GL_{nk}(\mathbb{Z}[t,t^{-1}])$, where the first map is Cohen's map and the second is the standard representation, ρ. The images of the generators of B_n under $\phi(n,k)$, are given by*

$$\phi(n,k)(\sigma_i) = I_{k(i-1)} \oplus \begin{pmatrix} 0 & t^k I_k \\ I_k & 0 \end{pmatrix} \oplus I_{nk-k(i+1)} \quad , \qquad i = 1, 2, \dots, n-1.$$

After obtaining explicitly the images of the generators of B_n under $\phi(n,k)$, we now study the irreducibility of $\phi(n,k)$.

Theorem 3.4. *The linear representation, $\phi(n,k) : B_n \to B_{nk} \to GL_{nk}(\mathbb{Z}[t,t^{-1}])$, obtained by composing Cohen's map with the standard representation is the direct sum of k copies of the standard representation of B_n after replacing t by t^k.*

Proof. We first prove that there exists k non trivial invariant subspaces each of dimension n. Let U_j be the subspace generated by $e_j, e_{j+k}, e_{j+2k}, \dots, e_{j+(n-1)k}$ for $1 \le j \le k$. We show that U_j is an invariant subspace, for all j. Using the result of Lemma 3.3 and the fact that each matrix $\phi(n,k)(\sigma_i)$ is a monomial matrix, we consider the cycle decomposition of each matrix $\phi(n,k)(\sigma_i)$, for $1 \le i \le n-1$, as follows:

$$(1)(2)\dots(k(i-1))$$

$$(k(i-1)+1, ik+1)(k(i-1)+2, ik+2)\dots(ki, ik+k)$$

$$(k(i+1)+1)\dots(kn).$$

Using the above cycle decomposition and the result of Lemma 3.3 and Lemma 2.7, we compute $\phi(n,k)(\sigma_i)(e_{j+lk})$, for $0 \le l \le n-1$.
Since $1 \le j \le k$, it follows that $j + (i-2)k \le k(i-1)$. Then, the images of the first $i-1$ vectors in U_j under $\phi(n,k)(\sigma_i)$ are as follows:

$$\phi(n,k)(\sigma_i)(e_j) = e_j$$

$$\phi(n,k)(\sigma_i)(e_{j+k}) = e_{j+k}$$

$$\vdots$$

$$\phi(n,k)(\sigma_i)(e_{j+(i-2)k}) = e_{j+(i-2)k}.$$

Also, we have that $1 + (i-1)k \le j + (i-1)k \le ki$, and $1 + ik \le j + ik \le k(i+1)$. Then, the images of the vectors $e_{j+(i-1)k}$ and e_{j+ik} are as follows:

$$\phi(n,k)(\sigma_i)(e_{j+(i-1)k}) = e_{j+ik}$$

$$\phi(n,k)(\sigma_i)(e_{j+ik}) = t^k e_{j+(i-1)k}$$

Moreover, $1 + (i+1)k \le j + (i+1)k$, and $j + (n-1)k \le nk$. Then the images of the vectors $e_{j+(i+1)k}$ till $e_{j+(n-1)k}$ are as follows:

$$\phi(n,k)(\sigma_i)(e_{j+(i+1)k}) = e_{j+(i+1)k}$$

$$\vdots$$

$$\phi(n,k)(\sigma_i)(e_{j+(n-1)k}) = e_{j+(n-1)k}.$$

We observe that $\phi(n,k)(\sigma_i)(e_{j+lk}) \in U_j$ for all $0 \le l \le n-1$, which completes the proof .
Second, we consider, U, the new basis formed by the generating vectors of U_j. In other words, consider the new basis

$$U = \{e_1, e_{1+k}, e_{1+2k}, \dots e_{1+(n-1)k}, e_2, e_{2+k} \dots e_{2+(n-1)k}, \dots e_k, e_{2k}, \dots e_{nk}\}.$$

We have $e_{j+lk} \in U_j$, for $0 \leq l \leq n-1$. We computed the vectors $\phi(n,k)(\sigma_i)(e_{j+lk})$, for $1 \leq j \leq k$. Thus, we determine the matrices $\phi(n,k)(\sigma_i)$ with respect to the new basis U. For each j, we obtain a block of the following form:

$$\left(\begin{array}{c|c|c} I_{i-1} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & t^k \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_{n-i-1} \end{array}\right),$$

which is, by Definition 2.2, the standard representation after replacing t by t^k. For $j = 1, \ldots, k$, we then obtain the required result. □

4. Cohen's map and Welded braid group

We now consider the welded braid group. We extend the work done on composing Cohen's map with representations of the braid group to the welded braid group. Using Cohen's map on k strings, we define the map $\psi : wB_n \to wB_{nk}$. We prove ψ is a well defined homomorphism, then we compose it with the standard representation of wB_{nk}.

Proposition 4.1. *For all natural numbers $i \in \{1, \ldots n-1\}$, the map*

$$\psi : wB_n \to wB_{nk} \ \ \textit{defined by:}$$

$$\psi(\sigma_i) = (\sigma_{ki}\sigma_{ki+1}\ldots\sigma_{ki+k-1})(\sigma_{ki-1}\sigma_{ki}\ldots\sigma_{ki+k-2})\ldots(\sigma_{ki-k+1}\sigma_{ki-k+2}\ldots\sigma_{ki})$$

and

$$\psi(\tau_i) = (\tau_{ki}\tau_{ki+1}\ldots\tau_{ki+k-1})(\tau_{ki-1}\tau_{ki}\ldots\tau_{ki+k-2})\ldots(\tau_{ki-k+1}\tau_{ki-k+2}\ldots\tau_{ki})$$

is a well defined homomorphism.

Proof. To show it is a well-defined homomorphism, we prove it respects the relations of wB_n (see Definition2.3) . It is enough to show that $\psi(\tau_i)^2 = 1$ and $\psi(\tau_i)\psi(\sigma_{i+1})\psi(\sigma_i) = \psi(\sigma_{i+1})\psi(\sigma_i)\psi(\tau_{i+1})$.

(i) $\psi(\tau_i)^2 = 1$

Each $\psi(\tau_i)$ is the composition of k factors of which each is of the form

$$\tau_{ki-j+1}\tau_{ki-j+2}\ldots\tau_{ki-j+k},$$

starting from $j = 1$ till $j = k$. Each one of these factors is a composition of k consecutive terms (consecutive generators) itself. In each $\psi(\tau_i)$, for $1 \leq r \leq k$, let f_r be the r^{th} factor appearing in the product of words above. For $1 \leq m \leq k$, let t_m represent the m^{th} term in the factor f_r. Using properties (5) and (3), we compose terms of the second generator $\psi(\tau_i)$ (that comes from the right of $\psi(\tau_i)^2$) with some identical terms in the first $\psi(\tau_i)$ to obtain one. Precisely, starting from f_1 till f_k and for each f_r starting from t_1 till t_k, we compose the term t_m in the factor f_r of the second generator $\psi(\tau_i)$ with the term t_{k-r+1} in the factor f_{k-m+1} of the first $\psi(\tau_i)$ to obtain the desired result.

(ii) $\psi(\tau_i)\psi(\sigma_{i+1})\psi(\sigma_i) = \psi(\sigma_{i+1})\psi(\sigma_i)\psi(\tau_{i+1})$.

$$\begin{aligned}\psi(\tau_i)\psi(\sigma_{i+1})\psi(\sigma_i) = {} & (\tau_{ki}\tau_{ki+1}\ldots\tau_{ki+k-1})\ldots(\tau_{ki-k+1}\tau_{ki-k+2}\ldots\tau_{ki}) \\ & (\sigma_{ki+k}\ldots\sigma_{ki+2k-1})\ldots(\sigma_{ki+2}\ldots\sigma_{ki+k+1})(\sigma_{ki+1}\ldots\sigma_{ki+k}) \\ & (\sigma_{ki}\sigma_{ki+1}\ldots\sigma_{ki+k-1})(\sigma_{ki-1}\sigma_{ki}\ldots\sigma_{ki+k-2})\ldots(\sigma_{ki-k+1}\sigma_{ki-k+2}\ldots\sigma_{ki}).\end{aligned}$$

We show that the last term in $\psi(\tau_i)$, τ_{ki}, becomes τ_{ki+k} after its action on $\psi(\sigma_{i+1})\psi(\sigma_i)$. To do so, we take τ_{ki} the last term in $\psi(\tau_i)$ and σ_{ki} the first term in $\psi(\sigma_i)$, using relations (6) and (1), to obtain $\tau_{ki}\sigma_{ki+1}\sigma_{ki}$. That is,

$$\begin{aligned}\psi(\tau_i)\psi(\sigma_{i+1})\psi(\sigma_i) = {} & (\tau_{ki}\tau_{ki+1}\ldots\tau_{ki+k-1})\ldots(\tau_{ki-k+1}\tau_{ki-k+2}\ldots\tau_{ki-1}) \\ & (\sigma_{ki+k}\ldots\sigma_{ki+2k-1})\ldots(\sigma_{ki+2}\ldots\sigma_{ki+k+1})\ \underbrace{\tau_{ki}(\sigma_{ki+1}\sigma_{ki}}\sigma_{ki+2}\ldots\sigma_{ki+k})\end{aligned}$$

$(\sigma_{ki+1}\sigma_{ki+2}\dots\sigma_{ki+k-1})(\sigma_{ki-1}\sigma_{ki}\dots\sigma_{ki+k-2})\dots(\sigma_{ki-k+1}\sigma_{ki-k+2}\dots\sigma_{ki})$.

Then, by relation (7), we replace $\tau_{ki}\sigma_{ki+1}\sigma_{ki}$ by $\sigma_{ki+1}\sigma_{ki}\tau_{ki+1}$, to obtain the following:

$$\psi(\tau_i)\psi(\sigma_{i+1})\psi(\sigma_i) = (\tau_{ki}\tau_{ki+1}\dots\tau_{ki+k-1})\dots(\tau_{ki-k+1}\tau_{ki-k+2}\dots\tau_{ki-1})$$
$$(\sigma_{ki+k}\dots\sigma_{ki+2k-1})\dots(\sigma_{ki+2}\dots\sigma_{ki+k+1})\ (\underbrace{\sigma_{ki+1}\sigma_{ki}\tau_{ki+1}}\sigma_{ki+2}\dots\sigma_{ki+k})$$
$$(\underbrace{\sigma_{ki+1}}\sigma_{ki+2}\dots\sigma_{ki+k-1})(\sigma_{ki-1}\sigma_{ki}\dots\sigma_{ki+k-2})\dots(\sigma_{ki-k+1}\sigma_{ki-k+2}\dots\sigma_{ki}).$$

By a similar work, we now take σ_{ki+1}, the first term of $\psi(\sigma_i)$. Using relation (1), we now change the place of the first term of $\psi(\sigma_i)$, namely σ_{ki+1}, and write it in the second line of the equation above. That is,

$$\psi(\tau_i)\psi(\sigma_{i+1})\psi(\sigma_i) = (\tau_{ki}\tau_{ki+1}\dots\tau_{ki+k-1})\dots(\tau_{ki-k+1}\tau_{ki-k+2}\dots\tau_{ki-1})$$
$$(\sigma_{ki+k}\dots\sigma_{ki+2k-1})\dots(\sigma_{ki+2}\dots\sigma_{ki+k+1})\ (\sigma_{ki+1}\sigma_{ki}\ \underbrace{\tau_{ki+1}\sigma_{ki+2}\sigma_{ki+1}}\dots\sigma_{ki+k})$$
$$(\sigma_{ki+2}\dots\sigma_{ki+k-1})(\sigma_{ki-1}\sigma_{ki}\dots\sigma_{ki+k-2})\dots(\sigma_{ki-k+1}\sigma_{ki-k+2}\dots\sigma_{ki}).$$

Again by relation (7), we replace $\tau_{ki+1}\sigma_{ki+2}\sigma_{ki+1}$ by $\sigma_{ki+2}\sigma_{ki+1}\tau_{ki+2}$. We repeat the process for a total of k times, we obtain the following :

$$\psi(\tau_i)\psi(\sigma_{i+1})\psi(\sigma_i) = (\tau_{ki}\tau_{ki+1}\dots\tau_{ki+k-1})\dots(\tau_{ki-k+1}\tau_{ki-k+2}\dots\tau_{ki-1})$$
$$(\sigma_{ki+k}\sigma_{ki+k+1}\dots\sigma_{ki+2k-1})\dots(\sigma_{ki+2}\sigma_{ki+3}\dots\sigma_{ki+k+1})$$
$$\sigma_{ki+1}\sigma_{ki}\sigma_{ki+2}\sigma_{ki+1}\sigma_{ki+3}\sigma_{ki+2}\dots\sigma_{ki+k}\sigma_{ki+k-1}\tau_{ki+k}$$
$$(\sigma_{ki-1}\sigma_{ki}\dots\sigma_{ki+k-2})\dots(\sigma_{ki-k+1}\sigma_{ki-k+2}\dots\sigma_{ki}).$$

Using the relations (6) and (1), we rearrange the terms to obtain the desired result.
Similarly, τ_{ki-1} becomes τ_{ki+k-1} after its action on $\psi(\sigma_{i+1})\psi(\sigma_i)$. Following these steps for all the $\tau's$, the proof is completed.

A similar proof is done by replacing the relations (1)(6) and (7) by (3) (6) and (8) respectively, which shows $\psi(\sigma_i)\psi(\tau_{i+1})\psi(\tau_i) = \psi(\tau_{i+1})\psi(\tau_i)\psi(\sigma_{i+1})$. □

We now present the following example to demonstrate the proof (ii) in Proposition4.1.

Example 4.2. For $k = 2$, we consider the map

$$\psi : wB_n \to wB_{2n}$$

defined in Proposition 4.1. We verify, for this particular case, the proof of (ii) in Proposition4.1, namely

$$\psi(\tau_i)\psi(\sigma_{i+1})\psi(\sigma_i) = \psi(\sigma_{i+1})\psi(\sigma_i)\psi(\tau_{i+1}).$$

$$\psi(\tau_i)\psi(\sigma_{i+1})\psi(\sigma_i) =$$
$$= (\tau_{2i}\tau_{2i+1})(\tau_{2i-1}\ \underbrace{\tau_{2i}})(\sigma_{2i+2}\sigma_{2i+3})(\sigma_{2i+1}\sigma_{2i+2})(\underbrace{\sigma_{2i}}\ \sigma_{2i+1})(\sigma_{2i-1}\sigma_{2i}).$$

Rearranging the terms and using relations (6) and (1), we get,

$$\psi(\tau_i)\psi(\sigma_{i+1})\psi(\sigma_i) = \tau_{2i}\tau_{2i+1}\tau_{2i-1}\sigma_{2i+2}\sigma_{2i+3}\,\underbrace{\tau_{2i}\sigma_{2i+1}\sigma_{2i}}\,\sigma_{2i+2}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}$$
$$= \tau_{2i}\tau_{2i+1}\tau_{2i-1}\sigma_{2i+2}\sigma_{2i+3}\,\underbrace{\sigma_{2i+1}\sigma_{2i}\tau_{2i+1}}\,\sigma_{2i+2}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i} \quad \text{(using relation (7))}$$
$$= \tau_{2i}\tau_{2i+1}\tau_{2i-1}\sigma_{2i+2}\sigma_{2i+3}\sigma_{2i+1}\sigma_{2i}\,\underbrace{\tau_{2i+1}\sigma_{2i+2}\sigma_{2i+1}}\,\sigma_{2i-1}\sigma_{2i}$$
$$= \tau_{2i}\tau_{2i+1}\tau_{2i-1}\sigma_{2i+2}\sigma_{2i+3}\sigma_{2i+1}\sigma_{2i}\,\underbrace{\sigma_{2i+2}\sigma_{2i+1}\tau_{2i+2}}\,\sigma_{2i-1}\sigma_{2i} \quad \text{(using relation (7))}$$
$$= \tau_{2i}\tau_{2i+1}\tau_{2i-1}\sigma_{2i+2}\sigma_{2i+3}\sigma_{2i+1}\sigma_{2i+2}\sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}\tau_{2i+2} \qquad \text{(A)}$$

Similarly, we move the term τ_{2i-1} in equation (A) few places and proceed as above to get

$$\tau_{2i}\tau_{2i+1}\sigma_{2i+2}\sigma_{2i+3}\sigma_{2i+1}\sigma_{2i+2}\sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}\tau_{2i+1}\tau_{2i+2}$$

Doing the same for τ_{2i+1}, and τ_{2i} we get

$$\sigma_{2i+2}\sigma_{2i+3}\sigma_{2i+1}\sigma_{2i+2}\sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}\tau_{2i+2}\tau_{2i+3}\tau_{2i+1}\tau_{2i+2}$$
$$= \psi(\sigma_{i+1})\psi(\sigma_i)\psi(\tau_{i+1}).$$

Using this result, we aim to construct a linear representation of wB_n of degree nk. In Section 3, we studied the composition of the Cohen's map with the standard representation of B_n. In a similar way, we compose ψ with the standard representation of wB_n.

Proposition 4.3. *The linear representation,*

$$\alpha(n,k) : wB_n \to wB_{nk} \to GL_{nk}(\mathbb{Z}[t,t^{-1}]),$$

obtained by composing the map ψ with the standard representation of the welded braid group is the direct sum of k copies of the standard representation of wB_n after replacing t by t^k .

Proof. Using Theorem 3.4, we have that the images of the generators σ_i under $\alpha(n,k)$ are the direct sum of k copies of the standard representation of B_n. By substituting $t = 1$ in the matrices corresponding to σ_i the same result holds for the permutation matrices corresponding to τ_i. □

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